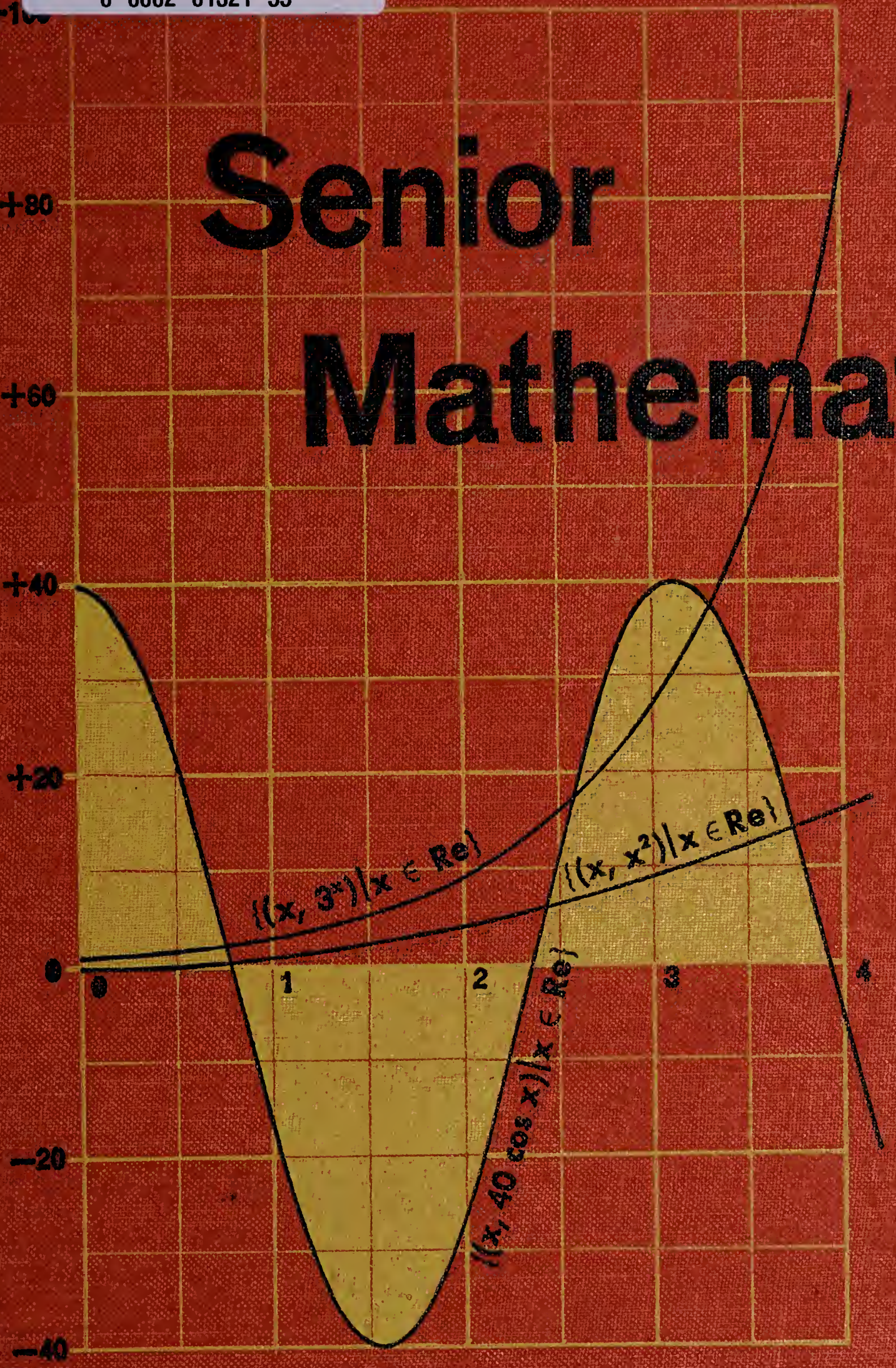
University Of Alberta
0 0002 01321 55

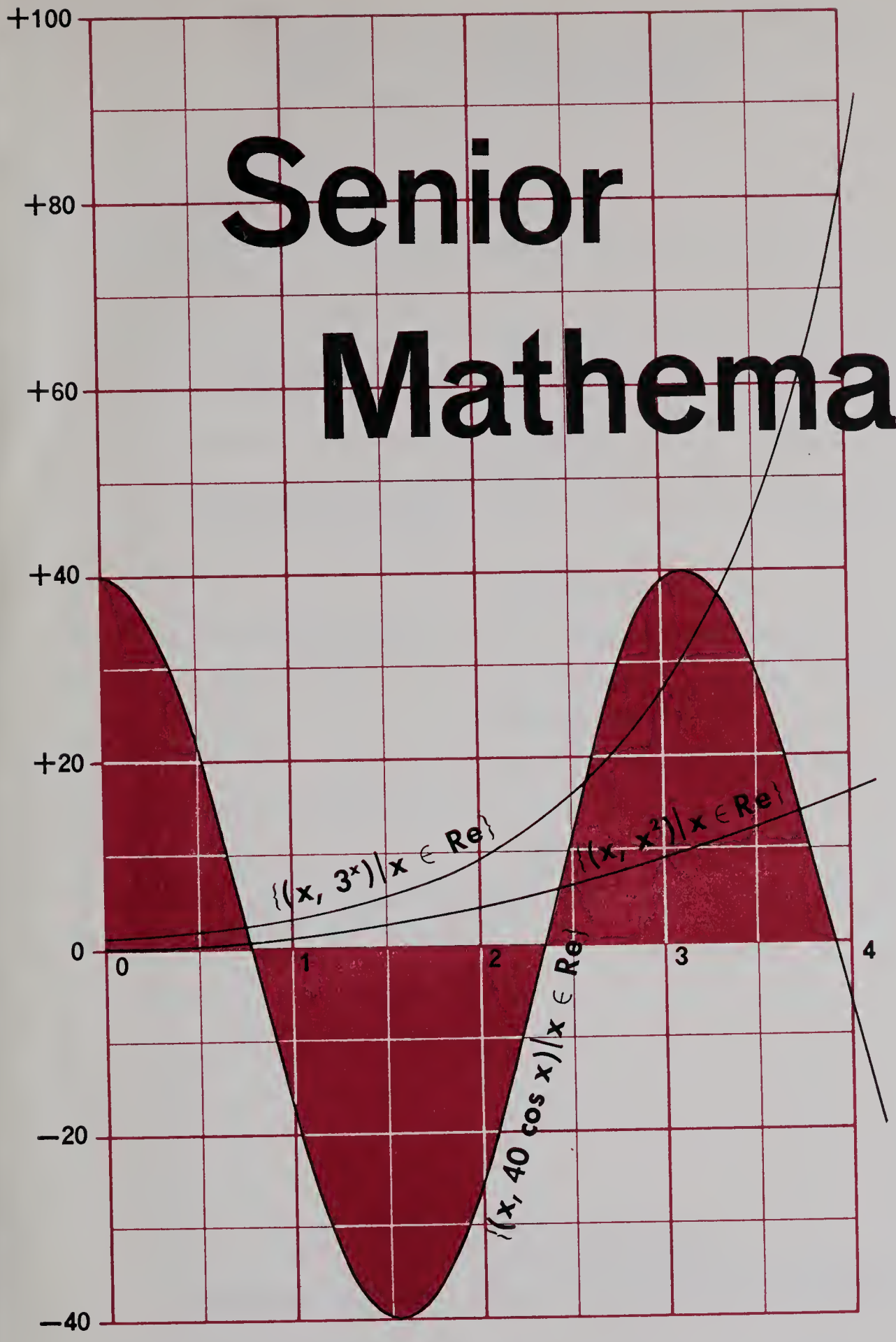
Senior Mathematics

2



ELLIOTT
FRYER
GARDNER
HILL

Ex libris
universitatis
albertaensis



Senior Mathematics

2

H. A. ELLIOTT
K. D. FRYER
J. C. GARDNER
N. J. HILL

HOLT, RINEHART AND WINSTON OF CANADA, LIMITED
TORONTO

Senior Mathematics 2 is the second book in the Holt, Rinehart and Winston Canadian senior mathematics programme.

The authors of the programme are:

H. A. ELLIOTT
Head of Mathematics Dept., Royal Military College, Kingston

J. W. FENCOTT
Head of Mathematics Dept., R. H. King Collegiate, Scarborough

K. D. FRYER
Associate Chairman, Mathematics Dept., University of Waterloo

J. C. GARDNER
Vice-Principal, Woodroffe High School, Ottawa

N. J. HILL
Head of Mathematics Dept., Prince Edward Collegiate, Picton

Copyright © 1965 by
HOLT, RINEHART AND WINSTON OF CANADA, LIMITED

Printed in Canada

1 2 3 4 5 — 69 68 67 66 65

LIBRARY
UNIVERSITY OF ALBERTA

Contents

1. Relations - - - - - - - - - - 1
2. Functions - - - - - - - - - - 31
3. Exponential Functions - - - - - - - - 49
4. Logarithmic Functions - - - - - - - - 77
5. Simple and Compound Interest - - - - - - 109
6. The Quadratic Function - - - - - - - - 123
7. The Quadratic Equation - - - - - - - - 149
8. Polynomial Equations - - - - - - - - 187
9. Elementary Theory of Equations and Complex Numbers - - 211
10. The Circle - - - - - - - - - - 236
11. Analytic Geometry of the Circle - - - - - - 267
12. The Trigonometric Functions - - - - - - - 281
13. The Solution of Triangles - - - - - - - 311
14. Sequences and Series - - - - - - - - 329

Appendix A - - - - - - - - - - 363

Appendix B - - - - - - - - - - 368

Appendix C - - - - - - - - - - 375

Review Tests - - - - - - - - - - 381

Mathematical Tables - - - - - - - - 391

Answers - - - - - - - - - - - 404

Index - - - - - - - - - - - 453

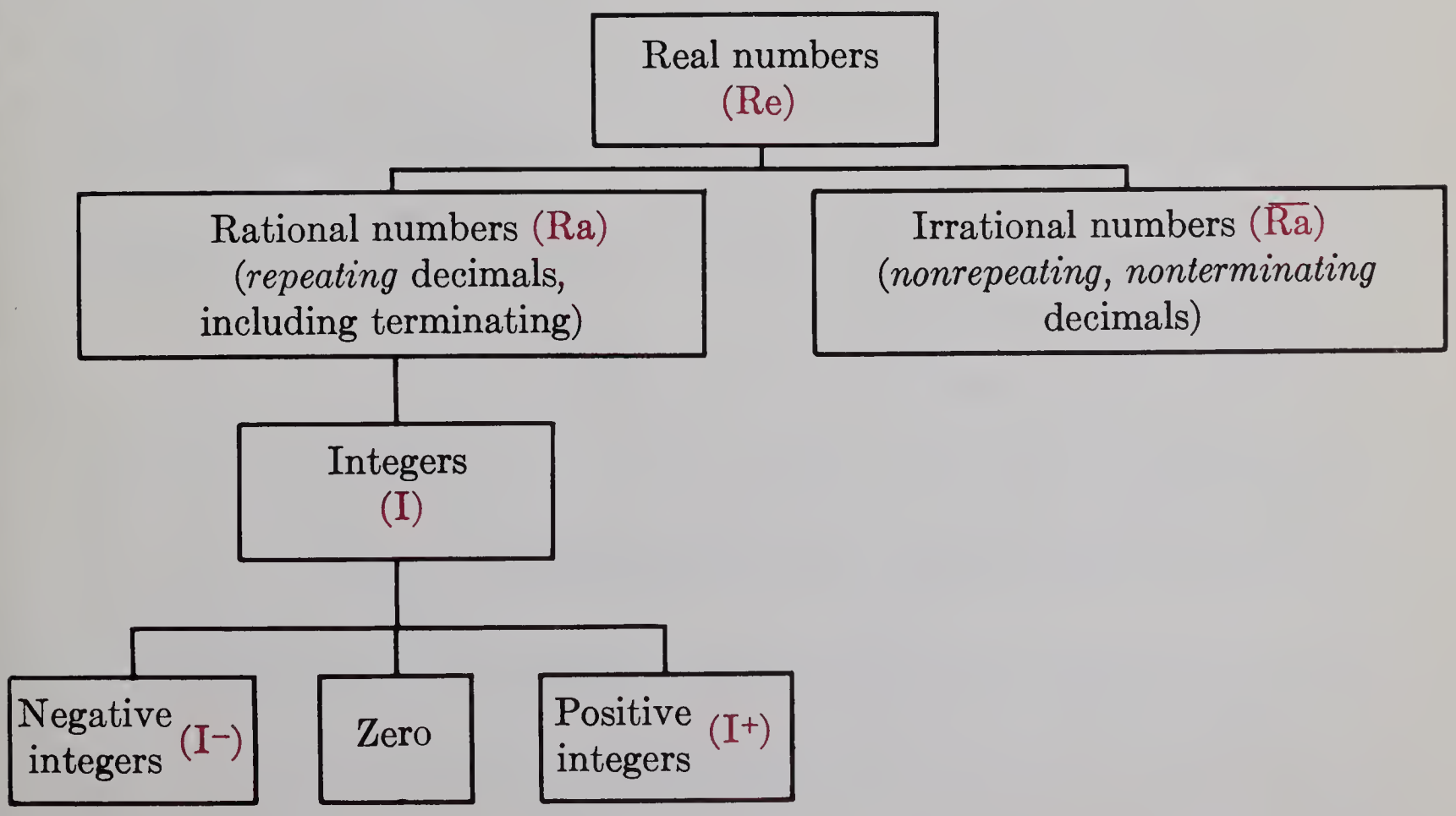
2795566

Chapter 1

Relations

1.1 The Real-Number System

The set of real numbers contains several important subsets as shown in the chart:



If a and b are real numbers, their additive and multiplicative properties are the following:

Property	Addition	Multiplication
Closure	$a+b$ is a real number	ab is a real number
Commutativity	$a+b=b+a$	$ab=ba$
Associativity	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Distributivity	$a(b+c)=ab+ac$	
Identity element	$0\ ;\ a+0=a$	$1\ ;\ a(1)=a$
Inverse element	$-a\ ;\ a+(-a)=0$	$\frac{1}{a}\ ;\ a\left(\frac{1}{a}\right)=1,\ a\neq 0$

EXERCISE 1—1

These questions deal with the set Re and its subsets Ra, $\overline{\text{Ra}}$, and I, as well as the set of natural numbers N and the set of whole numbers W.

1. With respect to the sets listed above, make lists to show which sets do, and which sets do not, possess
   (i) an additive inverse for all elements,
   (ii) a multiplicative inverse for all nonzero elements.
2. Consider the set $\text{E}=\{\ldots -4, -2, 0, +2, +4, \ldots\}$.
   Which of the eleven properties shown in the table are possessed by E?
3. If $\text{O}=\{2n+1 | n\in \text{I}\}$, which of the eleven properties are possessed by O?
4. The equation $x+7=5$ has no solution in N but has a solution in I. Construct an equation with integral coefficients which has no solution in I but has a solution in Ra.
5. Find a rational number between the rationals a and b.
6. Construct an irrational number between the rationals 0.71524 and 0.71525.
7. Write $3.\dot{7}1\dot{2}$ as a rational numeral of the form $\frac{a}{b}$.
8. If α is an irrational number give an indirect proof to show that $\alpha+\frac{a}{b}$ is irrational for all rational numbers $\frac{a}{b}$.

9. If α is irrational, prove that $\frac{a}{b}\times\alpha$ is also irrational for all rationals $\frac{a}{b}$ except one. What is the exception?

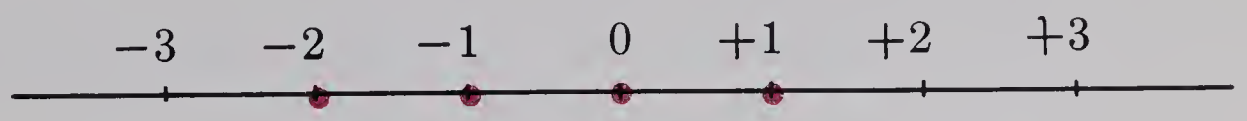
1.2 The Number Line

We have found it convenient to associate members of a set of numbers with points on a line. The numbers are called the *coördinates* of the points.

Example 1

Graph the set of integers which are greater than -3 and less than $+2$.

Solution:



Consider the open sentences

$$2x+5=3,\ x\in \mathrm{I},\ \text{and}\ 10x-4\geq 7,\ x\in \mathrm{Re}.$$

x is a real *variable*, that is, a member of the replacement set (I and Re respectively in these examples.) To *solve* an open sentence is to find the member or members of the replacement set which make the sentence true. If the sentence is an equation such as $2x+5=3$, we say that a value of x that makes the sentence express a true statement is a *root* of the equation and is a member of its *solution set*. Each member of the solution set *satisfies* the equation. Values of x which satisfy an inequality form the solution set of the inequality.

Example 2

Solve $2x+5=3,\ x\in \mathrm{I}$.

Solution:

$$2x+5=3$$
$$\therefore 2x+5-5=3-5 \qquad \text{(explain)}$$
$$2x=-2$$
$$\frac{2x}{2}=\frac{-2}{2} \qquad \text{(explain)}$$
$$x=-1$$

(Some of the steps in this solution and in the solutions for the examples which follow are not usually written out and may be omitted by the student.)

Verification:

$$\text{L.S.}=2(-1)+5 \qquad \text{R.S.}=3$$
$$=-2+5$$
$$=3$$

$\therefore$ the solution set is $\{-1\}$.

Example 3

Find the solution of the inequality $10-4x \geq 7$, $x \in \text{Re}$.

Solution:

$$10-4x \geq 7$$
$$\therefore (-10)+10-4x \geq (-10)+7 \quad \text{(explain)}$$
$$-4x \geq -3$$
$$(-\tfrac{1}{4})(-4x) \leq (-\tfrac{1}{4})(-3) \quad \text{(explain)}$$
$$x \leq \tfrac{3}{4}$$

The solution may be demonstrated graphically.

Example 4

Solve $|3x-2|=4$.

Solution: Recall that the absolute value of x, written $|x|$, has been defined as

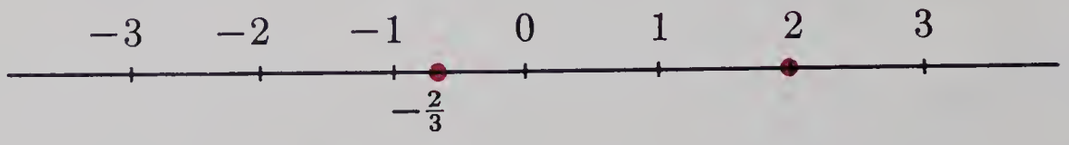
$$|x| = x \text{ if } x \geq 0$$
$$= -x \text{ if } x < 0.$$

Thus $3x-2=4$ or $-(3x-2)=4$.

$\therefore 3x-2=4$ or $3x-2=-4$.

$3x=6$ or $3x=-2$

$x=2$ or $x=-\tfrac{2}{3}$



Example 5

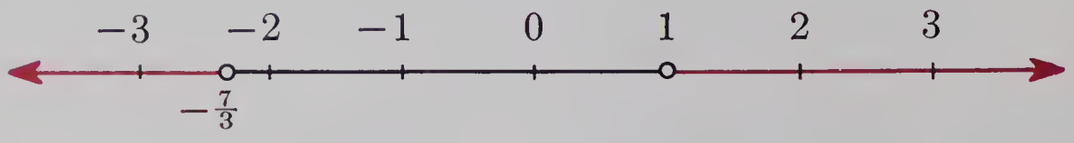
Find $\{x \in \text{Re} \mid |2+3x| > 5\}$.

Solution: Either $2+3x>5$ or $-(2+3x)>5$

$3x>3$ or $-2-3x>5$

$x>1$ or $-3x>7$

$x<-\tfrac{7}{3}$



The solution set is indicated in the diagram, open circles indicating the exclusion of the values 1 and $-\tfrac{7}{3}$.

EXERCISE 1—2

(Unless otherwise indicated, the variables are members of Re.)

1. Solve and verify:

   (a) $7x-5=37$ (b) $5(x-1)=3(x-4)$

   (c) $3(2x-1)^2=4(x-2)(3x+5)$ (d) $5(3x+2)^2=3(15x^2-4)$

   (e) $3[2-(x+5)]=7(x-6)$ (f) $3(x-2)(x+2)=(3x-7)(x+2)$

   Graph the solution set in each case.

2. Graph the set $\{x|2x-3>-1\}$.

3. Find the real numbers that satisfy both inequalities

   $$2-3x<1 \text{ and } 2-3x>-1.$$

   Note that these inequalities may be combined and written as

   $$-1<2-3x<1.$$

4. Find x so that

   (a) $-2<2x+5<7$ (b) $-3<9-3x<6$

   (c) $-5\leq 1-2x\leq 9$ (d) $7>3x-2>13.$

5. Graph the solution sets in (4a) and (4b).

6. Find the solution set for each of the following:

   (a) $|x+2|=5$ (b) $|3x-1|=2$

   (c) $|7-2x|=1$ (d) $|5-4x|=-3$

   (e) $|-2x+5|<1$ (f) $|x|=2$ and $|x|<3$

7. Graph the solution set in each part of (6).

8. Find the sets of real numbers satisfying these inequalities:

   (a) $|4-3x|<7$ (b) $|12-5x|\leq 17$

   (c) $|3-4x|<-9$ (d) $|4a+1|\geq 3$

   (e) $|-3a+7|>10$ (f) $|x-5|>-2x$

9. Graph the following sets:

   (a) $\{x|3x<2 \text{ and } 5x+1>-9\}$

   (b) $\{x|5-3x\geq -4 \text{ or } 7-2x\leq -3\}$

   (c) $\{x|5x<1 \text{ and } |3x+1|<-2\}$

   (d) $\{x|4x>9 \text{ or } 4x<9\}$

   (e) $\{x|3x+7>4 \text{ and } 5-x<3+3x\}$

   (f) $\{x|2+3x<5 \text{ and } |x|=3\}$

1-3 Equations Involving Fractional Coefficients

To solve a new problem, mathematicians often attempt to alter the problem so that it resembles a problem already solved. Because the real numbers are closed with respect to multiplication, we may multiply each member of an equation by the lowest common denominator of any fractions involved as coefficients. This produces an equivalent equation in which the coefficients are integral and we proceed as in Section 1-2.

Example 1

Solve for x: $\frac{3x}{5}-\frac{x}{7}=\frac{16}{7}$, $x\in \mathrm{Re}$.

Solution: Since $\frac{3x}{5}-\frac{x}{7}=\frac{16}{7}$,

$$35\left(\frac{3x}{5}-\frac{x}{7}\right)=35\left(\frac{16}{7}\right)$$

$$21x-5x=80 \qquad \text{(explain)}$$

$$16x=80 \qquad \text{(explain)}$$

$$x=5.$$

Example 2

Find the subset of real numbers that are members of the set

$$\left\{x \;\middle|\; \left|\frac{3x}{4}-\frac{x-1}{3}\right|=2\right\}.$$

Solution:

$$\left|\frac{3x}{4}-\frac{x-1}{3}\right|=2$$

$$\therefore \quad \frac{3x}{4}-\frac{x-1}{3}=2 \quad \text{or} \quad -\frac{3x}{4}+\frac{x-1}{3}=2.$$

$$12\left(\frac{3x}{4}-\frac{x-1}{3}\right)=12(2) \quad \text{or} \quad 12\left(-\frac{3x}{4}+\frac{x-1}{3}\right)=12(2)$$

$$9x-4x+4=24 \quad \text{or} \quad -9x+4x-4=24$$

$$5x=20 \quad \text{or} \quad -5x=28$$

$$x=4 \quad \text{or} \quad x=-\frac{28}{5}$$

The solution set is $\{4, -\frac{28}{5}\}$.

EXERCISE 1—3

Solve the following:

1. $\frac{x}{3}+\frac{x}{4}=7$

2. $\frac{3x}{2}=x+8$

3. $\frac{m+5}{6}-\frac{m+3}{4}=\frac{m-1}{9}$ (Verify.)

4. $2-\frac{r+2}{10}=\frac{r-1}{5}$

5. $\frac{x-6}{4}-\frac{x+5}{3}<\frac{x-13}{5}$ (Graph the solution set.)

6. $\left|\frac{x-5}{13}\right|=5$

7. $\left|\frac{4x}{5}-\frac{2x-7}{7}\right|=2$

8. Find the value of x so that the product $(x-7)(x+6)$ exceeds $(x-5)(x+3)$ by $\frac{1}{2}$.

9. Ken can wash a car in 1 hour and Jim can wash it in 40 minutes. How long will it take if they work together?

1 4 Equations Involving Literal Coefficients

When we say that, if $3x=12$,

then $x=4$,

we are multiplying both $3x$ and 12 by $\frac{1}{3}$ and using the facts that each real number (except zero) has a multiplicative inverse and the real numbers are closed with respect to multiplication.

To discuss the general first degree equation

$$ax+b=0 \quad (a,\ x,\ b\in \mathrm{Re}),$$

we must consider the following cases:

(1) $a,\ b\neq 0$

Solving for x, we have $ax=-b$

$$x=-\frac{b}{a}.$$

The equation has *one root.*

(2) $a\neq 0,\ b=0$

The equation is $ax=0.$

$$\therefore x=\frac{0}{a}.$$

$$x=0$$

The equation has *one root.*

(3) $a=0,\ b\neq 0$

The equation is $0x=b.$

There is no real x which satisfies this equation.
$\therefore$ the equation has *no real roots.*

(4) $a=b=0$

The equation is $0x=0.$

Any real x satisfies this equation; x is *indeterminate.*
The equation has Re as solution set but *no unique solution.*

Example 1

Solve for x: $cx+d=e\ (c\neq 0).$

Solution:

If $cx+d=e,$
then $cx=e-d.$ (explain)
$\therefore x=\dfrac{e-d}{c}.$ (explain)

Example 2

Solve for w: $p=2(l+w).$

Solution:

If $p=2(l+w),$
then $p=2l+2w.$
$2w=p-2l$
$\therefore w=\dfrac{p-2l}{2}.$

Example 3

Solve for W: $s=\dfrac{\mathrm{W}}{\mathrm{W}-w}\quad(\mathrm{W}\neq w).$

Solution:

If $s=\dfrac{\mathrm{W}}{\mathrm{W}-w},$
then $s(\mathrm{W}-w)=\mathrm{W}.$
$s\mathrm{W}-sw=\mathrm{W}$
$s\mathrm{W}-\mathrm{W}=sw$
$\mathrm{W}(s-1)=sw$ (explain)
$\therefore \mathrm{W}=\dfrac{sw}{s-1}.$ (explain)

What real number must be excluded as a value for s?

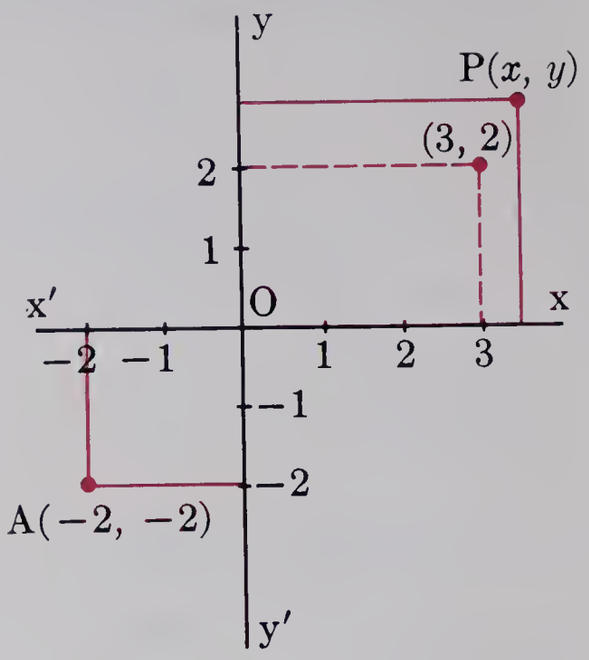
EXERCISE 1—4

A literal equation that represents some scientific law is called a *formula*. Solve the following formulae for the variable indicated:

1. $d=vt$ (t)
2. $\mathrm{A}=lw$ (w)
3. $c=\pi d$ (π)
4. $i=prt$ (r)
5. $v=u+at$ (u)
6. $v=u+at$ (a)
7. $s=\frac{1}{2}at^2$ (a)
8. $\mathrm{A}=\pi r^2$ (r)
9. $s=\frac{n}{2}(a+l)$ (l)
10. $\mathrm{A}=\pi r^2 h$ (h)
11. $\mathrm{V}=\frac{4}{3}\pi r^3$ (r)
12. $\mathrm{V}=\frac{1}{3}\pi r^2 h$ (h)
13. Solve $\mathrm{C}=\frac{5}{9}(\mathrm{F}-32)$ for F. If C represents a reading on the Centigrade scale of 20°, find the corresponding Fahrenheit reading F.
14. If $\mathrm{I}=\frac{n\mathrm{E}}{\mathrm{R}+nr}$, solve for n.
15. If $y=5x+7$, solve for x.
16. If $5x+3y=15$, solve for y.
17. If $ax+by=c$, solve for y.
18. Given that $\frac{1}{\mathrm{R}}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$, where R is the total resistance of three resistances r_1, r_2, and r_3 in parallel in an electric circuit, find R if $r_1=2$, $r_2=6$, and $r_3=3$.
19. In (18), if $\mathrm{R}=3$, $r_1=6$, and $r_2=24$, find the value of r_3.
20. In (18), if $r_1=12$, $r_2=8$, and $r_3=36$, find the value of R.
21. If $y-5=6(x+2)$, solve for x.
22. If $px_1+qx_2=r$, solve for x_1.

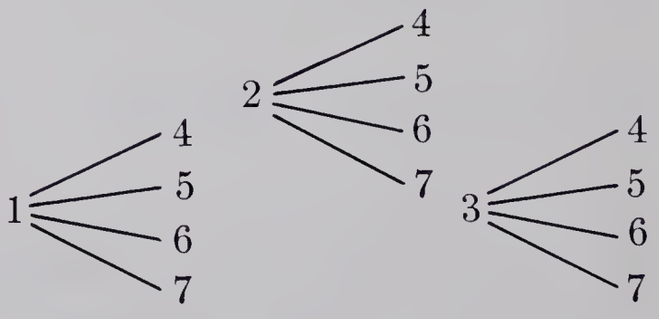
15 Cartesian Coördinates: Cartesian Product Sets

You will recall that if a second number line is drawn at right angles to the original line through the zero point or *origin*, each point in the plane may be located by its directed distances from these *number lines*, xOx′ and yOy′, called axes. The point (+3, +2) is found by travelling 3 units to the right from the origin along the x-axis and then 2 units upward parallel to the y-axis; +3 and +2 are the coördinates of the point. The x coördinate is called the *abscissa* and the y coördinate is referred to as the *ordinate* of the point. To avoid confusion, the abscissa is always written first, so that the coördinates of a point always form an *ordered pair*.



We sometimes speak rather casually of "the point (−2, −2)" when we mean "the point whose coördinates are (−2, −2)." It is also permissible to speak of the point A(−2, −2). Generally, *any* point in the plane may be represented by P(x, y). The coördinates and the plane itself are described as *Cartesian* in honour of René Descartes, a 17th-century French mathematician.

Now consider sets A = {1, 2, 3} and B = {4, 5, 6, 7}. Let us form the set $\{(x, y) | x \in A, y \in B\}$. This is the set of all ordered pairs such that the first element x is chosen from A and the second element y from B. The diagram shows how the pairs may be determined.

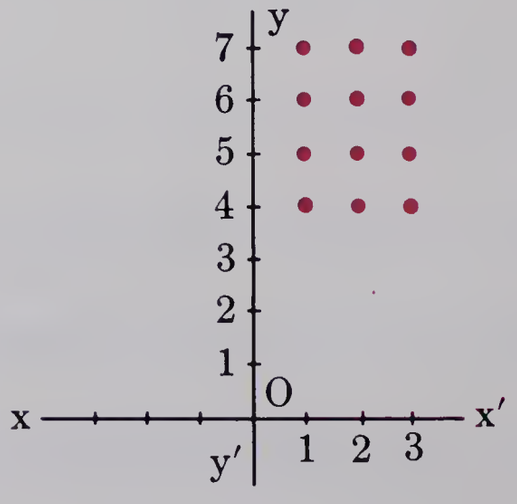


The twelve possible pairs form the set

{(1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7)}.

This set is called the *Cartesian Product* of sets A and B and is denoted by $A \times B$ (read "A cross B").

The graph of A×B is shown at the right.



$A \times B$ is a subset of $I \times I$, which in turn is a subset of $Re \times Re$, the set of all ordered pairs of real numbers. The graph of $Re \times Re$ is the set of all points in the plane; that is, *there is a one-to-one correspondence between the elements of $Re \times Re$ and the points in the Cartesian plane.*

We normally consider points with coördinates chosen for ease in plotting (often integral). Note that this is only for convenience. It would be awkward (but theoretically possible) to locate the points with coördinates $(.75175117, .232332)$ or $(\sqrt{5},\ 2-3\sqrt{3})$.

Notice that the Cartesian product of sets differs in at least one important respect from the product of real numbers. This "multiplication" of sets is not commutative. Consider $B \times A$ when A and B are as above. The number of elements remains the same but not the pairs themselves, since their components are interchanged.

EXERCISE 1—5

1. Find $A \times B$ if
   (a) $A = \{-2, 0, 2\}$ and $B = \{3, 4, 5\}$,
   (b) $A = \{-3, -2, -1\}$ and $B = \{-1, 0, 1\}$,
   (c) $A = \{1, 2, 3\}$ and $B = \{2, -2, 3, -3\}$,
   (d) $A = \{4\}$ and $B = \{1, 2, 3, 4, 5\}$,
   (e) $A = \{2, 4, 6, 8\}$ and $B = \{-3, 0, 3\}$.
2. Graph the sets in each part of (1).
3. Form the set $A \times A$ in (1b). Graph $A \times A$.
4. Find $\{(x, y) | x \in S, y \in T\}$ if
   (a) $S = \{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $T = \{0, 2\}$,
   (b) $S = \{a, b, c\}$ and $T = \{c, d, e, f\}$,
   (c) $S = \{x \in N | x^2 < 25\}$ and $T = \{x \in N | 3x < 12\}$,
   (d) $S = \{x \in I | -1 < x < 3\}$ and $T = \{x \in I |\ |x| \leq 2\}$.

   Draw the graph of $S \times T$ in (c).
5. Under what conditions does $(x, y) = (y, x)$? Does $\{x, y\} = \{y, x\}$?
6. What is $\phi \times \phi$?
7. Find $\{(x, y) | x \in S, y \in T\}$ if $S = \{2, 4, 6\}$ and $T = \{3, 5, 7\}$. From $S \times T$ select the subset of pairs for which $y = x + 1$.
8. Graph $B \times A$ for parts (a) and (b) of (1). Compare your graphs with those in (2).

16 Binary Relations

Frequently we make statements involving *two* people, such as

"X is more attractive than Y."

Many ordered pairs may be determined by this defining sentence, for example: (Sue, Sally) [Sue is more attractive than Sally]; (Sally, Eva); (Eva, Ruby); (Ruby, Ringo); etc.

In mathematics it is equally common to associate pairs of numbers by a sentence such as "$y=x+2$". This determines an infinite set of ordered pairs $\{(-2, 0), (-1, 1), (0, 2), \ldots\}$. Such a connection between two numbers is called a *binary relation*. If the equation (or inequality) is linear (first degree in x and y), the relation is described as a linear relation. The linear relation determined by $y=x+2$ may be denoted by

$\{(x, y)|y=x+2; x, y \in \mathrm{I}\}$, that is, by the set of ordered pairs.

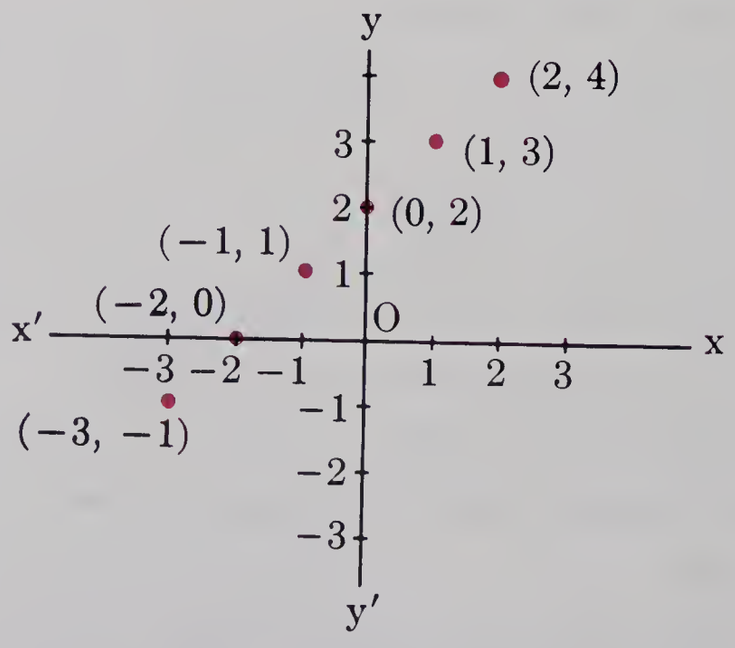
Notice that three things are involved:

(1) There is a set (in this case I) to which x belongs.
There is a set (in this case I) in which y is determined.

(2) There is a defining rule, in this case the equation $y=x+2$.

(3) There is a set of ordered pairs determined by the defining rule.

The graph of a relation is the set of points whose coördinates are given by the set of ordered pairs in the relation. A partial graph of the relation defined by $y=x+2$ $(x, y \in \mathrm{I})$ is shown.



Partial graph of relation defined by $y=x+2$ $(x, y \in I)$

Note: This relation involves a set of ordered pairs which is a subset of $\mathrm{I}\times\mathrm{I}$.

A relation determines a subset of a Cartesian product.

Example 1

If $A=\{1, 2\}$ and $B=\{3, 4, 5\}$, find $\{(x, y)|y=2x+1;\ x\in A,\ y\in B\}$.

Solution: The set of ordered pairs in this relation is $\{(1, 3), (2, 5)\}$. This set is a subset of the six pairs in $A\times B$.

The *domain* of a relation is the set of all first members of the ordered pairs in the relation.

The *range* of a relation is the set of all second members of the ordered pairs in the relation.

In Example 1, the domain is $\{1, 2\}$ and the range is $\{3, 5\}$.

Example 2

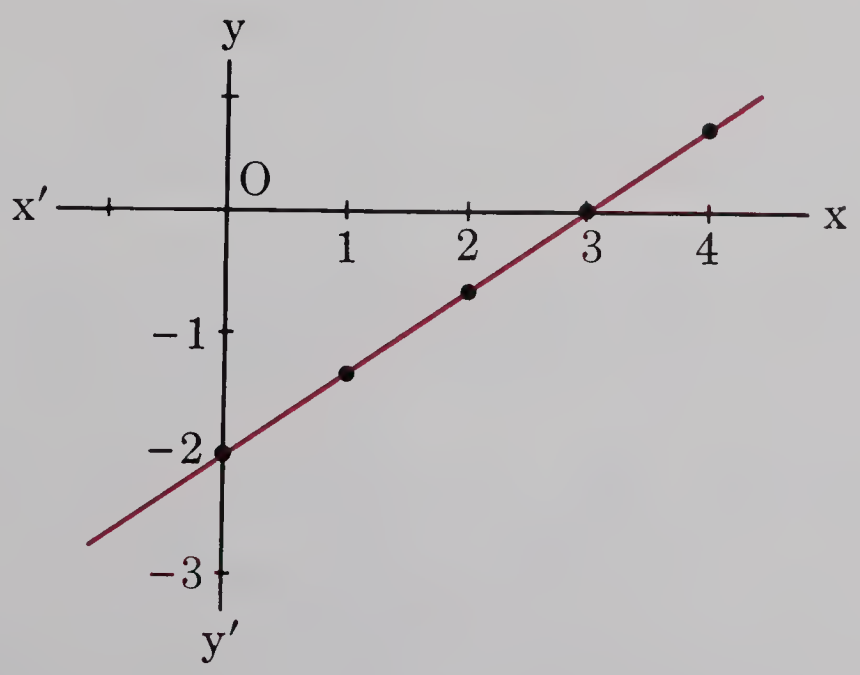
Draw the graph of the relation

$$S=\{(x, y)|2x-3y=6;\ x, y\in \mathrm{Re}\}.$$

Solution: Since $2x-3y=6$, therefore $y=\dfrac{2x-6}{3}$. The following partial table of values may be calculated for x and y:

x	0	1	2	3	4
$y=\dfrac{2x-6}{3}$	-2	$-\frac{4}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$

Points with coördinates $(0, -2)$, $(1, -\frac{4}{3})$, etc., are now plotted. Since x and y are real numbers, any number of intermediate points may be plotted so that it seems reasonable to join the points we have plotted. The line segment formed is a partial graph. The graph appears to be a line extending indefinitely in both directions. The domain and range of the relation are each Re.



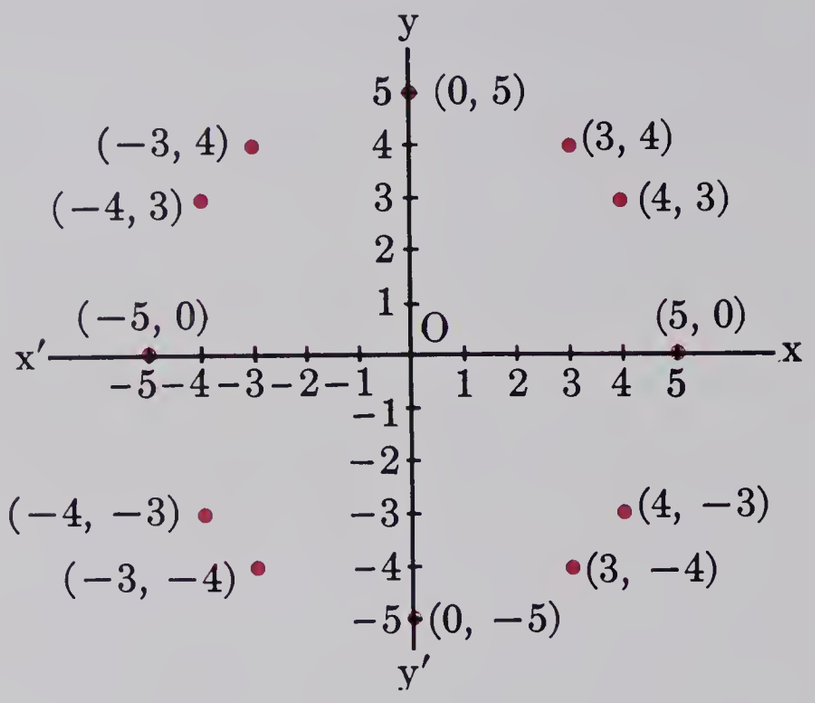
Graph of the relation defined by $2x-3y=6$

Example 3

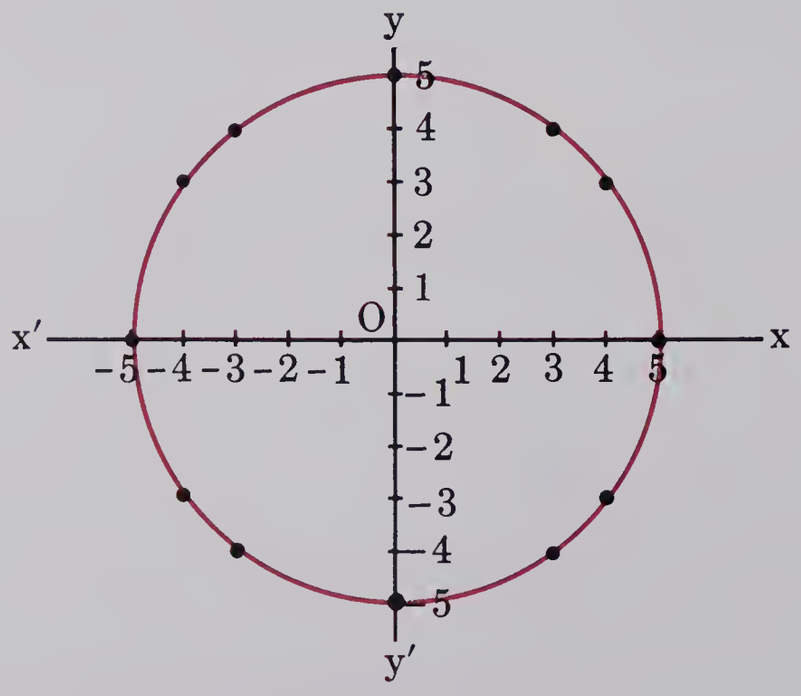
Graph the relation defined by

$$x^2+y^2=25 \quad (x, y \in \mathrm{Re}).$$

Solution: Certain integral values for x and y are fairly obvious and we obtain the following graph which is symmetric with respect to both coördinate axes.



As in Example 2, since x and y are real numbers, intermediate points may be plotted, and it seems reasonable to join the above 12 points by a smooth curve. The *complete* graph appears to be a circle with centre at the origin and radius 5.



EXERCISE 1—6

(All variables represent real numbers unless otherwise stated.)

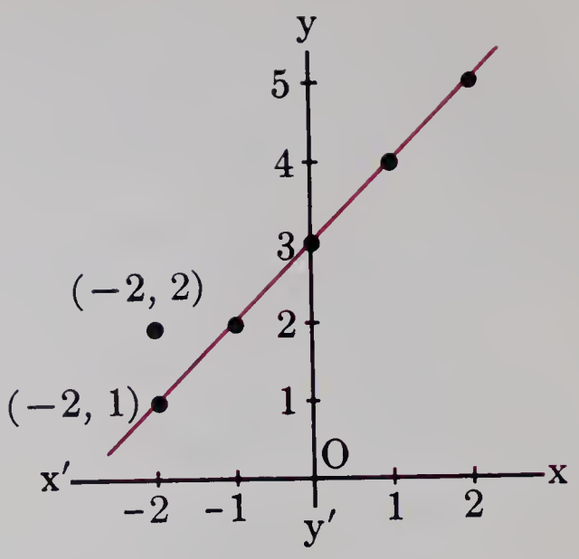
1. Draw the graph of the binary relation defined by
   (a) $y=-3$ (b) $x=5$ (c) $y=0$ (d) $x=0$.
2. State the domain and range for each relation in (1).
3. Given the relation defined by $y=3x+2$.
   (a) What is the domain?
   (b) What is the range?
   (c) What is the value of x when $y=0$? What point on the graph is thus determined?
   (d) What is the value of y when $x=0$? What point on the graph is thus determined?
   (e) Draw a partial graph as in Example 2 of this section.
4. Given the relation denoted by $\{(x, y)|3x+5y=15\}$, determine which of the following ordered pairs are members of the relation:
   (a) $(1, 2)$ (b) $(0, 3)$ (c) $(-5, 6)$ (d) $(\frac{8}{3}, \frac{7}{5})$
5. Draw the graph of $\{(a, b)|b=3a-1\}$.
6. Draw the graph of $\{(x, y)|y=2x+3; x, y\in \mathrm{I}\}$.
7. Graph the relations defined by
   (a) $y=|x|$ (b) $x^2+y^2=4$ (c) $|x+y|=5$ (d) $xy=0$.
8. Draw the graph of $\{(x, y)|x^2+y^2=36; x, y\in \mathrm{Re}\}$.
9. Repeat (8) with the added restriction $|x|\leq 4$.
10. Graph the relation in $\mathrm{I}\times\mathrm{I}$ denoted by the following:
    $\{(x, y)|y=x^2; -3\leq x\leq 3\}$.
11. Graph the relation defined by $y=x^3$ where $x\in \mathrm{Re}$ and $|x|\leq 3$.
12. Draw a partial graph of the relation defined by $y=\frac{1}{x}$, $x, y\in \mathrm{Re}$.
13. (a) Obtain a partial graph of S in $\mathrm{Re}\times\mathrm{Re}$ where $\mathrm{S}=\{(x, y)|y=x^2\}$.
    (b) By shading, indicate on the graph the region defined by
    (i) $y>x^2$, (ii) $y<x^2$.
14. Graph the relation which determines the set
    $\{(x, y)|x^2+y^2\leq 25; x\geq 0, y\geq 0, x, y\in \mathrm{Re}\}$.

17 Equations of Straight Lines

The equation $y=x+3$ $(x, y \in \text{Re})$ determines an infinite set of ordered pairs, including

$(-1, 2), (0, 3), (1, 4), (2, 5), \ldots$

When these points are plotted as shown, it then seems reasonable to join them, since x and y are real numbers and an unlimited number of intermediate points can be found. The graph or rather the *partial graph* obtained appears to be a line. $y=x+3$ is called the *equation of the line.*



Notice that the point $(-2, 1)$ has coördinates that satisfy the equation; $(-2, 1)$ is *on* the line.

$(-2, 2)$ has coördinates that do not satisfy the equation; $(-2, 2)$ is *not* on the line.

Example 1

Draw the graph defined by $3x+2y=8$, $x \in \text{Re}$.

Solution:

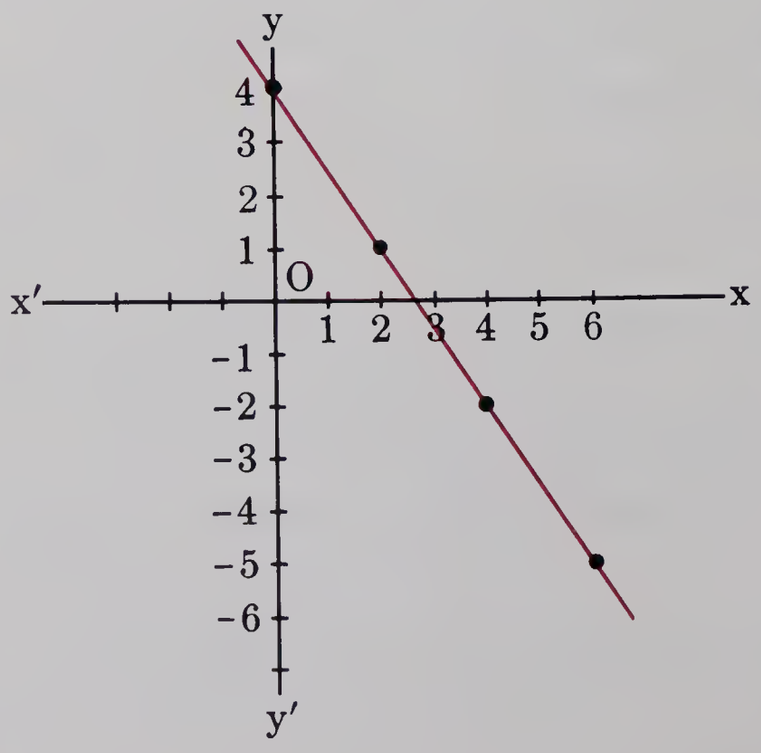
If $3x+2y=8$,

then $2y=8-3x$.

$y=4-\frac{3}{2}x$

We now assign arbitrary values to x, calculate the corresponding values of y, and then plot the points:

x	$4-\frac{3}{2}x$
2	1
4	−2
6	−5



We assume the truth of the following statement:

All points (x, y) with coördinates that satisfy an equation of the form $ax+by=c$ $(a, b, c \in \text{Re})$ lie on a line.

$ax+by=c$, the equation of the line, is called a *linear equation* in x and y. The real variables x and y occur to no more than the first power and the graph is a line. This suggests an alternative solution for Example 1:

Let $x=0$ in $3x+2y=8$ and solve for y:

$$2y=8$$
$$\therefore\ y=4.$$

Then the point with coördinates $(0, 4)$ is on the line. This value 4 for y is called the *y-intercept* of the line.

Now let $y=0$ in $3x+2y=8$ and solve for x:

$$3x=8$$
$$\therefore\ x=\tfrac{8}{3}.$$

Then $(\frac{8}{3}, 0)$ is on the line. The line may now be drawn and a third point may be used as a check. The value $\frac{8}{3}$ for x is called the *x-intercept* of the line.

EXERCISE 1—7

(All variables belong to Re.)

1. Find the x- and y-intercepts of the line defined by $5x+3y=15$. Draw the line. Test to see if the points with the following coördinates are on the line:
   (a) $(6, -5)$ (b) $(-6, 15)$ (c) $(2, 3)$ (d) $(5, -3)$
2. Draw the graphs of the relations defined by the following:
   (a) $2x+3y=-4$ (b) $3x+8y=16$ (c) $x+y=2$
   (d) $4x-3y=12$ (e) $y=3x-5$ (f) $y=3x$
3. Draw the following lines, using one pair of axes:
   (a) $2x+y=6$ (b) $2x+y=2$ (c) $2x+y=4$ (d) $2x+y=0$
   These lines belong to the same *family of lines*. How are they related? $2x+y=k$ or $2x+y+c=0$ represents all of these lines and is called the general equation of this family of lines.
4. Repeat (3) for the following family of lines:
   (a) $3x-4y=-12$ (b) $3x-4y=0$
   (c) $3x-4y=6$ (d) $3x-4y=12$
5. Sketch the family of lines whose equation is $4x+3y=k$. What relationship appears to exist between these lines and those in (4)?
6. Is it possible to find points with integral coördinates on the lines represented by (a) $4x+4y=7$? (b) $5x-15y=2$? Explain.

18 The Intersection of Straight Lines: Intersection Sets of Linear Relations

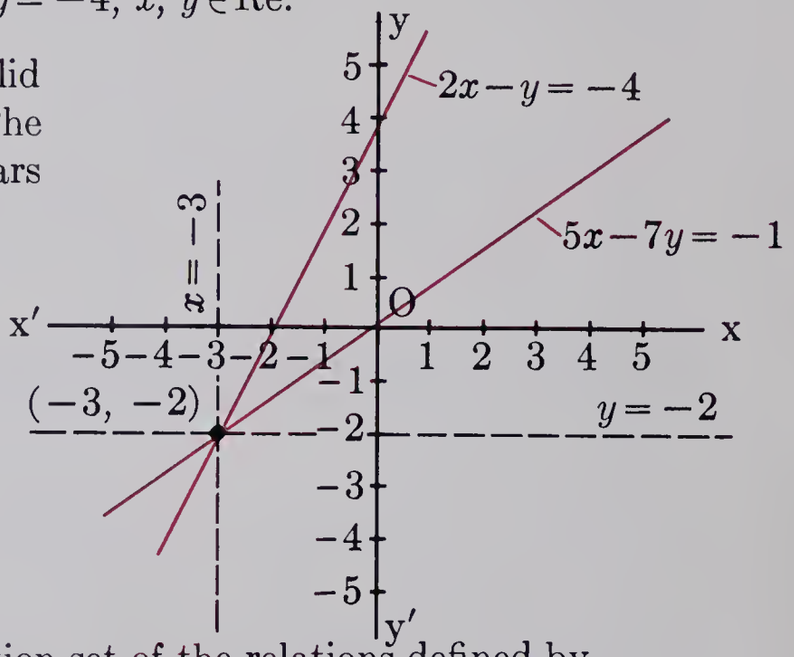
The solution set of a real linear equation is the set of ordered pairs (x, y) that satisfy the equation and that are the coördinates of points on the line defined by the equation.

This set of ordered pairs is the set determined by the linear relation defined by the equation concerned. The intersection of the solution sets of two linear equations is the pair (x, y) common to both sets. In Examples 1, 2, and 3, we consider a pair of equations representing two distinct non-parallel lines. In this case, the equations are said to be *consistent*, and there is only one element, a unique ordered pair (x, y), in the intersection set determined by the two linear relations concerned; x and y are the coördinates of the point of intersection.

Example 1

Find graphically the point of intersection of the lines whose equations are $5x-7y=-1$ and $2x-y=-4$, $x, y \in \text{Re}$.

Solution: The lines are drawn as solid lines in the diagram. The point of intersection appears to be $(-3, -2)$.



Example 2

Find algebraically the intersection set of the relations defined by

$$5x-7y=-1 \qquad (1)$$
$$2x-y=-4; \qquad (2)$$

that is, find $\{(x, y)|5x-7y=-1\} \cap \{(x, y)|2x-y=-4\}$, $x, y \in \text{Re}$.

Solution: *Method 1*—Substitution.

From equation (2) we have $y=2x+4$. (3)
Any point of this line has coördinates of the form $(x, 2x+4)$. Since we are looking for a point common to both lines, these coördinates must also satisfy equation (1).

Replacing y by $2x+4$ in (1), we have

$$5x-7(2x+4)=-1.$$
$$\therefore\ 5x-14x-28=-1.$$
$$\therefore\ -9x=27.$$
$$\therefore\ x=-3. \qquad (4)$$

Replacing x by -3 in (3), we have

$$y=-6+4.$$
$$\therefore\ y=-2. \qquad (5)$$

Thus the point of intersection of the two lines concerned has coördinates $(-3, -2)$,
or the required solution set is $\{(-3, -2)\}$.

Note: Equations (4) and (5) have the same solution set as (1) and (2). We say that the system (4) and (5) is *equivalent* to the system (1) and (2). Replacing (1) and (2) by (4) and (5) is equivalent geometrically to the replacement of the original lines by lines parallel to the axes through the point of intersection $(-3, -2)$. These lines are those defined by $x=-3$ and $y=-2$ and are shown as broken lines in the diagram.

Method 2—Elimination.

Since $2x-y=-4$, (2)
$\therefore\ 7(2x-y)=7(-4)$ (explain)
or $14x-7y=-28$. (6)

Now consider the equivalent system

$14x-7y=-28$ (6)
and $5x-7y=-1$. (1)

Subtracting,

*(6)$-$(1) $9x=-27$ (Re is closed under subtraction)
or $x=-3$. (4)

Finally, we have another equivalent system

$x=-3$ (4)
and $2x-y=-4$. (2)

Substituting -3 for x in (2),

$$-6-y=-4.$$
$$\therefore\ -y=2$$
$$\text{or}\qquad y=-2.$$

$\therefore$ the solution set is $\{(-3, -2)\}$.

We have eliminated the variable y in the above example. The solution is ordinarily written down as in Example 3.

*We do not subtract *equations*. (6)$-$(1) is a symbolic representation for the fact that we subtract the real numbers equated in (1) from the corresponding real numbers in (6).

Example 3

Find the solution set for the equations $3x-2y=5$
and $4x-3y=12$, $x, y \in \text{Re}$.

Solution:

	$3x-2y=5$	(1)
	$4x-3y=12$	(2)
*(1)$\times 3$	$9x-6y=15$	(3)
(2)$\times 2$	$8x-6y=24$	(4)
(3)$-$(4)	$x=-9$	(5)

Substitute -9 for x in (1):

$\therefore \ -27-2y=5.$

$\therefore \ -2y=32.$

$\therefore \ y=-16.$ (6)

Verification:

in (1)	in (2)
L.S. $=3(-9)-2(-16)$	L.S. $=4(-9)-3(-16)$
$=-27+32$	$=-36+48$
$=5$	$=12$
R.S. $=5$	R.S. $=12$

The solution set is $\{(-9, -16)\}$.

Note: The identifying numbers (5) and (6) would not ordinarily be needed but have been used here so that we may draw attention to the fact that (1) and (2), (3) and (4), (5) and (6), are all examples of the equivalent systems which may be found in this example. Can you name other equivalent systems from these six equations?

Example 4

Discuss the solution set of $3x-2y=5$ (1)
and $6x-4y=10$ (2), where $x, y \in \text{Re}$.

Solution: $6x-4y=10$ may be written as

$$2(3x-2y)=2(5).$$

Therefore any pair (x, y) satisfying equation (1) must also satisfy equation (2). These equations represent the same line. The solution set has an infinite number of elements (x, y) and the equations are described as *dependent*.

Generally, equations of the form $ax+by+c=0$
and $kax+kby+kc=0 \quad (k \neq 0)$
are dependent.

*We do not multiply *equations*. See footnote in Example 2.

Example 5

Find the solution set for the equations

$$4x-3y=12$$
$$\text{and } 4x-3y=24.$$

Solution: Any pair of real numbers x and y, such that $4x-3y=12$, cannot also be such that $4x-3y=24$; hence there is no solution for this system, that is

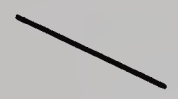
$$\{(x, y)|4x-3y=12\}\cap\{(x, y)|4x-3y=24\}=\phi.$$

The equations are *inconsistent.* Plotting, we obtain parallel lines; that is, the lines have no point in common.

Generally, two equations of the form $ax+by+c=0$ and $kax+kby+d=0$ are inconsistent if $d\neq kc$.

If $d=kc$, then the equations are dependent.

Summary

Systems of Two Linear Equations		
Graph	*Equations*	*Solution Set*
Two distinct intersecting lines	Consistent	Contains one pair (x, y)
 Two coincident lines	Dependent	Contains an infinite number of pairs (x, y)
Two parallel lines	Inconsistent	ϕ

EXERCISE 1—8

(Unless otherwise stated, the variables are members of Re.)

Classify the following systems as consistent, inconsistent, or dependent. State the geometrical interpretation in each case:

1. $x+y=3$, $x+y=5$
2. $x-y=3$, $2x-2y=6$
3. $3x+y=7$, $6x-2y=14$
4. $2x-7y=1$, $2x-y=5$
5. $x+2y=4$, $3x-6y=8$
6. $x-y-7=0$, $-x+y=-7$
7. For what value of k are the equations $a-3b=4$ and $ka-3kb=-8$ dependent?

In the following, use the method of substitution to solve the systems that are consistent:

8. $x+y=2$, $2x-y=7$
9. $x-y=0$, $3x-y=5$
10. $2x-3y=7$, $2x-3y=3$
11. $x+y=4$, $3x+y=7$
12. $7a+8b=9$, $a+5b=9$
13. $x-3y=5$, $2x+y=12$
14. $x-5y=0$, $3x-y=14$
15. $3x-y=6$, $6x-2y=12$
16. $8m+5n=21$, $m+n=3$
17. $m+3n=4$, $2m+6n=1$
18. Solve graphically the system in (12).

In the following, use the method of elimination to solve the systems that are consistent:

19. $x+2y=14$, $3x+2y=22$
20. $2a+b=19$, $4a+3b=17$
21. $5x-6y=0$, $10x-12y=2$
22. $x-y=4$, $\frac{x}{2}-\frac{y}{3}=0$
23. $\frac{2x}{5}+\frac{3y}{8}=\frac{7}{2}$, $\frac{x}{5}-\frac{y}{8}=\frac{1}{2}$
24. $\frac{x}{5}-\frac{y}{3}=1$, $\frac{x}{5}+\frac{y}{3}=3$
25. $3x+5y=15$, $\frac{2x}{5}-\frac{y}{3}=2$
26. $\frac{3x}{2}-\frac{4y}{5}=2$, $7x-\frac{y-2}{5}=27$
27. $4a+3b=25$, $\frac{3a+5b}{7}=3\frac{6}{7}$
28. $4x-3y=12$, $x-\frac{3y}{4}=3$
29. Construct a graphical solution for the system in (22).
30. Solve for x and y: $\frac{x+5}{3}+\frac{y+3}{2}=7$, $\frac{2x-3}{5}-\frac{y+4}{3}=-2$.
31. Solve for x and y: $\frac{1}{x}+\frac{1}{y}=4$, $\frac{1}{x}-\frac{1}{y}=8$.

    Hint: Solve for $\frac{1}{x}$ and $\frac{1}{y}$, then x and y.
32. Solve for x and y: $\frac{3}{x}-\frac{5}{y}=-4$, $\frac{6}{x}+\frac{25}{y}=57$.
33. For what rational value of k will the system $3x+7y=k$, $6x+14y=20$, be dependent?
34. For what rational values of k will the following system have no solution?

    $3x-y=2$, $15x-5y=k$
35. For what rational values of k will the system $4x+2y=5$, $32y=80+kx$, have no solution?

1 9 Systems of Linear Equations With Literal Coefficients

As in Section 1-4, it will be necessary to place certain restrictions on the coefficients so that the problem of division by zero will not be allowed to rear its forbidden head!

Example 1

Solve the system $ax+by=m$
$cx+dy=n,$

for x and y.

Solution:

	$ax+by=m$	(1)
	$cx+dy=n$	(2)
$(1)\times d$	$adx+bdy=dm$	(3)
$(2)\times b$	$bcx+bdy=bn$	(4)
$(3)-(4)$	$adx-bcx=dm-bn$	
or	$(ad-bc)x=dm-bn$	(explain)
	$x=\dfrac{dm-bn}{ad-bc}$	(explain)

This is the value of x, subject to the condition that $ad-bc\neq 0$. To find y, it is probably easier to begin again and eliminate x:

$(1)\times c$	$acx+bcy=cm$	(3)
$(2)\times a$	$acx+ady=an$	(4)
$(3)-(4)$	$(bc-ad)y=cm-an$	
	$y=\dfrac{cm-an}{bc-ad}$	
or	$y=\dfrac{an-cm}{ad-bc}$	(if $ad-bc\neq 0$)

EXERCISE 1–9

Solve the following systems of equations; in each case, state any necessary restrictions on the coefficients: (All variables belong to Re.)

1. $px+qy=2$
   $px-qy=6$

2. $ax+by=k$
   $x+y=0$

3. $mx+ny=1$
   $px-qy=2$

4. $3ax+2by=2$
   $ax-3by=8$

5. $ax-by=a$
   $ax-y=ab$

6. $x+a_1y=b_1$
   $x-a_2y=b_2$

7. $ax+by=a^2+b^2$
   $x+y=a+b$

8. $\dfrac{x}{m}+\dfrac{y}{n}=a, \quad \dfrac{x}{m}-\dfrac{y}{n}=b$

9. $\dfrac{x}{y}=\dfrac{p}{q}, \quad \dfrac{x+1}{y+1}=\dfrac{p+1}{q+1}$

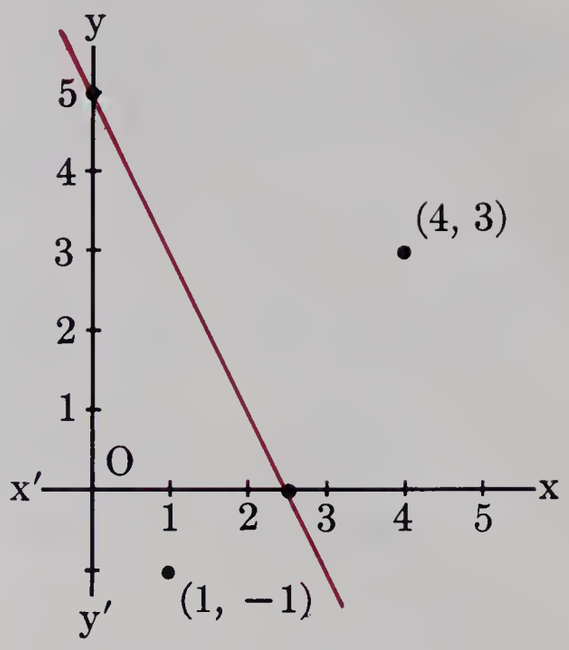
10. $\dfrac{b}{x-a}=\dfrac{d}{y-c}, \quad x+y=e$

Example 1

Draw the graph of the relation in Re×Re denoted by

$$P=\{(x, y)|2x+y\geq 5\}.$$

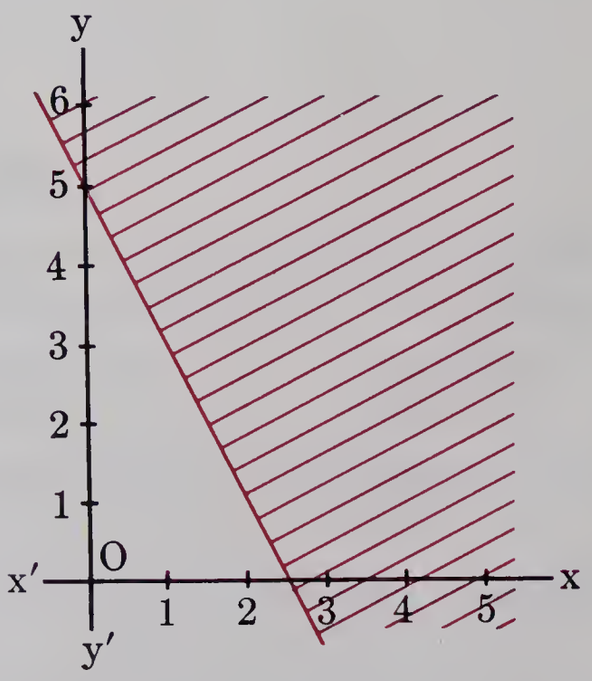
Solution: Consider the equation $2x+y=5$, obtained from the rule of formation for the relation denoted by P. Setting $y=0$ and $x=0$ in turn, we obtain the x-intercept $\frac{5}{2}$ and the y-intercept 5 respectively. The graph must contain the points $(\frac{5}{2}, 0)$ and $(0, 5)$. The line whose equation is $2x+y=5$, or $y=5-2x$, is drawn as shown. The coördinates of each pointon the line are of the form $(x, 5-2x)$.



Graph defined by $2x+y=5$

Now if $2x+y>5$,
then $y>5-2x$.

Consider a single point (4, 3) in the *half plane* to the right of the line. Here $5-2x$ has the value $5-8=-3$. Since $3>-3$, then $y>5-2x$ at this point. A similar result will be obtained for any point in the right half plane. The graph of the region defined by $2x+y\geq 5$ includes the points in the *closed half plane* consisting of the line $2x+y=5$ and all points to the right of it. (Use the point $(1, -1)$ to verify that points in the left open half plane are those for which $2x+y<5$ or $y<5-2x$.)



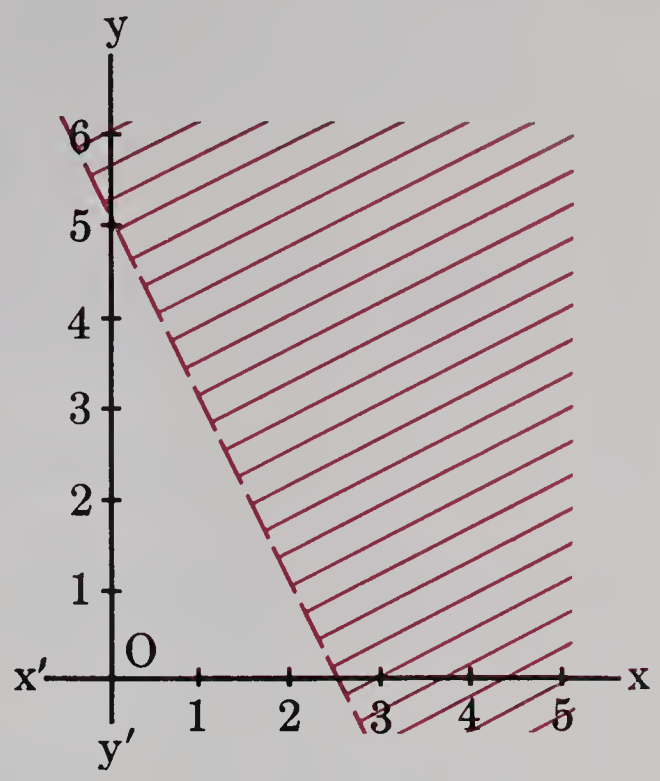
Graph of $2x+y\geq 5$

Example 2

Draw the graph of the relation in Re×Re denoted by

$$Q=\{(x, y)|2x+y>5\}.$$

Solution: This problem differs from Example 1 in that equality is now excluded. To indicate that the line $2x+y=5$ is *not* part of the graph, it is customary to use a broken line as in the diagram.



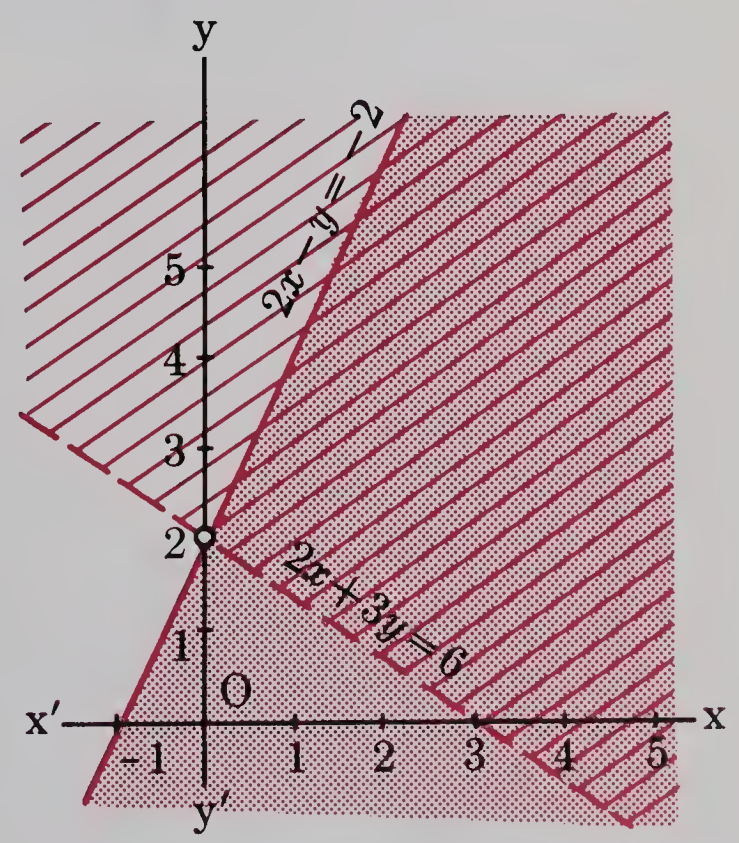
Graph of $2x+y>5$

Example 3

Draw the graph of the relation denoted by

$$S=\{(x, y)|2x+3y>6 \text{ and } 2x-y\geq -2\}.$$

Solution: Draw the lines represented by the equalities $2x+3y=6$ (broken line) and $2x-y=-2$ (solid line). The region required is that portion of the plane to the right of each line and including the part of the boundary defined by $2x-y=-2$. This is the region covered by the crosshatching in the diagram. A small circle indicates that the point (0, 2) is excluded. Why?



Graph of $2x+3y>6$ and $2x-y\geq -2$

Example 4

Graph the region defined by $x^2+y^2>25$.

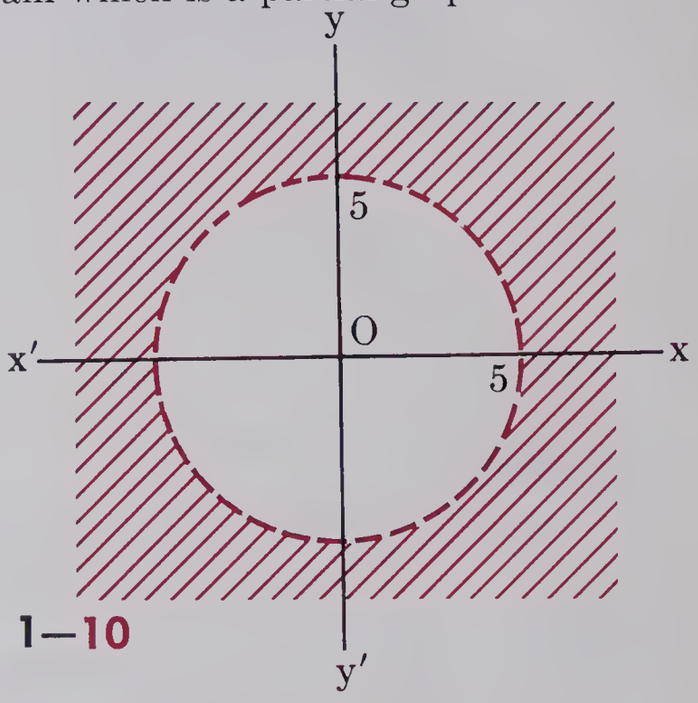
Solution: In Example 3 (page 14), we found that $x^2+y^2=25$ defined a circle with centre (0, 0) and radius 5. This circle divides the entire plane into 3 regions: (1) all points *inside* the circle ($x^2+y^2<25$),

(2) all points *on* the circle ($x^2+y^2=25$),

(3) all points *outside* the circle ($x^2+y^2>25$).

To show that the circle itself is excluded, we use a broken boundary line in the following diagram which is a partial graph.

Graph of $x^2+y^2>25$



EXERCISE 1—10

(Unless otherwise stated, all variables belong to Re.)

1. With respect to the diagram, describe the following regions:

   (a) $x>0$

   (b) $y<0$

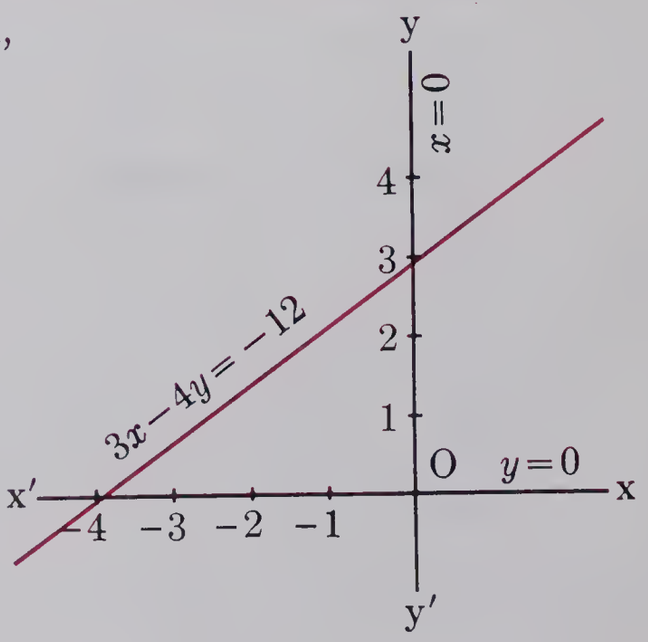
   (c) $3x-4y>-12$

   (d) $3x-4y\leq-12$

   (e) $3x-4y>-12$, $x<0$

   (f) $3x-4y<-12$, $y<0$

   (g) $3x-4y>-12$, $x<0$, $y>0$



2. Draw the graphs of the relations in Re$\times$Re denoted by the following:

$$T=\{(x, y)|2x+5y>10\}$$
$$U=\{(a, b)|3a-b\leq 6\}$$
$$V=\{(m, n)|5m-3n\geq 15\}$$
$$W=\{(x, y)|3x+2y>6,\ x>0\}$$
$$X=\{(x, y)|x+y=2,\ x>0,\ y>0\}$$

3. Draw the graph of
$$\{(x, y)|3x+2y>6\}\cap\{(x, y)|x>1\}.$$

4. Draw the graph of
$$\{(x, y)|3x-2y<5\}\cup\{(x, y)|3x+2y>13\}.$$

5. Draw the graph of
$$\{(x, y)|3x-2y>5\}\cap\{(x, y)|3x+2y<13\}\cap\{(x, y)|y>0\}.$$

6. Is there a region defined by
$3x+2y>13,\ x<0,\ y<0$? Explain.

7. Graph the relation defined by $2x+3y-6=0,\ x>0,\ y>0$. State the intercepts, domain, and range.

8. Graph the quadratic (or second degree) relations defined by
(a) $x^2+y^2\leq 25$, (b) $x^2+y^2\leq 25,\ x\geq 0$.

9. Draw the graph of $\{(x, y)|y\geq x^2\}$.

10. Draw the graph of $\{(x, y)|y\geq x^2;\ x\geq 0\}$.

11. Draw the graph of $\{(x, y)|y\geq x^2\}\cap\{(x, y)|y<2x\}\cap\{(x, y)|x\geq 0\}$.

12. Draw the graph of $\{(x, y)|x^2+y^2<25\}\cap\{(x, y)|x^2+y^2>16\}$.

13. Draw the graph of $\{(x, y)|y\leq\frac{1}{x};\ x, y>0\}$.

14. Draw the graph of $\{(x, y)|y^2<x;\ x, y>0\}$.

15. Draw the graph of $\{(x, y)|y^2<x\}\cap\{(x, y)|x^2+y^2<25\}$.

16. Draw the graph of $\{(x, y)|y>\frac{8}{x};\ x>0\}\cap\{(x, y)|x^2+y^2<25\}$.

17. Draw the graph of $\{(x, y)|y\geq x^2\}\cap\{(x, y)|y^2\leq x\}$.

18. Draw the graph of $\{(x, y)|y=x^2-4;\ |x|\leq 3\}$.

1.11 Chapter Summary

Properties of the Real Numbers		
Property	Addition	Multiplication
Closure	$a+b$ is a real number	ab is a real number
Commutativity	$a+b=b+a$	$ab=ba$
Associativity	$a+(b+c)=(a+b+c)$	$a(bc)=(ab)c$
Distributivity	$a(b+c)=ab+ac$	
Identity element	$0;\ a+0=a$	$1;\ a(1)=a$
Inverse element	$-a;\ a+(-a)=0$	$\frac{1}{a};\ a\left(\frac{1}{a}\right)=1,\ a\neq 0$

The number line. Solution of the linear equation $ax+b=0$. Graph of the solution set on the number line. Conditions for unique solution of $ax+b=0$.

Cartesian coördinates, ordered pairs. The Cartesian product $A\times B$.

Binary relations. Graph of a binary relation. Domain and range of a relation. Binary relation as a subset of a Cartesian product.

The straight line as the graph of a linear relation or as the graph of the set of ordered pairs defined by $ax+by+c=0$.

Intersection of straight lines. Graphical solution. Algebraic solution by substitution or elimination. Systems of equivalent equations. Consistent, dependent, and inconsistent systems of linear equations. (Finite, infinite, and null intersection sets of the linear relations defined by the equations.)

Relations defined by inequalities. Compound relations and their graphs.

REVIEW EXERCISE 1—11

(Unless otherwise stated the variables are members of Re.)

1. Consider the set $M=\{3n|n\in I\}$. Which of the eleven properties of the real numbers are possessed by the set M?

2. Graph the set $\{x|5-4x<-3\}$.

3. Graph the solution set of the inequality $|3-2x|>7$.

4. Solve for x and verify: $3ax+b=2b-ax$.

5. If $v^2=u^2+2as$, solve for a.

6. If $P=\{2, 4, 6\}$ and $Q=\{3, 5, 7\}$, list the members of
   (a) $P\times Q$ (b) $Q\times P$.

7. Sketch the family of lines represented by $x-2y+k=0$.

8. Solve for x and y (any method):
$$5x-2y=23$$
$$3x+7y=-19.$$

9. Solve for x and y: $\frac{3x}{5}-\frac{4y}{3}=-9$
$$\frac{3x}{2}-\frac{7y}{3}=-21.$$

10. Solve for a and b in terms of m and n:
$$ma-nb=9$$
$$na-mb=3.$$

11. If $A=\{1, 2, 3, 4, 5\}$, list the ordered pairs of the relation in $A\times A$ defined by $y=2x-1$. List the domain and range of the relation.

12. Draw a partial graph of the set
$$\{(x, y)|y=-x;\ x, y\in I\}.$$

13. Draw a partial graph of the relation whose rule of formation is
$$y=|x|.$$

14. Indicate on the graph in (13) the region containing points with coördinates (x, y) such that $y>|x|$.

15. Draw the graph of the relation whose rule of formation is
$$x^2+y^2=2\tfrac{1}{4}.$$

16. Repeat (15) if the defining rule is
$$9x^2+9y^2=4.$$

17. Use the results of (15) and (16) to construct the graph of the region defined by $x^2+y^2<2\frac{1}{4}$ and $9x^2+9y^2>4$.

18. Graph the relations denoted by
    (a) $\{(x, y)|3x-4y=12;\ x, y \in \text{Re}\}$
    (b) $\{(a, b)|b=2a+3;\ a, b \in \text{I},\ |a| \leq 3\}$.

19. Graph the relation $\{(x, y)|x-y>1\} \cap \{(x, y)|x-y<5\}$ in $\text{Re} \times \text{Re}$.

20. Graph the region determined by
    $y<2x^2,\ x>0,\ x \in \text{Re}$.

21. Graph the intersection set of
    $\{(x, y)|2x+5y>10\}$ and $\{(x, y)|x^2+y^2<25\}$.

22. Graph the set
    $\{(x, y)|xy=8;\ x, y \in \text{Re},\ x<0,\ y<0\}$.

23. Construct the graph of the region defined by
    $y<(x-1)(x-4),\ -1 \leq x \leq 6$.

24. Draw the graph of the relation which involves the intersection set
    $\{(x, y)|2x+3y \leq 6\} \cap \{(x, y)|4x+y \leq 8\} \cap \{(x, y)|x \geq 0\} \cap \{(x, y)|y \geq 0\}$.

25. On the boundary of, or in the region graphed in, (24), find the point (x, y) whose coördinates appear to offer the maximum value of the expression $x+4y$.
    Hint: Sketch the family of lines $x+4y=k$ and note the member of this family with greatest value of k that intersects the region in (24).

Chapter 2

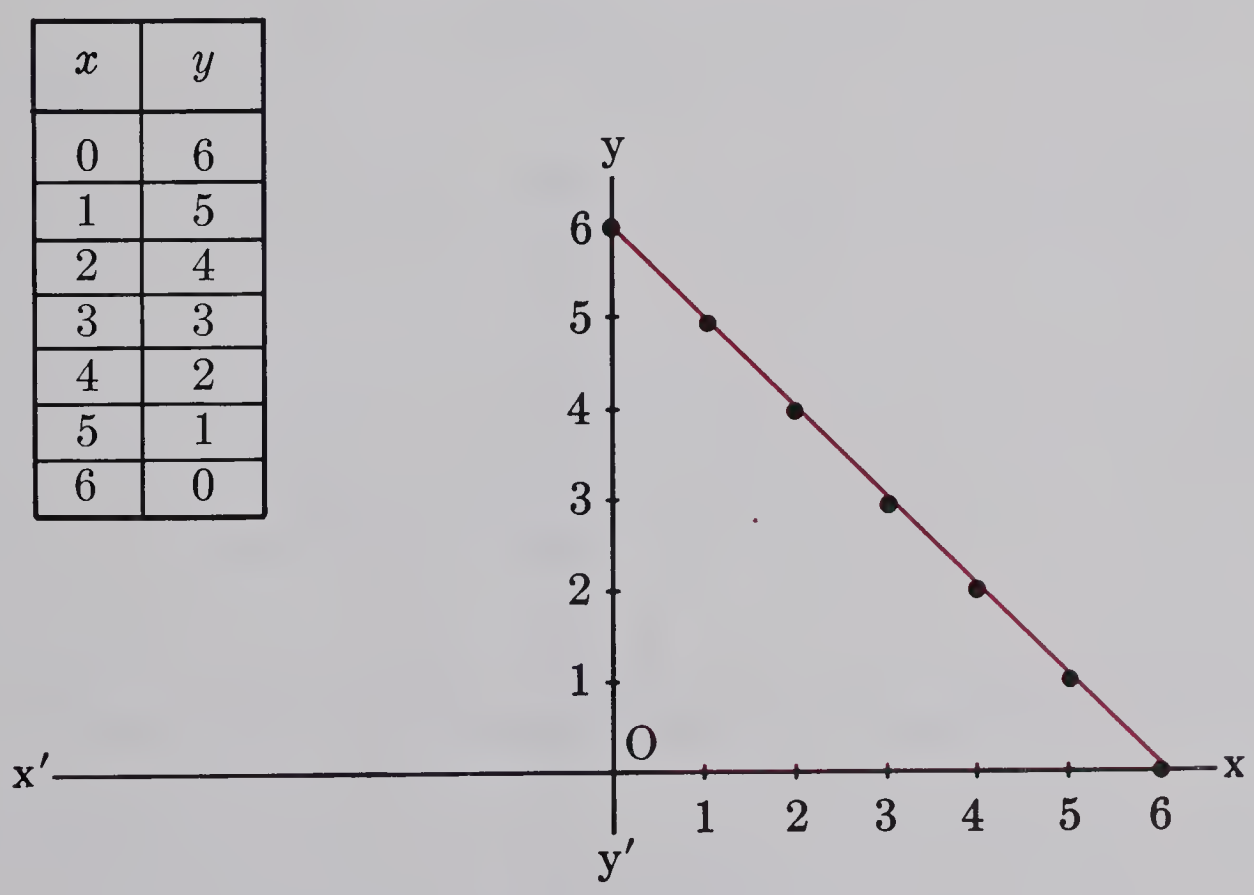
Functions

2 1 Functions As Single-Valued Binary Relations

Consider the relations which determine the sets R_1, R_2, R_3, and R_4 below:

(a) $R_1 = \{(x, y) | y = 6 - x;\ 0 \leq x \leq 6,\ x, y \in \text{Re}\}$:

x	y
0	6
1	5
2	4
3	3
4	2
5	1
6	0

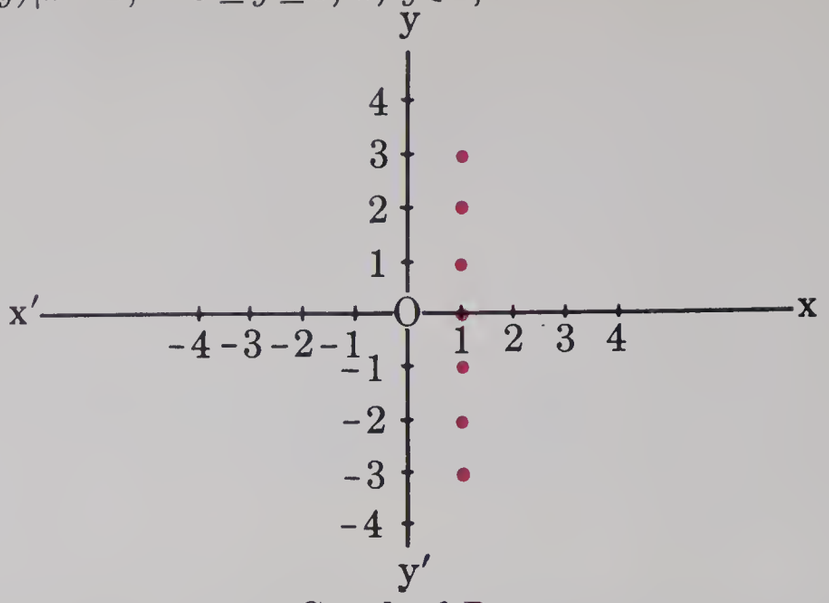


Graph of R_1

The domain of R_1 is $\{x \in \text{Re} | 0 \leq x \leq 6\}$; the range of R_1 is $\{y \in \text{Re} | 0 \leq y \leq 6\}$.

Note that for each x in the domain there is exactly one y in the range. A line parallel to the y-axis meets the graph in at most one point.

(b) $R_2 = \{(x, y) | x = 1;\ -3 \leq y \leq 3,\ x, y \in I\}$:



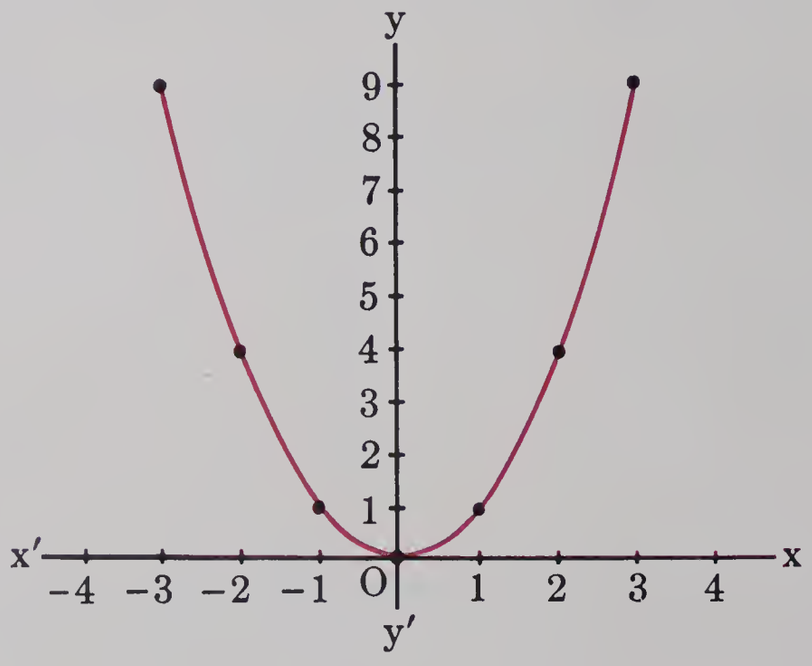
Graph of R_2

Domain $= \{1\}$ Range $= \{-3, -2, -1, 1, 2, 3\}$

Note that for the value of x in the domain there are seven values of y in the range. There is a line parallel to the y-axis intersects the graph in seven points at the value $x = 1$.

(c) $R_3 = \{(x, y) | y = x^2;\ -3 \leq x \leq 3,\ x, y \in \text{Re}\}$:

x	y
0	0
1	1
2	4
3	9
−1	1
−2	4
−3	9

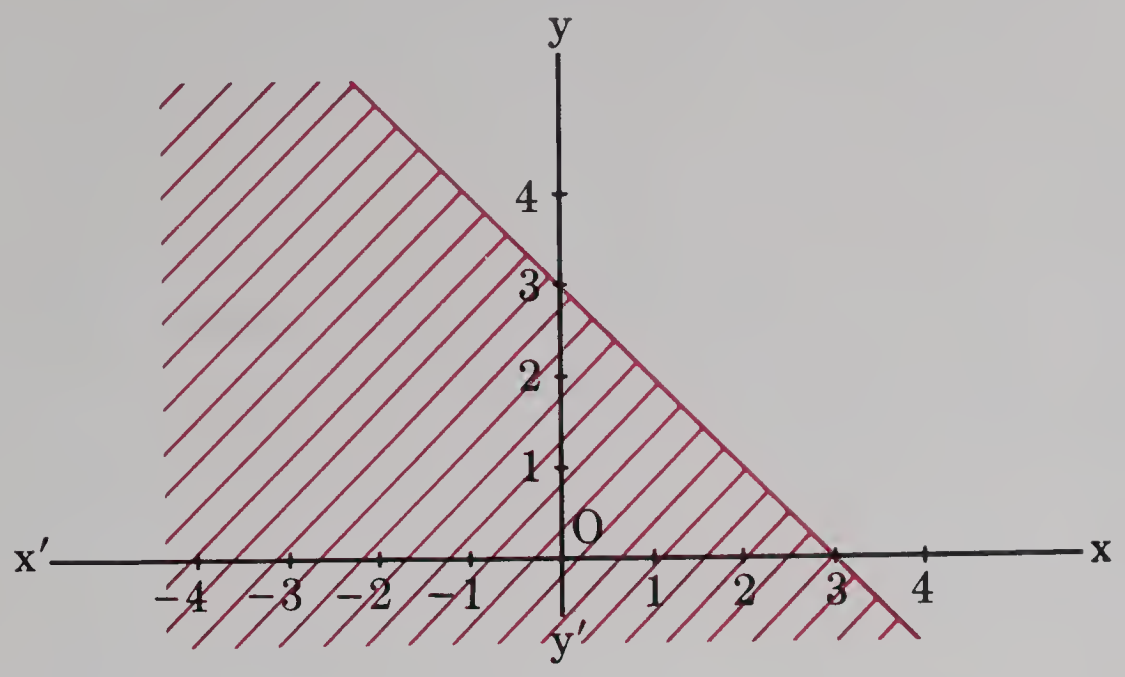


Graph of R_3

Domain $=$ Re Range $= \{y \in \text{Re} | y \geq 0\}$

Note that to each element in the domain there corresponds exactly one element in the range. A line parallel to the y-axis cuts the graph in at most one point. The graph in the diagram is of course only a partial graph for the finite domain $-3 \leq x \leq 3$.

(d) $R_4 = \{(x, y) | x + y \leq 3,\ x, y \in \text{Re}\}$:



Graph of R_4

Domain = Re

Range = Re

Note that more than one element in the range corresponds to each element of the domain. A line parallel to the y-axis cuts the graph in more than one point.

R_1 and R_3 are examples of sets determined by a special type of binary relation called a *function*. *A function in a given domain of one variable is a single-valued binary relation* in that domain in the sense that *to each x in the domain there corresponds a single value of y in the range.*

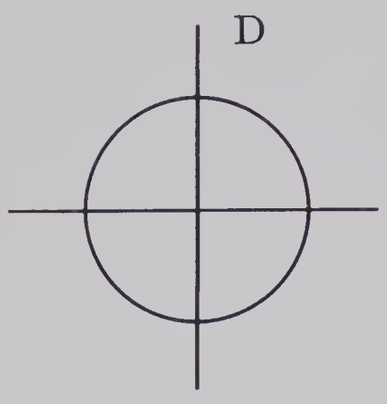
DEFINITION: A function in the domain D is a relation in D such that for each element in D, there corresponds one and only one element in the range.

GEOMETRICAL TEST: A relation is a function if and only if any line parallel to the y-axis meets the graph of the relation in at most one point.

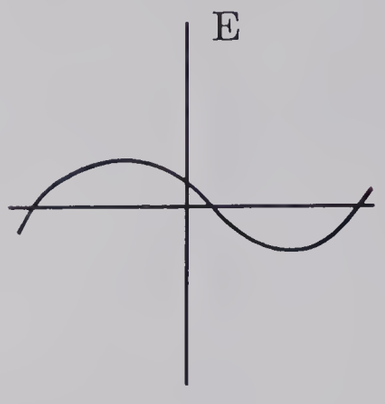
Note that a function determines a set of ordered pairs (x, y) in which no two members have the same x. In this sense we could say that a function *is* such a set of ordered pairs. Certainly these ordered pairs are a representation of the function. However such a set of ordered pairs need not always be interpreted as a function. For example, rational numbers may be written as ordered pairs.

EXERCISE 2—1

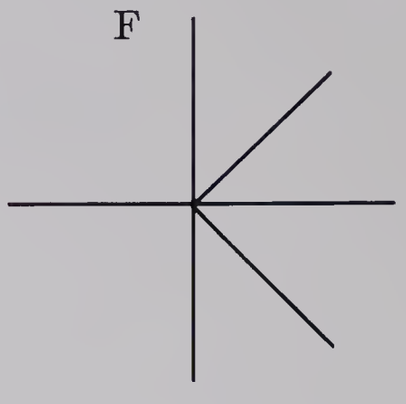
1. State which of the graphs of the relations shown represent functions:



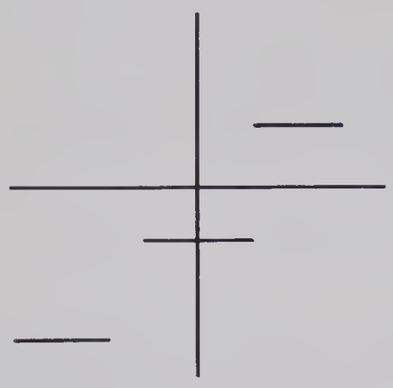
A



B

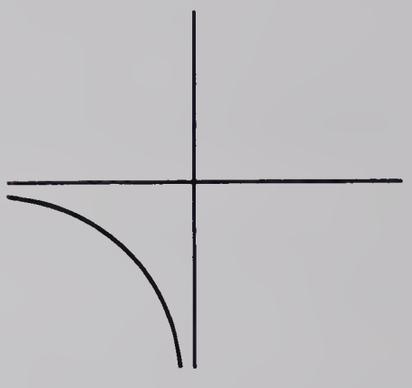


C



D

E



F

2. Which of the following sets may represent functions? Explain.
   (a) $R_1 = \{(2, 1), (2, 2), (2, 3), (2, 4)\}$
   (b) $R_2 = \{(4, 1), (5, 2), (6, 3), (-4, 1), (-5, 2)\}$
   (c) $R_3 = \{(-4, 9), (-3, 2), (-2, 0), (0, 2), (-2, -2)\}$
   (d) $R_4 = \{(5, 4), (6, 4), (7, 4), (8, 4)\}$

3. A function is defined by the following table:

x	2	0	-2
y	0	1	2

   State the domain, the range, and the ordered pairs in this function.

4. If the following table defines a function, what variable must represent the first member of each pair?

a	2	2	2
b	3	4	5

   State the domain, the range, and the ordered pairs in the function.

(Unless otherwise stated, the variables are members of Re.)

5. If $A=\{-2, -1, 0, 1, 2\}$, graph the following sets determined by relations in $A\times A$ and list those relations that are functions:

   (a) $\{(x, y)|y=x\}$ (b) $\{(x, y)|y\neq x\}$ (c) $\{(x, y)|y=|x|\}$

   (d) $\{(x, y)|\ |y|=x\}$ (e) $\{(x, y)|y=x^2\}$ (f) $\{(x, y)|y^2=x\}$

6. A rectangle has a perimeter of 20 units but the dimensions vary.

   (a) Express width w in terms of length l.

   (b) Does this rule define a relation that is a function?

   (c) List the domain and range of the relation given that the maximum length is 8 units.

7. A boy drives his scooter 200 miles per week during the summer.

   (a) Express the total distance driven (s) in terms of the number of weeks involved (n).

   (b) Does the relation define a function? List the domain, the range, and the ordered pairs involved if the total number of weeks is 8.

8. A particle falls $16t^2$ feet through the atmosphere in t seconds. Draw the graph of the relation which determines the set

$$\{(t, d)|d=16t^2;\ 0\leq t\leq 5,\ t\in \mathrm{Re}\}.$$

   (a) Is this relation a function? Explain.

   (b) From the graph, estimate the number of feet which a particle falls in $3\frac{1}{2}$ seconds.

   (c) From the graph, estimate how long it would take a particle to fall 81 feet.

9. The inverse of a relation R is the relation obtained by interchanging the first and second elements in every ordered pair.

   (a) Graph the inverse of each relation in (5).

   (b) Which of the inverse relations are functions?

10. A relation involves a set of ordered pairs (a, b) such that b is the wife of a. Suppose the domain for a is the set of all husbands who obey the laws of the country. Is this relation a function if the law is based on a civilization which is (a) monogamous?
    (b) polygamous?
    (c) polyandrous? Explain.

11. Graph the function whose rule of formation is $y=x^3$, $x\in \mathrm{Re}$, $|x|\leq 2$.

12. Graph the function whose rule of formation is $y=\frac{1}{x}$, $x\in \mathrm{Re}$, $|x|\leq 6$.

2.2 Functional Notation

It is common practice to use a letter such as f (or g) to represent a function. The single value of y which corresponds to any x in the domain is called $f(x)$. Read "f at x", or "the value of f at x", or "f of x". Note that $f(x)$ does *not* denote a product of f and x. $f(x)$ identifies the element of the range corresponding to x in the domain. For the function f determining the set

$$\{(x, y) | y = x+3;\ x \in \text{Re}\},$$

we may say $f(x) = x+3$; $f(x)$ is the *value of the function at (or for) x*. Sometimes we speak of the "function $x+3$" when we mean the function f as defined above. The "f at x" or "f of x" notation provides another method for denoting the function f. We may speak of the function f defined by

$$f(x) = x+3,\ x \in \text{Re}.$$

Each ordered pair in the set determined by the function is of the form $(x, f(x))$. A particular pair would be $(2, f(2))$, that is, $(2, 2+3)$ or $(2, 5)$.

Example 1

For the function f defined by $f(x) = 3x-2$, $x \in \text{Re}$, find

(a) $f(-3)$ (b) $f(a+1)$ (c) $f[f(x)]$.

Solution: (a) $f(-3) = 3(-3)-2$
$= -11$

(b) $f(a+1) = 3(a+1)-2$
$= 3a+3-2$
$= 3a+1$

(c) $f[f(x)] = 3(3x-2)-2$
$= 9x-6-2$
$= 9x-8$

Example 2

If g is defined by $g(x) = \dfrac{1}{x+1}$, $x \in \text{Re}$, $x \neq -1$, find

(a) $g(\frac{1}{4})$ (b) $g\left(\dfrac{1}{x}\right)$.

Solution: (a) $g(\frac{1}{4}) = \dfrac{1}{\frac{1}{4}+1} = \dfrac{1}{\frac{5}{4}} = \frac{4}{5}$

(b) $g\left(\dfrac{1}{x}\right) = \dfrac{1}{\frac{1}{x}+1} = \dfrac{1}{\frac{1+x}{x}} = \dfrac{x}{1+x}$

Example 3

If h is defined by $h(x)=(x+1)^2$, find

(a) $h(2)$ (b) $h(k+1)$ (c) $h(x^2)$.

Solution: (a) $h(2)=(2+1)^2=9$

(b) $h(k+1)=(k+1+1)^2$
$=(k+2)^2$
$=k^2+4k+4$

(c) $h(x^2)=(x^2+1)^2$
$=x^4+2x^2+1$

EXERCISE 2—2

(All variables represent real numbers unless otherwise stated.)

1. If f determines the set $\{(x, y)|y=x+5\}$, compute the following values of the function:

   (a) $f(3)$ (b) $f(0)$ (c) $f(\frac{1}{2})$ (d) $f(-5)$

   (e) $f\left(\frac{1}{x}\right)$ (f) $f[f(x)]$ (g) $f(k+1)$ (h) $f(2r-5)$.

2. If $f(x)=3x$ and $g(x)=x^2-2$, find

   (a) $f(x+1)$ (b) $g(x+1)$ (c) $g(\sqrt{2})$

   (d) $f[g(x)]$ (e) $g[f(-1)]$ (f) $g[f(x+1)]$.

3. In what sense may we write "$f=\{(x, y)|y=3x\}$" in (2)?

4. If $p(x)=4x^3+3x^2+2$, compute

   (a) $p(0)$ (b) $p(2)$ (c) $p(-2)$ (d) $p(-1)$.

   List four ordered pairs in the function defined by $p(x)$.

5. Graph the following functions:

   (a) $f=\{(x, f(x))|f(x)=x\}$ (b) $g=\{(x, g(x))|g(x)=2x+1\}$

   (c) $h=\{(x, h(x))|h(x)=0\}$ (d) $k=\{(x, k(x))|k(x)=-x\}$

6. (a) Graph the function f whose rule of formation is $f(x)=x^2-5x+6$, $0\leq x\leq 6$, $x\in \mathrm{I}$.

   (b) For what value of x is $f(x)=0$?

   (c) How are these zeros of the function connected with the x-intercepts of the graph?

7. Graph the following functions. In each case state the range.

   (a) $f=\{(x, f(x))|f(x)=-x;\ |x|\leq 5\}$

   (b) $g=\{(x, g(x))|g(x)=x^3;\ |x|\leq 3\}$

   (c) $h=\{(x, h(x))|h(x)=(x-3)^2;\ 0\leq x\leq 6\}$

8. Graph the inverse of each function in (7). Is the inverse in each case a function?

2 3 Linear Functions Developed From Problems

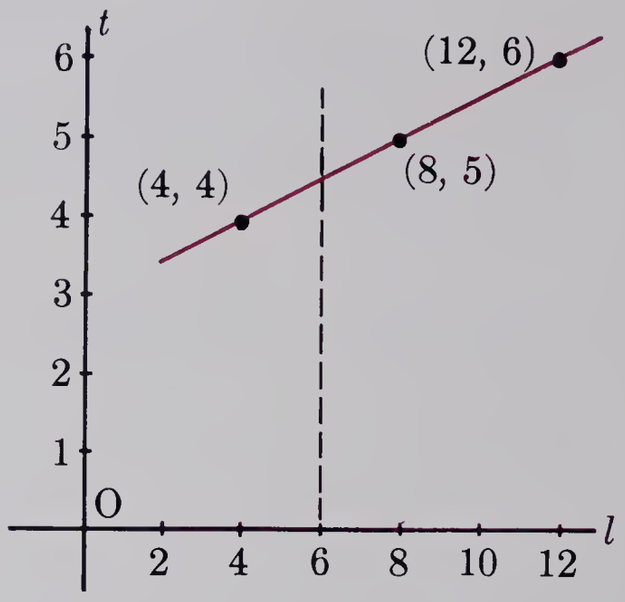
Example

With a certain arrangement of pulleys, the tension (t) required to raise a load (l) may be expressed as $t=\frac{1}{4}l+3$. Graph the function defined by this equation. (Use $l=4,\ 8,\ 12$.) Interpolate to find the tension required to raise a load of 6 units.

Solution: This function involves the set $L=\{(l,\ t)|t=\frac{1}{4}l+3;\ l\in Re^+\}$. The graph of this function may be drawn using the table of values below.

l	4	8	12
t	4	5	6

From the graph, we interpolate or "read between" the plotted points to estimate that $t=4\frac{1}{2}$ when $l=6$.
Therefore a tension of $4\frac{1}{2}$ units is required when the load is 6 units.



Graph of L

EXERCISE 2—3

1. The heights of five boys are: Tom, 5 ft. 7 in.; Jerry, 5 ft. 10 in.; Chris, 6 ft. 0 in.; Paul, 6 ft. 1 in.; and Geoff, 6 ft. 3 in. List the ordered pairs $(x,\ y)$ of boys in the function defined by the rule "y is two inches taller than x". State the domain and range of the function.

2. A man buys a used car for \$50 down and pays \$10 per week for one year. Write the rule connecting total amount paid (t) with number of weeks (w). List the first three and the last three ordered pairs $(w,\ t)$ determined by this rule. State the domain and range of this function.

3. $d=vt$ where d represents distance in miles, v speed in m.p.h., and t time in hours. If $v=40$, then a function involving pairs $(t,\ d)$ is defined by $d=40t$. Using the horizontal axis for time and the vertical axis for distance, plot the graph of the function f so that $f(t)=40t,\ 0\leq t\leq 5$. This graph may be thought of as a representation of the distance travelled by a car travelling for 5 hours at 40 miles per hour.

4. A second car leaves 1 hour after the car in (3) and travels at 60 miles per hour.
   (a) Using the same set of axes as in (3), plot the graph of the function g so that $g(t) = 60t - 60$, $1 \leq t \leq 5$.
   (b) Write the coördinates of the point of intersection of the two graphs. How many hours does the first car travel before it is overtaken by the second?
   (c) From the graph, interpolate to find out how many miles separate the two cars after the first has travelled for 5 hours.

5. Two cars set out at the same time, travelling in the same direction on the same road. The second, travelling at 60 m.p.h. starts 80 miles behind the first, which has a speed of 40 m.p.h. Write defining rules for the two distance-time functions involved. Then draw distance-time graphs similar to those in (3) and (4) for the domain $0 \leq t \leq 5$. In how many hours does the second car overtake the first?

6. Hooke's Law states that the extension of an elastic string is directly proportional to the tension applied. A certain elastic cord 10 inches long is extended 1 inch by a 4-lb. tension.
   (a) Write the defining rule for the function involving pairs (t, l), where l represents total length in inches and t represents tension in pounds.
   (b) Plot the graph of the function using points whose abscissae (t) are 0, 10, and 20.
   (c) From the graph, estimate the total length for a tension of 15 lb.

7. The normal weight of a healthy adult male is calculated from

$$w = \frac{11(h-40)}{2},$$

   where w is weight in pounds, and h represents height in inches. Use set-builder notation to represent the function. In the function so defined, list the pairs (h, w) associated with men 5 ft. 6 in. and 5 ft. 10 inches tall.

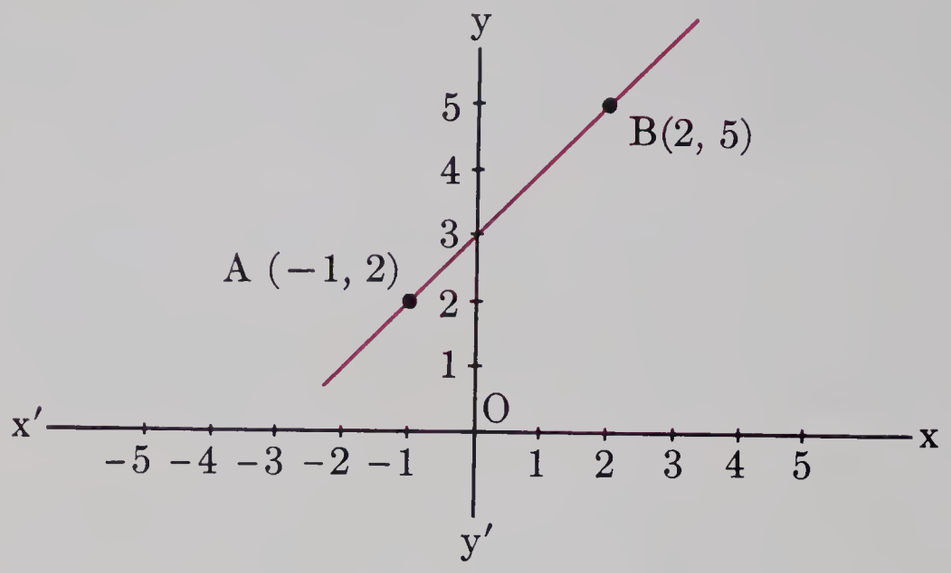
8. The cost of the fall prom at the school is the total of fixed charges (orchestra \$150, decorations \$50) and a charge of 50 cents per person for refreshments.
   (a) If C is the total cost in dollars, and n is the number of students who attend, write the equation expressing C in terms of n.
   (b) Use set-builder notation to represent the function involving ordered pairs of the form (n, C).
   (c) If the 350 students who attend pay \$1.25 each, how much profit, if any, will be realized by the students' council?

2 4 Slope of a Line

In Section 1-7, we drew the graph defined by

$$y=x+3 \ (x, y\in \mathrm{Re}).$$

A partial graph is the line shown below.

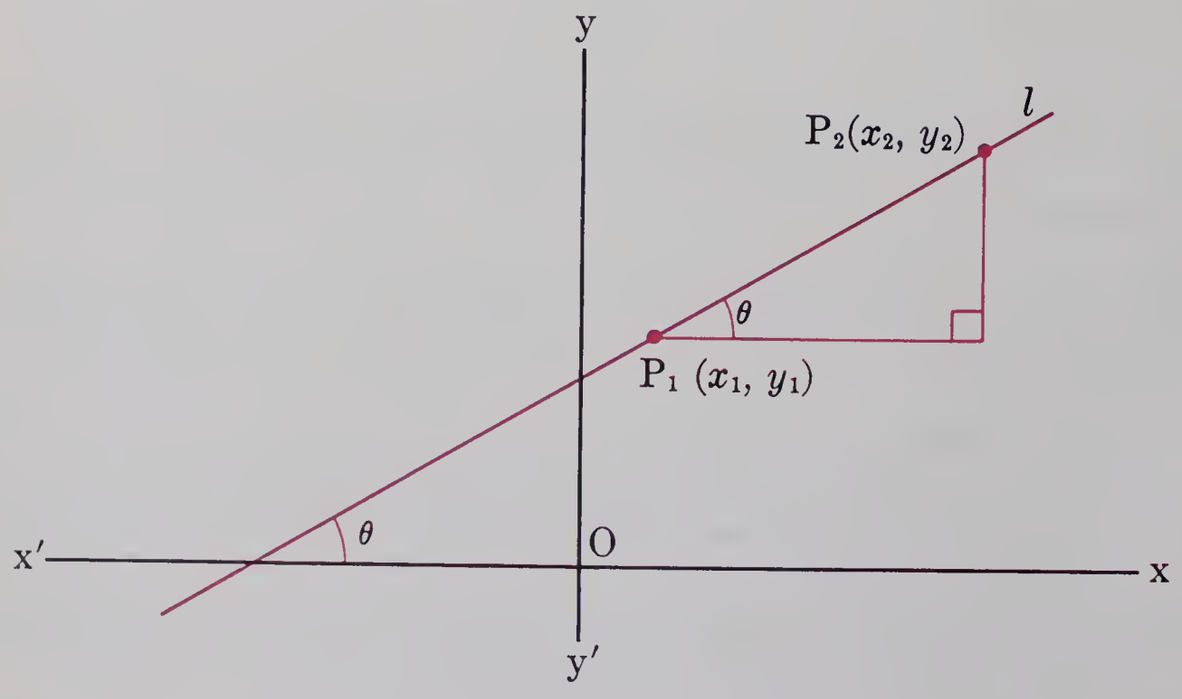


$$\{(x, y)|y=x+3;\ x, y\in \mathrm{Re}\}$$

We assumed that any equation of the form

$$ax+by+c=0 \ (a, b, c\in \mathrm{Re})$$

has a line as its graph. Suppose that (x_1, y_1) and (x_2, y_2) are the coördinates of any two points on the line defined by $ax+by+c=0$, whose angle of inclination, that is, the angle the line makes with the x-axis, is θ as shown:



The *slope* of a line is a number that measures the "steepness" of the line. Slope m of line l may be defined in at least two equivalent ways.

DEFINITION 1: slope = tangent of the angle of inclination

$$m = \tan\theta$$

Note that if P_1A and AP_2 are drawn parallel to the coördinate axes, then $\angle P_2P_1A = \theta$. (Explain.)

The coördinates of A are (x_2, y_1).

Then
$$\begin{aligned} m &= \tan\theta \\ &= \tan \angle P_2P_1A \\ &= \frac{AP_2}{P_1A} \\ &= \frac{y_2 - y_1}{x_2 - x_1}. \end{aligned}$$

Note also that

(1) if P_1P_2 is parallel to the x-axis, then $y_2 = y_1$ and $m = 0$;

(2) if P_1P_2 is parallel to the y-axis, then $x_2 = x_1$ and m is not defined. (Explain.)

The change in ordinate between P_1 and P_2 is called $\triangle y$, read "delta y". ($\triangle y$ is a single symbol representing a single real number; it is not the product of two numbers $\triangle$ and y.) This change in ordinate is also called the *rise* from P_1 to P_2.

The corresponding change in abscissa is $\triangle x$. This difference $x_2 - x_1$ is also called the *run* from P_1 to P_2.

$\triangle y$ and $\triangle x$ represent the change in coördinates between the points with coördinates (x, y) and $(x + \triangle x, y + \triangle y)$.

In the diagram, we see that the ratio of rise to run is

$$\frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This suggests a second definition.

DEFINITION 2: $\text{slope} = m = \dfrac{\triangle y}{\triangle x} \quad (\triangle x \neq 0)$

Note that since
$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{-(y_2 - y_1)}{-(x_2 - x_1)} \\ &= \frac{y_1 - y_2}{x_1 - x_2}, \end{aligned}$$

the same value is obtained if the two points are considered in reverse order.

Example 1

Find the slope of the line joining A (2, 5) and B (−1, 2).

Solution: Slope of line segment AB $= m_{AB}$

$$= \frac{\triangle y}{\triangle x}$$

$$= \frac{2-5}{-1-2}$$

$$= \frac{-3}{-3}$$

$$= 1.$$

Repeat Example 1 with the two points considered in the reverse order.

Example 2

Find (to 3 figures) the slope of a line whose inclination is

(a) 50° (b) 150°.

Solution: (a) $m = \tan 50°$

$= 1.19$ (tables)

(b) The tangent of an obtuse angle is the negative of the tangent of its supplement.

$\therefore m = \tan 150°$

$= \tan (180° - 30°)$

$= -\tan 30°$

$= -.577.$ (tables)

In defining slope, we arbitrarily chose the line segment P_1P_2. $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ were *any* two points on the line l. The slope is a constant for all segments of line l.

All segments of a line have the same slope.

This fundamental property of a straight line gives us a useful test for collinearity.

Example 3

Prove that points A(−1, −2), B(0, 1), and C(2, 7) are collinear.

Solution: $m_{AB} = \frac{1+2}{0+1}$ $\qquad$ $m_{BC} = \frac{7-1}{2-0}$

$= 3$ $\qquad$ $= 3$

$\because m_{AB} = m_{BC},$

$\therefore$ A, B, and C are collinear.

EXERCISE 2—4

1. Find the value of m for the lines determined by the following pairs of points:
   (a) P(3, 4), Q(1, 0) (b) A(0, 2), B (3, −2)
   (c) M(−3, 2), N(−4, 1) (d) R(0, 5), S(−1, −7)
   (e) T(−3, −2), U(−5, −6) (f) V(4, 3), W(5, 3)
   (g) D(−3, 0), E(5, 0) (h) F(2, 3), G(2, 8).

2. Through the following points draw lines with the slope indicated:
   (a) A(3, 1), $m=\frac{5}{3}$ (b) B(2, −2), $m=-3$
   (c) A(3, 1), $m=-\frac{3}{5}$ (d) B(2, −2), $m=\frac{1}{3}$.

3. Find (to 3 figures) the slope of a line whose inclination is:
   (a) 35° (b) 60° (c) 140°.

4. Find (to the nearest degree) the inclination of a line whose slope is:
   (a) 1 (b) 0 (c) .810
   (d) −5.67 (e) .231 (f) −.781.

5. Determine by slopes whether or not the following sets of points are collinear:
   (a) (1, 4), (−1, 0), (3, 8) (b) (0, −3), (1, −4), (3, 0)
   (c) (3, 0), (−1, −4), (1, −2) (d) (2, 4), (1, 2), (−1, −4).

6. If the slope of the line joining A (1, 2) to B (x_1, 5) is −2, find x_1.

7. If the slope of the line joining P (−5, y_2) to Q (−7, 2) is $\frac{3}{4}$, find y_2.

8. If the set of points (0, 4), (3, 1), and (5, y_1) is collinear, find y_1.

9. If the line segment determined by A($-\frac{1}{2}$, $-\frac{1}{4}$) and B (3, y_1) has a slope of 3, find y_1.

10. The points whose coördinates are (3, −2), (x_1, y_1), and (−5, −7) are collinear. Express x_1 in terms of y_1.

11. Find the value of m if the points with coördinates (−4, −2), (m, m) and (5, 3) are collinear.

12. Find (to the nearest degree) an angle of intersection between two lines whose slopes are:
   (a) .839 and 5.67 (b) −.577 and −3.73.

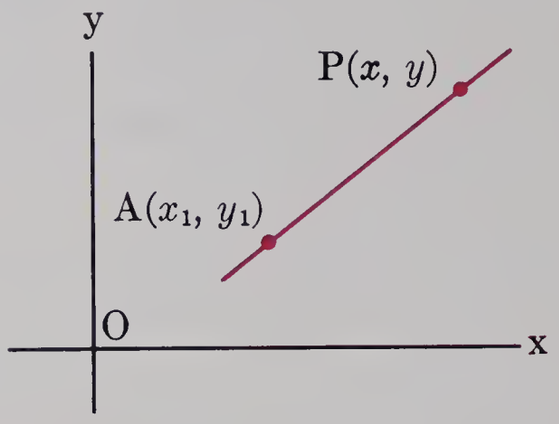
2 5 Forming Equations of a Line

We have agreed that an equation of the form $ax+by+c=0$ represents a line. It is always possible to form the equation of any line if we know two conditions which determine the line. Given a point and a slope, the line through the given point and possessing the given slope is uniquely determined.

If a line passes through the fixed point A with slope m, and $P(x, y)$ is any other point on the line, then $\triangle x=x-x_1$ and $\triangle y=y-y_1$.

$$\therefore m=\frac{\triangle y}{\triangle x}$$

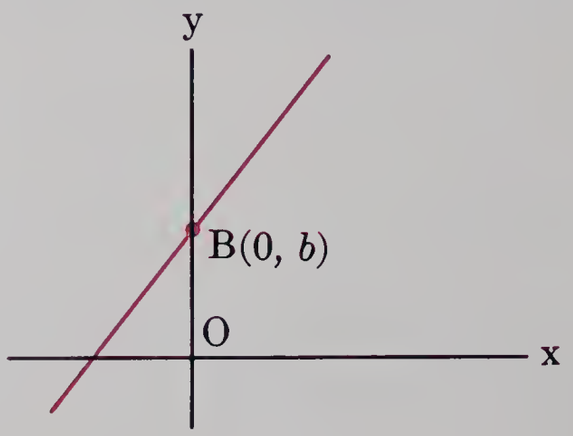
$$=\frac{y-y_1}{x-x_1} \text{ (if } \triangle x\neq 0)$$



$$\therefore y-y_1=m(x-x_1)$$

This is called the slope-point form of the equation of a line.
$y-y_1=m(x-x_1)$ is the equation of the locus of the set of points P.

Note that if $\triangle x=x-x_1=0$, then $x=x_1$ which is the equation of a line parallel to the y-axis.

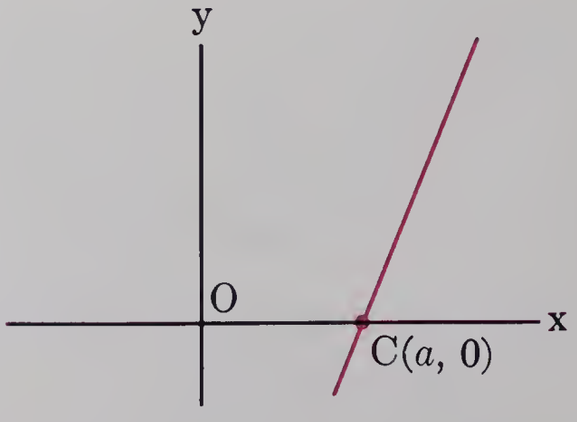


Now if the fixed point is $B(0, b)$, then the equation of the line with slope m becomes $y-b=m\ (x-0)$

or $y-b=mx$.

$$\therefore y=mx+b.$$

b is the y-intercept of the line.
$y=mx+b$ is called the slope y-intercept form of the line equation.



Finally, if the fixed point is $C(a, 0)$ and the slope is m, then the equation of the line is

$y-0=m\ (x-a)$

or $y=m\ (x-a)$.

a is the x-intercept of the line.
$y=m(x-a)$ is the slope x-intercept form of the equation of the line.

Example 1

Find the equation for the line through A $(-3, 1)$ with a slope of 5.

Solution: The required equation is

$$y-1=5(x+3)$$
$$\therefore y-1=5x+15$$
$$\text{or } 5x-y+16=0.$$

Example 2

Find the equations for the lines defined by

(a) $m=2, b=-3,$ (b) $m=\frac{3}{4}, a=2.$

Solution: (a) $y=2x-3$

(b) $y=\frac{3}{4}(x-2)$

$$\therefore 4y=3x-6$$
$$\text{or } 3x-4y-6=0.$$

Note: $y=3x+b$ represents all lines with slope 3; b is called the *parameter* of this *family* of lines. Each value of the parameter selects a *member* of the family.

$y=m(x-5)$ defines a family of lines with x-intercept 5 where the parameter m represents slope. What family of lines is represented by

$$y=mx-3? \quad y=m(x+5)?$$

EXERCISE 2—5

1. State the equations of the lines uniquely determined by:
   (a) $m=3, b=2$ (b) $m=1, b=1$
   (c) $m=-3, b=4$ (d) $m=4, b=-5.$

2. State the slope and y-intercepts of the lines represented by:
   (a) $y=5x+3$ (b) $y=x-4$
   (c) $y=-3x-2$ (d) $y=-x+7$
   (e) $2x+y=4$ (f) $3x+y-7=0.$

3. Find an equation for each of the lines determined by:
   (a) $P(3, 2), m=\frac{1}{4}$ (b) $A(-4, -5), m=1$
   (c) $Q(-7, 2), m=\frac{3}{5}$ (d) $R(4, 4), m=0.$

4. Find equations for the lines with:
   (a) $m=2, a=1$ (b) $m=-3, a=-3$
   (c) $m=1, a=\frac{5}{2}$ (d) $\theta=80°, a=\frac{2}{3}$ (use tables).

5. Sketch the family of lines represented by

   (a) $y=mx+5$ (b) $y=3x+b$

   (c) $y-3=m(x-1)$ (d) $y+5=m(x-2)$

   (e) $y=m(x+2)$ (f) $y=mx$.

6. From the family of lines defined by $y=3x+b$, select the line which passes through $(\frac{1}{2}, -\frac{1}{4})$.

7. From the family of lines represented by $y-5=m(x+3)$, select the member whose angle of inclination is 135°.

8. (a) Write an equation of the family of lines with slope -3.

   (b) From the family in (a), select the member whose y-intercept is 5.

9. (a) Write an equation of the family of lines passing through $(2, -5)$.

   (b) From the family in (a), select the member passing through $(1, -1)$.

10. If A is $(-1, 3)$, find the coördinates of two points 10 units from A, and on a line through A with slope $\frac{4}{3}$.

11. From the family of lines in (9), select a member whose intercepts are equal.

26 Reduction to the Form y = mx + b

A linear equation such as $5x+3y=15$ may be reduced to the form $y=mx+b$. Dividing by 3, we obtain

$$\tfrac{5}{3}x+y=5$$

or

$$y=-\tfrac{5}{3}x+5.$$

Comparing with $y=mx+b$, we conclude that the slope of the line is $-\frac{5}{3}$ and the y-intercept is 5.

We have been able to calculate the y-intercept by setting $x=0$ in the defining equation, but now we also can read the slope from the equation.

If the equation is in the form $ax+by+c=0$,

then $$\text{slope}=-\frac{\text{coefficient of } x}{\text{coefficient of } y},$$

and $$m=-\frac{a}{b} \qquad (b\neq 0).$$

EXERCISE 2—6

1. State the slope of the lines represented by:
   (a) $x-5y+3=0$ (b) $3x+2y=6$
   (c) $2x+2y-1=0$ (d) $x-y+7=0$
   (e) $3x-7y+8=0$ (f) $mx-2ky+r=0$ $(k\neq 0)$
   (g) $5x+5y-7=0$ (h) $(h+1)x-y+k=0$

2. Reduce the following equations to the form $y=mx+b$:
   (a) $3x+5y=20$ (b) $4x-6y=7$
   (c) $4x-4y=9$ (d) $3x-4y=12$
   (e) $x-y-1=0$ (f) $px-qy+r=0$ $(q\neq 0)$
   (g) $2x+y-11=0$ (h) $(r+s)x-ty+w=0$ $(t\neq 0)$

3. Write the value of m for each part of (2).

4. Determine the value of the parameter m of the family $mx-y+4=0$ which selects the line
   (a) passing through (3, 3),
   (b) with slope 2,
   (c) parallel to the line defined by $3x-5y+7=0$.

5. What is the value of k if the lines represented by the following equations have the slope indicated?
   (a) $5kx-3y+7=0$ $(m=2)$
   (b) $3x-ky+3=0$ $(m=-4)$
   (c) $(k^2-4)x-(k+2)y+1=0$ $(m=5,\ k\neq -2)$

6. What is the value of k if
   (a) $3kx-4y+7=0$ is parallel to $5x-3y-5=0$?
   (b) $2kx-5y+1=0$ is parallel to $4x-ky+9=0$?
   (c) $(3k-1)x-(2k+1)y-1=0$ is parallel to $(5-3k)x-(2-k)y+3=0$?

2 7 Chapter Summary

Functions as single-valued binary relations. Graph of a function. Functional notation. Linear functions arising from problems.

Slope of the line, angle of inclination, $\triangle y$, $\triangle x$, rise and run. Forming equations of the straight line:

$y-y_1=m(x-x_1)$ (slope-point)
$y=mx+b$ (slope y-intercept)
$y=m(x-a)$ (slope x-intercept)

Reduction from the form $ax+by+c=0$ to the form $y=mx+b$.

REVIEW EXERCISE 2–7

1. If $U=\{\ldots, -6, -3, 0, 3, 6, \ldots\}$, can the following relation in $U\times U$ be a function?

$$\{(x, y)|y=|x|\} \quad \text{Explain.}$$

2. What is the inverse of the relation in (1)? Is this relation a function? Explain.

3. Graph the function whose rule of formation is

$$y=\frac{x^2}{2}, \quad |x|\leq 4, \quad x\in \text{Re}.$$

4. If f determines the set $\{(x, y)|y=4x-5;\ x\in \text{Re}\}$, find

(a) $f(3)$ (b) $f(0)$ (c) $f(x+1)$

(d) $f[f(x)]$ (e) $f(x^2)$ (f) $[f(x-1)]^2$.

5. If $g=\{(x, y)|y=x^3;\ x\in \text{Re}\}$, find

(a) $g(0)$ (b) $g(-1)$ (c) $g[g(x)]$

(d) $g[g(2)]$ (e) $g\left(\frac{1}{x+a}\right)$ (f) $g(-\frac{1}{7})$.

6. State the slopes and intercepts of the lines defined by
(1) $3x-5y=15$ and (2) $7x+y=8$.

7. Find (to the nearest degree) the inclination of the lines in (6).

8. Determine whether or not $(3, 4)$, $(1, 2)$, and $(-4, -3)$ are collinear.

9. If $(x_1, 2)$, $(3, 4)$, and $(-6, -1)$ are collinear, find x_1.

10. If $(3, 0)$, $(2, y)$, and $(-2, -5)$ are collinear, find a value for y.

11. Find equations of the lines with

(a) $m=4$, $b=-1$; (b) $m=-1$, $a=2$;

(c) $\theta=45°$, $b=\frac{2}{5}$; (d) $\theta=135°$, $a=-3$.

12. Reduce $7x-11y=21$ to the form $y=mx+b$. Write a general equation for all lines having the same slope as this line.

13. Discuss the function f denoted by

$$f=\{(x, y)|y=\frac{3}{x^2+1};\ x\in \text{Re}\},$$

under the headings intercepts, domain, range, and symmetry. Graph the function for values of x such that $|x|\leq 5$.

Chapter 3

Exponential Functions

3 1 Products and Quotients of Positive Integral Powers of the Same Base

In earlier grades we have learned that

$$2^3 = 2\times 2\times 2 = 8, \text{ and}$$
$$2^7 = 2\times 2\times 2\times 2\times 2\times 2\times 2 = 128.$$

In the second example the number 7 is called the index or *exponent* and 2 is the *base*; the *exponent* 7 tells us how many times the *base* 2 occurs as a *factor* in the *power* 128. Similarly, in the other example, 2^3, 2 is the *base*, 3 is the *exponent*, and 8 is the *power*.

Any real number may be used as the base:

$$3^4 = 3\times 3\times 3\times 3 = 81 \qquad (-7)^3 = (-7)\times(-7)\times(-7) = -343$$
$$(\sqrt{2})^5 = \sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2} = 2\times 2\times\sqrt{2} = 4\sqrt{2}$$
$$\pi^2 = \pi\times\pi$$

DEFINITION: If $a \in \mathrm{Re}$ and $m \in \mathrm{N}$, then $a^m = a\times a\times \ldots \times a$ (m factors); a is called the base, m the exponent, and a^m the power.

Let us now consider the product of two powers of the same base; for example,

$$\begin{aligned}(2^3)(2^5) &= (2\times 2\times 2)\times(2\times 2\times 2\times 2\times 2)\\ &= 2\times 2\times 2\times 2\times 2\times 2\times 2\times 2\\ &= 2^8\end{aligned}$$

We notice immediately that $3+5=8$. Thus, in this case, when we multiply two powers of the same base, the exponent of the product is the sum of the

exponents of the two powers in the product. Let us verify this with some other bases.

$$\begin{aligned} 3^2\times 3^3 &= (3\times 3)\times(3\times 3\times 3)\\ &= 3\times 3\times 3\times 3\times 3\\ &= 3^5 (=243)\end{aligned}$$

$$\begin{aligned}(-5)^3\times(-5)^4 &= [(-5)\times(-5)\times(-5)]\times[(-5)\times(-5)\times(-5)\times(-5)]\\ &= (-5)^7 (=-78{,}125)\end{aligned}$$

The result is true in these cases. Is the result true in general? The examples just studied indicate how we can prove that it is.

THEOREM: The product of two positive integral powers of the same base is equal to the power of the same base that has its exponent equal to the sum of the exponents of the given powers.

In symbols the theorem is:

THEOREM: If $a \in \text{Re}$, and m and $n \in \text{N}$, then $a^m \times a^n = a^{m+n}$. (product rule)

Given: Two numbers a^m and a^n, where $a \in \text{Re}$, and m and $n \in \text{N}$.

Required: To prove that $a^m \times a^n = a^{m+n}$.

Proof:

$a^m = a\times a\times \ldots \times a$ (m factors) (definition)

$a^n = a\times a\times \ldots \times a$ (n factors) (definition)

$$\begin{aligned}\therefore a^m\times a^n &= (a\times a \ldots \times a)\times(a\times a \ldots \times a)\\ &\qquad\quad m \text{ factors} \qquad\qquad n \text{ factors}\\ &= a\times a\times \ldots \times a\times a\times a \ldots \times a\\ &\qquad\qquad (m+n) \text{ factors}\\ &= a^{m+n}. \qquad \text{(definition)}\end{aligned}$$

Corollary: If $a^m, a^n, \ldots, a^r$ are several powers of a, then $a^m \times a^n \times \ldots \times a^r = a^{m+n \ldots +r}$.

Since we know that division is the *inverse operation* to multiplication, we would expect to find a somewhat similar rule for the division of two powers. Let us test this guess by a few examples:

$$\begin{aligned} 2^5 \div 2^3 &= (2\times 2\times 2\times 2\times 2)\div(2\times 2\times 2)\\ &= 2\times 2\times[(2\times 2\times 2)\div(2\times 2\times 2)]\\ &= 2\times 2\\ &= 2^2\end{aligned}$$

$$\therefore 2^5 \div 2^3 = 2^{5-3} = 2^2 = 4.$$

$$\begin{aligned} a^6 \div a^2 &= (a\times a\times a\times a\times a\times a)\div(a\times a)\\ &= (a\times a\times a\times a)\times(a\times a)\div(a\times a)\\ &= a\times a\times a\times a\\ &= a^4\end{aligned}$$

$$\therefore a^6 \div a^2 = a^{6-2} = a^4.$$

$$(-3)^3 \div (-3)^7 = \frac{(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}$$

$$= \frac{1}{(-3)\times(-3)\times(-3)\times(-3)}$$

$$= \frac{1}{(-3)^4}$$

$$\therefore (-3)^3 \div (-3)^7 = \frac{1}{(-3)^{7-3}} = \frac{1}{(-3)^4} = \frac{1}{81}.$$

THEOREM: If $a \in \text{Re}\ (a \neq 0)$, and m and $n \in \text{N}$,

$$\text{then } a^m \div a^n = \begin{cases} a^{m-n} & \text{if } m > n; \\ 1 & \text{if } m = n; \\ \dfrac{1}{a^{n-m}} & \text{if } m < n. \end{cases}$$

(quotient rule)

(The proof is required in Exercise 3-1, (36).)

In the corollary for products, we considered $a^m \times a^n \ldots \times a^r$ and tacitly assumed that $m \neq n \neq \ldots \neq r$. However, the corollary does not require this assumption; it is still true when $m = n = \ldots = r$, and this leads to an interesting result because, in this case, we have a *power* of a *power* of a *base*. Let us look at an example to see what this implies.

From the definitions, $(7^3)^2 = 7^3 \times 7^3 = (7\times7\times7)\times(7\times7\times7)$

$$= 7\times7\times7\times7\times7\times7$$

$$= 7^6,$$

or we could have obtained the result more directly from our product rule:

$$(7^3)^2 = 7^3 \times 7^3 = 7^{3+3} = 7^6.$$

Also, $[(-5)^2]^4 = (-5)^2 \times (-5)^2 \times (-5)^2 \times (-5)^2 = (-5)^{2+2+2+2} = (-5)^8,$

$$(3^5)^3 = 3^5 \times 3^5 \times 3^5 = 3^{5+5+5} = 3^{15}.$$

We notice that $6 = 3\times2$, $8 = 2\times4$, and $15 = 5\times3$,
and therefore we guess that $(a^m)^n = a^m \times a^m \ldots a^m = a^{mn}$.
(n factors)

THEOREM: If $a \in \text{Re}$, and m and $n \in \text{N}$, then $(a^m)^n = a^{mn}$. (power rule)

Given: $a \in \text{Re}$ and $m, n \in \text{N}$.

Required: To prove that $(a^m)^n = a^{mn}$.

Proof: $(a^m)^n = a^m \times a^m \times \ldots \times a^m$ (definition)
(n factors)

$= a^{m+m \ldots +m}$ (n terms) (product rule)

$= a^{nm}$

EXERCISE 3—1

1. Evaluate:

(a) 3^5	(b) 7^3	(c) 12^2
(d) $(-5)^4$	(e) $(\sqrt{2})^6$	(f) $(-1)^5$
(g) $(-3)^3$	(h) 5^3	(i) $(-9)^2$
(j) 4^4	(k) $(-\sqrt{3})^4$	(l) 2^{10}

Evaluate the following expressions using the theorems of this chapter. Check the result by calculating each term separately and then performing the multiplications or divisions required.

2. $6^2 \times 6^3$
3. $(-3)^3 \times (-3)^2$
4. $(-2)^3 \times (-2)^4$
5. $(5)^3 \times (5)^2$
6. $(\frac{1}{2})^2 \times (\frac{1}{2})^3$
7. $(.4)^3 \div (.4)^5$
8. $(.3)^4 \times (.3)^2$
9. $(\frac{1}{3})^3 \times (\frac{1}{3})^3$
10. $7^4 \div 7^2$
11. $(\frac{2}{3})^5 \div (\frac{2}{3})^2$
12. $(\sqrt{2})^4 \times (\sqrt{2})^6$
13. $\pi^3 \div \pi^2$
14. $(\sqrt{5})^7 \times (\sqrt{5})^4$
15. $\left(\frac{1}{\sqrt{3}}\right)^6 \times \left(\frac{1}{\sqrt{3}}\right)^2$
16. $\left(\frac{1}{\pi}\right)^2 \div \left(\frac{1}{\pi}\right)^3$
17. $(\sqrt{\frac{2}{5}})^2 \times (\sqrt{\frac{2}{5}})^4$
18. $x^5 \times x^2 \div x^3$
19. $(a^4 \div a^7) \times a^3$
20. $12^3 \times 12^2 \div 12^4$
21. $(.3)^5 \times (.3)^2 \div (.3)^4$
22. $(5^2)^3 \div 5^4$
23. $(7^3)^2 \div 7^4$
24. $(6^3 \div 6^2) \times 6^4$
25. $5^2 \times 5^3 \div 5^4$
26. $4^3 \times 4^2 \div 4^5$
27. $(\frac{1}{2})^4 \times (\frac{1}{2})^3 \div (\frac{1}{2})^7$
28. $(2x^2)^4 \div (3x^3)^3$
29. $(a^5)^3 \times (2a^3)^4 \div (a^8)^2$
30. $(p^2)^3 \times (p^3)^2$
31. $(4q^2)^2 \div (2q^3)^3$
32. $(3^3) \div (3^2)^2$
33. $(4^2)^3 \div (4^3)^2$
34. $[(.3)^2]^2 \div (.3)^3$
35. $(\frac{5}{7})^5 \div [(\frac{5}{7})^2]^2$
36. Using the definitions of this section, prove that

$$\begin{aligned} a^m \div a^n &= a^{m-n} && m > n, \\ &= 1 && m = n, \\ &= \frac{1}{a^{n-m}} && m < n. \end{aligned}$$

3 2 Products and Quotients of Positive Powers of Different Bases

Let us examine the products $2^3 \times 3^3$ and $a^5 \times b^5$.

$$\begin{aligned} 2^3 \times 3^3 &= (2\times2\times2)\times(3\times3\times3) \\ &= (2\times3)\times(2\times3)\times(2\times3) \\ &= 6\times6\times6 \\ &= 6^3. \end{aligned}$$

$$\begin{aligned} a^5 \times b^5 &= (a\times a\times a\times a\times a)\times(b\times b\times b\times b\times b) \\ &= (a\times b)\times(a\times b)\times(a\times b)\times(a\times b)\times(a\times b) \\ &= (a\times b)^5. \end{aligned}$$

These results suggest the following theorem:

THEOREM: If $a, b \in \mathrm{Re}$, and $n \in \mathrm{N}$,

then $a^n \times b^n = (a\times b)^n$. (power of product rule)

(The proof is required in Exercise 3-2, (44).)

Similarly for quotients:

THEOREM: If $a, b \in \mathrm{Re}$ $(b \neq 0)$ and $n \in \mathrm{N}$,

then $a^n \div b^n = \left(\frac{a}{b}\right)^n$. (power of quotient rule)

Given: $b \neq 0$, and $a, b \in \mathrm{Re}$, $n \in \mathrm{N}$.

Required: To prove that $a^n \div b^n = \left(\frac{a}{b}\right)^n$.

Proof:

$$\begin{aligned} a^n \div b^n &= (a\times a\times \ldots \times a)\div(b\times b\times \ldots \times b) && \text{(definition)} \\ &= \frac{a\times a\times \ldots \times a}{b\times b\times \ldots \times b} && (n \text{ factors}) \\ &= \frac{a}{b}\times\frac{a}{b}\times \ldots \times\frac{a}{b} && (n \text{ factors}) \\ &= \left(\frac{a}{b}\right)^n && \text{(definition)} \end{aligned}$$

Example 1

$$\begin{aligned} 3^4 \div 5^4 &= (3\times3\times3\times3)\div(5\times5\times5\times5) \\ &= \frac{3\times3\times3\times3}{5\times5\times5\times5} \\ &= \frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5} \\ &= \left(\frac{3}{5}\right)^4 \end{aligned}$$

Example 2

Verify that $\left(-\frac{3}{4}\right)^2 \times \left(+\frac{2}{3}\right)^2 = \left(-\frac{1}{2}\right)^2$.

Solution: $\left(-\frac{3}{4}\right)^2 = \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right)$ $\qquad$ $\left(+\frac{2}{3}\right)^2 = \left(+\frac{2}{3}\right) \times \left(+\frac{2}{3}\right)$

$= +\frac{3 \times 3}{4 \times 4}$ $\qquad$ $= +\frac{2 \times 2}{3 \times 3}$

$= +\frac{9}{16}$ $\qquad$ $= +\frac{4}{9}$

$\therefore \left(-\frac{3}{4}\right)^2 \times \left(+\frac{2}{3}\right)^2 = \frac{9}{16} \times \frac{4}{9}$ and $\left(-\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right)$

$= \frac{4}{16}$ $\qquad$ $= +\frac{1 \times 1}{2 \times 2}$

$= \frac{1}{4}$ $\qquad$ $= \frac{1}{4}$

$\therefore \left(-\frac{3}{4}\right)^2 \times \left(+\frac{2}{3}\right)^2 = \left(-\frac{1}{2}\right)^2$.

Example 3

Simplify $\left(\frac{2}{3}\right)^2 \times \pi^3 \div 2^3$.

Solution: $\left(\frac{2}{3}\right)^2 \times \pi^3 \div 2^3 = \frac{2^2}{3^2} \times \pi^3 \times \frac{1}{2^3} = \frac{2^2 \times \pi^3}{3^2 \times 2^3} = \frac{\pi^3}{3 \times 3 \times 2} = \frac{\pi^3}{18}$

Example 4

Simplify $\frac{2^4 x^5 y^3}{6^2 x^3 y^7}$.

Solution: $\frac{2^4 x^5 y^3}{6^2 x^3 y^7} = \frac{2^4}{(2 \times 3)^2} \times \frac{x^5}{x^3} \times \frac{y^3}{y^7} = \frac{2^4}{2^2} \times \frac{1}{3^2} \times x^2 \times \frac{1}{y^4} = \frac{2^2}{3^2} \times x^2 \times \frac{1}{y^4} = \frac{4x^2}{9y^4}$

Note: In these examples we have written out almost every step in the simplification. In practice, as many steps as possible are performed mentally.

EXERCISE 3—2

Evaluate the following, using the theorems for exponents, and check your results.

1. $2^3 \times 3^3$
2. $4^2 \times 5^2$
3. $(.3)^3 \times (.5)^3$
4. $2^4 \div 5^4$
5. $7^3 \div 2^3$
6. $4^6 \div 2^6$
7. $(-4)^2 \times (-3)^2$
8. $(-2)^3 \times (3)^3$
9. $(-3)^2 \times (-5)^2$
10. $\left(\frac{1}{3}\right)^3 \times \left(\frac{1}{2}\right)^3$
11. $\left(\frac{2}{3}\right)^2 \div \left(\frac{3}{5}\right)^2$
12. $\left(-\frac{2}{5}\right)^2 \div \left(\frac{3}{4}\right)^2$
13. $2^3 \times 3^3 \div 4^3$
14. $(5^2 \div 2^2) \div 3^2$
15. $(15^3 \div 3^3) \times 2^3$
16. $(-2)^2 \times 5^2 \times 3^2$
17. $(-2)^3 \times (-3)^3 \div (4)^3$
18. $(-2a)^2 \times b^2 \div c^2$
19. $(x^3 \div y^3) \times 3^3$
20. $(1.25)^2 \div (.25)^2$
21. $(-.4)^3 \times (-.3)^3$
22. $(2p)^3 \times (3q)^3 \div r^3$
23. $[(21^3 \div 7^3) \times 2^3] \div 3^3$
24. $\left(\frac{1}{4}\right)^3 \times 2^3 \times \left(\frac{3}{5}\right)^3$
25. $\left(-\frac{1}{2}\right)^2 \times \left(\frac{1}{3}\right)^2 \div \left(\frac{5}{12}\right)^2$

Simplify the following:

26. $\frac{x^5y^2}{x^3y^4}$
27. $\frac{(x)^3}{y} \cdot \frac{(y)^4}{z}$
28. $(xy^2)(x^3y^2)$
29. $(-p^2q^3)(pq)^4$
30. $\frac{(3a^2b)^2}{(ab^2)^3}$
31. $\frac{2^3g^2h^4}{(gh^3)^2}$
32. $(x+y)^3 \times (x-y)^3$
33. $(x^3)(y^2)^3(z)^3$
34. $\frac{p^3q^2+p^2q^3}{p+q}$
35. $(pq^2r)^5 \div (p^2r)^3$
36. $(p^2q^3)^2 \div (p^2q)^3$
37. $(abc^2+a^2bc) \div abc$
38. $\frac{(a)^3}{b} \times \frac{(b^2a)^2}{c}$
39. $\left(\frac{x}{y^2}\right)^3 \div \left(\frac{xy}{z^2}\right)^2$
40. $(b^2)^3 \times c^3 \div (bc)^4$
41. $(p^2q+qp^2)^3 \div (p+q)^3$
42. $(xyz)^5 \times (xy^2)^3 \times z^4$
43. $(p^2q^3)^5 \div (pq)^3$
44. Prove the theorem $a^n \times b^n = (a \times b)^n$ where $a \in \mathrm{Re}$, $b \in \mathrm{Re}$, and $n \in \mathrm{N}$.

3 3 Real Bases and Integral Exponents

We have defined and used a^m when $a \in \text{Re}$ and $m \in \text{N}$. When we developed the number system, we extended our studies from the natural numbers to the integers, and we required the integers to obey the commutative, associative, and distributive rules obeyed by the natural numbers. Similarly, if we extend our work with exponents to include a^m, $a \in \text{Re}$ and $m \in \text{I}$, we wish to preserve all our rules for exponents.

Let us suppose that a^0 exists, and $a \neq 0$; then, if the product rule holds,

$$a^m \times a^0 = a^m;$$
$$\therefore \quad a^0 = a^m \div a^m; \qquad (a \neq 0)$$
$$\therefore \quad a^0 = 1.$$

For example,

$$(2^3)(2^0) = 2^{3+0};$$
$$\therefore \quad 8(2^0) = 8;$$
$$\therefore \quad 2^0 = 1.$$

Thus, if we *define*

$$a^0 = 1, \qquad (a \neq 0)$$

then a^0 satisfies *all* the exponent rules.

That a^0 satisfies all the exponent rules can easily be verified in any particular case, but it follows generally because the proofs of the other rules involve only the product rule and the standard rules of algebra.

In the same way, if we require a^{-n} to obey the product rule, then

$$a^n \times a^{-n} = a^{n-n}$$
$$= a^0$$
$$= 1.$$
$$\therefore \quad a^{-n} = \frac{1}{a^n}. \qquad (a \neq 0)$$

For example,

$$(3^4)(3^{-4}) = 3^{4-4}$$
$$= 3^0$$
$$= 1.$$
$$\therefore \quad 3^{-4} = \frac{1}{3^4}.$$

Therefore, if we *define* $$a^{-n} = \frac{1}{a^n}, \qquad (a \neq 0)$$

then a^{-n} satisfies all the exponent rules.

Summary

If $a, b \in \text{Re}$ $(a \neq 0, b \neq 0)$ and $m, n \in \text{I}$,

then (i) $a^m a^n = a^{m+n}$ (product rule)

(ii) $a^m \div a^n = a^{m-n}$ (quotient rule)

(iii) $(a^m)^n = a^{mn}$ (power rule)

(iv) $a^m b^m = (ab)^m$ (power of product rule)

(v) $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$. (power of quotient rule)

Example 1

Simplify (i) $a^3 \times a^{-2}$, (ii) $a^2 \div b^{-3}$.

Solution: (i) $a^3 \times a^{-2} = a^{3-2} = a^1 = a$

(ii) $a^2 \div b^{-3} = \dfrac{a^2}{b^{-3}} = a^2 b^{-(-3)} = a^2 b^3$

Example 2

Simplify $\dfrac{a^{-5} \times a^3}{a^4 \times a^{-2}}$ and then evaluate it when $a = 3$.

Solution: $\dfrac{a^{-5} \times a^3}{a^4 \times a^{-2}} = \dfrac{a^{-5+3}}{a^{4-2}} = \dfrac{a^{-2}}{a^2} = a^{-2-2} = a^{-4}$

When $a = 3$, $a^{-4} = 3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{3 \times 3 \times 3 \times 3} = \dfrac{1}{81}$.

Example 3

Simplify $\dfrac{a^3 \div b^{-2}}{c^{-2} \times d^2}$.

Solution: $\dfrac{a^3 \div b^{-2}}{c^{-2} \times d^2} = \dfrac{a^3 \times b^{-(-2)}}{d^2 \times c^{-2}} = \dfrac{a^3 \times b^2 \times c^2}{d^2} = \dfrac{a^3 b^2 c^2}{d^2}$

Example 4

Evaluate $(2^{-2})^3 \cdot (3^3)^{-1} \cdot 4^0$.

Solution: $(2^{-2})^3 \cdot (3^3)^{-1} \cdot 4^0 = 2^{-2(3)} \cdot 3^{3(-1)} \cdot 1$

$= 2^{-6} \cdot 3^{-3}$

$= \dfrac{1}{2^6} \cdot \dfrac{1}{3^3}$

$= \dfrac{1}{64} \cdot \dfrac{1}{27} = \dfrac{1}{1728}$

EXERCISE 3—3

Simplify by using the laws for exponents:

1. $x^3 \times x^{-2}$
2. $x^{-2} \times x^{-1}$
3. $x^2 \times x^{-2}$
4. $a^3 \div a^{-2}$
5. $a^{-1} \div a^{-2}$
6. $a^3 \div a^5$
7. $p^{-3} \times p^{-3}$
8. $p^{-2} \div p^3$
9. $p^{-3} \div p^2$
10. $q^5 \div q^3$
11. $q^5 \div q^{-5}$
12. $q^{-3} \div q^5$
13. $(x^2)^{-3}$
14. $(x^{-2})^3$
15. $(x^{-2})^{-3}$
16. $b^{-2} \cdot b^2$
17. $b^2 \cdot b^3$
18. $b^2 \cdot b^{-3}$
19. $2^3 \times 2^{-3}$
20. 125×5^{-3}
21. 16×2^{-2}
22. $2^{-2} \times 2^3$
23. $3^{-3} \times 3^4$
24. $5^2 \times 5^{-3}$
25. $2^{-3} \div 2^{-2}$
26. $4^{-1} \div 4^{-2}$
27. $3^2 \div 3^2$
28. $(12^2)^0$
29. $(3^0)^3$
30. $5^0 \times 3^0$
31. $3^5 \times 3^{-3}$
32. $7^3 \times 7^{-2}$
33. $4^2 \times 4^{-5}$
34. $2^7 \div 2^{-3}$
35. $3^{-2} \div 3$
36. $2^{-3} \times 2^3$
37. $5^{-5} \div 5^{-3}$
38. $9^{-2} \div 9^{-4}$
39. $(\frac{2}{3})^{-2} \times (\frac{2}{3})^3$
40. $(\frac{5}{6})^4 \div (\frac{5}{6})^{-4}$
41. $(.5)^{-2} \times (.5)^3$
42. $(.3)^{-3} \div (.3)^3$

Simplify:

43. $5^2y^{-3} \times 15^{-1}y^2$
44. $(12x)^2 \times (3x)^{-3}$
45. $\dfrac{5x^3y^{-4}}{2x^{-2}y^2}$
46. $\dfrac{\pi x^2 y}{\frac{4}{3}z \times 3}$
47. $(\frac{1}{2}x^3)^{-2} \times (2y^4)^3$
48. $(12x^2y)^{-2}$
49. $\left(\dfrac{x}{y}\right)^{-2} \times (x^{-1})^{-2}$
50. $\left(\dfrac{12xy}{z}\right)^{-2} \times \left(\dfrac{z}{x^2}\right)^2$
51. $(a^2bc^{-1})^2$
52. $(a^2b^{-1})^3$
53. $xy^{-1} + x^{-1}y$
54. $(abc)^3(ab^2)^{-3}$
55. $\dfrac{xy^{-3} + x^2y^{-2}}{y^{-4}}$
56. $(ab^{-2})^2 b^4 c^{-3} \left(\dfrac{b}{c^2}\right)^{-1}$
57. $(5x^2y)^{-1}(6x)^{-2} \div (36y^2x^3)^{-1}$
58. $(3pq)^{-2}(2q)^{-1}(5p)^{-2}$
59. $xy^{-2}z + x^{-1}y^{-1}z$
60. $(x^2 + y^{-2}) \div xy^{-1}$

3 4 Positive Real Bases and Rational Exponents

We extend the exponent rules to rational exponents by the same requirement that we used in the previous section.

Let us suppose that $a^{\frac{p}{q}}$ exists, $p, q \in \mathrm{I}$, $q \neq 0$, and that it obeys the power rule for exponents; then $(a^{\frac{p}{q}})^q = a^{\frac{p}{q} \times q} = a^p$.

Now $\sqrt[q]{c}$ is defined by the relation

$$\underbrace{\sqrt[q]{c} \times \sqrt[q]{c} \times \ldots \times \sqrt[q]{c}}_{(q \text{ factors})} = (\sqrt[q]{c})^q = \mathrm{c},$$

and, therefore, by this definition

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

For example, let $a=4$, $p=3$, and $q=2$;

then $\sqrt[2]{4^3} = \sqrt[2]{64} = 8,$

and thus we have $4^{\frac{3}{2}} = 8.$

Note that we have written $\sqrt[2]{\ }$ for the square root; normally we would use the usual form, $\sqrt{\ }$.

DEFINITION: If $a \in \mathrm{Re}^+$ (that is, $a \in \mathrm{Re}$ and $a > 0$),

then $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ where $p, q \in \mathrm{I}$, $q \neq 0$; that is, $\frac{p}{q} \in \mathrm{Ra}$.

Notice that we have restricted a in this case. This is essential if we confine our studies to the field of real numbers and assume that $\frac{p}{q}$ is *any* rational number.

$b^{\frac{1}{q}}$ does exist if $q \in \mathrm{I}$, $q \neq 0$, and $b \in \mathrm{Re}$, but it is not a real number in all cases. We shall consider this in the next section. If b is negative, $b^{\frac{1}{q}}$ can be a real number only if q is odd. A further study of this topic will have to wait until we study what are known as complex numbers.

The following examples illustrate the meaning of our definition and show how the rules of exponents are used.

Example 1

Evaluate $8^{\frac{1}{3}}$.

Solution: By definition $8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$

Example 2

Evaluate $9^{\frac{3}{2}}$.

Solution:

$$\begin{aligned} 9^{\frac{3}{2}} &= \sqrt[2]{9^3} \\ &= \sqrt[2]{729} \\ &= 27 \end{aligned}$$

We note that

$$27 = 3^3$$
$$= (\sqrt[3]{9})^3.$$

This result would suggest that $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$.
We can confirm this by using the power rule:

$$a^{\frac{p}{q}} = a^{\frac{1}{q} \times p} = (a^{\frac{1}{q}})^p$$
$$\therefore a^{\frac{p}{q}} = (\sqrt[q]{a})^p. \qquad \text{(by definition)}$$

This result is frequently more useful than the original definition for $a^{\frac{p}{q}}$, and may be regarded as an alternative definition.

Example 2b

Evaluate $9^{\frac{3}{2}}$.

Solution: $9^{\frac{3}{2}} = (\sqrt[2]{9})^3 = 3^3 = 27$

Example 3

Verify that $64^{\frac{1}{6}} \times 64^{\frac{1}{2}} = 64^{\frac{2}{3}}$.

Solution: By definition

$$64^{\frac{1}{6}} = \sqrt[6]{2\times2\times2\times2\times2\times2} = 2$$
$$64^{\frac{1}{2}} = \sqrt{8\times8} = 8$$
$$64^{\frac{2}{3}} = (\sqrt[3]{4\times4\times4})^2 = 4^2 = 16$$

and $64^{\frac{1}{6}} \times 64^{\frac{1}{2}} = 2\times8 = 16.$

Example 4

Verify that $(16^{\frac{1}{2}})^{\frac{3}{2}} = 16^{\frac{3}{4}}$.

Solution: By definition $16^{\frac{1}{2}} = \sqrt{16} = 4$

$$\therefore (16^{\frac{1}{2}})^{\frac{3}{2}} = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3$$

and also $16^{\frac{3}{4}} = (\sqrt[4]{2\times2\times2\times2})^3 = 2^3.$

Example 5

Simplify $a^{\frac{3}{4}} \times a^{\frac{5}{2}} \times a^{-\frac{7}{8}}$.

Solution:

$$a^{\frac{3}{4}} \times a^{\frac{5}{2}} \times a^{-\frac{7}{8}} = a^{(\frac{3}{4}+\frac{5}{2}-\frac{7}{8})}$$
$$= a^{\frac{6+20-7}{8}}$$
$$= a^{\frac{19}{8}}$$

Example 6

Simplify $\left(\dfrac{a^{\frac{2}{3}}b^{-\frac{1}{6}}}{a^{\frac{1}{2}}b^{\frac{1}{3}}}\right)^{\frac{3}{4}}$.

Solution:

$$\left(\frac{a^{\frac{2}{3}}b^{-\frac{1}{6}}}{a^{\frac{1}{2}}b^{\frac{1}{3}}}\right)^{\frac{3}{4}} = (a^{\frac{2}{3}}b^{-\frac{1}{6}}a^{-\frac{1}{2}}b^{-\frac{1}{3}})^{\frac{3}{4}}$$
$$= [a^{(\frac{2}{3}-\frac{1}{2})}b^{(-\frac{1}{6}-\frac{1}{3})}]^{\frac{3}{4}}$$
$$= (a^{\frac{1}{6}}b^{-\frac{1}{2}})^{\frac{3}{4}}$$
$$= a^{\frac{1}{6}\times\frac{3}{4}}b^{-\frac{1}{2}\times\frac{3}{4}}$$
$$= a^{\frac{1}{8}}b^{-\frac{3}{8}}$$

EXERCISE 3—4

(All variables are positive real numbers.)

Simplify:

1. $4^{\frac{1}{2}} \times 4^{\frac{3}{2}}$
2. $x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$
3. $(8)^{-\frac{1}{3}} \div (27)^{\frac{2}{3}}$
4. $y^{.3} \div y^{.5}$
5. $(5^{\frac{1}{3}})^3$
6. $(a^{.3})^{10}$
7. $(p^{-\frac{2}{3}})^3$
8. $(a^{\frac{1}{3}})^{\frac{3}{2}}$
9. $(a^2b)^{\frac{1}{2}}$
10. $(x^2y^3)^{\frac{1}{2}}$
11. $(27)^{\frac{1}{3}} \times 9^{\frac{1}{2}}$
12. $x^4 \times x^{1.2}$
13. $9^{\frac{1}{2}} \div 4^{\frac{3}{2}}$
14. $p^{.25} \div p^{-.50}$
15. $(2^{\frac{1}{2}})^6$
16. $(b^{-.2})^5$
17. $(q^{\frac{1}{4}})^{-3}$
18. $(b^{\frac{2}{3}})^{-\frac{1}{4}}$
19. $(p^9q^3)^{\frac{1}{3}}$
20. $(x^{\frac{1}{3}}y^{\frac{2}{3}})^3$
21. $(a^{-3} \times a^5)^{\frac{1}{2}}$

Simplify:

22. $(125y^{\frac{3}{2}})^{\frac{1}{3}}$
23. $(64x^4)^{\frac{1}{6}}$
24. $(12x^{\frac{1}{2}}y^{\frac{3}{2}}) \times (2xy^2)$
25. $3a^3b^2 \times 27^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}}$
26. $2x^{.4}y^{.9} \times 4^{.5}x^{.8}y^{.3}$
27. $9^{\frac{1}{2}}x^{-\frac{1}{2}}y^{\frac{3}{2}} \times 2x^{\frac{3}{2}}y^{-\frac{1}{2}}$
28. $4x^{1.2}y^{3.6} \div 2x^{.4}y^{.9}$
29. $2a^{1.3}b^{.6} \div 3a^{-.6}b^{.3}$
30. $\sqrt{2}b^{\frac{9}{4}}c^{\frac{1}{2}} \div b^{\frac{7}{4}}c^{\frac{1}{4}}$
31. $2\sqrt{2}p^{\frac{1}{3}}q^{-\frac{1}{5}} \div \sqrt{32}p^{\frac{1}{3}}q^{\frac{1}{5}}$
32. $(7\sqrt{3}x^2y)^{\frac{1}{2}} \div 3^{\frac{1}{4}}xy^2$
33. $(\pi^2rh)^{\frac{1}{2}} \div \frac{4}{3}\pi r^5$

Verify:

34. $(a+b)^{\frac{3}{2}}(a^2-b^2)^{\frac{1}{2}} = (a+b)^2(a-b)^{\frac{1}{2}}$
35. $(x-y)^{\frac{1}{2}}(x^2-y^2)^{\frac{3}{2}} = (x-y)^2(x+y)^{\frac{3}{2}}$
36. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x-y) = (x^{\frac{1}{2}}+y^{\frac{1}{2}})^2(x^{\frac{1}{2}}-y^{\frac{1}{2}})$
37. $(x^2-y^2)^{\frac{2}{3}}(x+y)^{\frac{1}{2}} = (x-y)^{\frac{2}{3}}(x+y)^{\frac{7}{6}}$
38. $\dfrac{5x^2(x^{\frac{1}{2}}+x^{-\frac{1}{2}})}{xy^2} = 5(x^{\frac{3}{2}}+x^{\frac{1}{2}})y^{-2}$
39. $\dfrac{2y(z^{\frac{2}{3}}+z^{\frac{5}{3}})}{z^{\frac{2}{3}}y^{\frac{3}{2}}} = 2y^{-\frac{1}{2}}(1+z)$
40. $x-5x^{\frac{1}{2}}+6 = (x^{\frac{1}{2}}-3)(x^{\frac{1}{2}}-2)$
41. $x^{\frac{4}{5}}+6x^{\frac{2}{5}}+8 = (x^{\frac{2}{5}}+2)(x^{\frac{2}{5}}+4)$

Evaluate: (Use square and cube root tables if necessary.)

42. $(27)^{\frac{1}{3}}$
43. $(25)^{-\frac{3}{2}}$
44. $(36)^{\frac{1}{6}} \times 6^{\frac{2}{3}}$
45. $8^{\frac{5}{3}} \times 9^{\frac{3}{2}}$
46. $(1024)^{-\frac{1}{5}}$
47. $(243)^{\frac{3}{5}}$
48. $(128)^{\frac{2}{7}}$
49. $(125)^{-\frac{5}{6}}$
50. $4^{\frac{5}{6}} \times 6^{\frac{2}{3}}$
51. $(5^3)^{-1.5} \div 5^{-\frac{3}{2}}$
52. $(3)^{-1.2} \times (3)^{2.7}$
53. $(.25)^{\frac{1}{4}} \div (.25)^{\frac{3}{4}}$

3 5 Principal Roots (Supplementary)

We noted in the previous section that when a is a positive real number, $\sqrt[q]{a}$ is defined as the *positive* number b such that $b\times b\times \ldots$ (q factors) $=a$. We now examine the situation that arises when a is negative. We shall use integers for ease in computation, but remember that the procedures apply to any real numbers.

For example, what is $\sqrt[3]{-27}$? If we let $b=\sqrt[3]{-27}$, then b is the solution of the equation

$$b\times b\times b=-27$$

and -3 is the *only real number* solution.

Therefore, $\sqrt[3]{-27}=-3.$

Similarly, $\sqrt[5]{-32}=-2.$

-3 is said to be the principal cube root of -27, and

-2, the principal fifth root of -32.

What is $\sqrt{-4}$? When we try to find the solution set of $b\times b=-4$, we see that no real number is a solution of this equation. Therefore, there is *no* real number which is $\sqrt{-4}$. Similarly, $\sqrt[4]{-81}$ is *not* real number.

Let us take another look at the roots of positive real numbers. To find $\sqrt{4}$ requires that we find the solution set of

$$b\times b=4,$$

and $b=+2$ *or* $b=-2$ are real number solutions to this equation. Only $+2$ is positive, and so by definition is $\sqrt{4}$; however, -2 is also a square root of 4 because $(-2)(-2)=4$. $\sqrt{4}=+2$ is said to be the *principal square root* of 4. Similarly, $\sqrt{25}=+5$ is the principal square root of 25, and $\sqrt[4]{81}=+3$ is the principal fourth root of 81.

When we examine cube roots, fifth roots, etc., of positive real numbers, a somewhat different situation arises. For example, what is $\sqrt[3]{125}$? To find $b=\sqrt[3]{125}$, we find the solution set of

$$b\times b\times b=125,$$

and $b=+5$ is the *only real number* solution.

Similarly, $b=\sqrt[3]{27}=+3$ and $b=\sqrt[5]{32}=+2$ are the *only real number* solutions of $b^3=27$ and $b^5=32$ respectively. $+3$ is called the *principal cube root* of 27, and $+2$ is the principal fifth root of 32.

We now state a definition of *principal roots* to meet the requirements in the above examples.

DEFINITIONS

If a is a *positive* real number and n is a natural number, then the *principal* n^{th} *root* of a is that *positive* number b which is such that $b^n=a$; the symbol $\sqrt[n]{a}$ is used for the principal root of a.

If a is a *negative* real number and n is an *odd* natural number, then the *principal n^{th} root* of a is that *negative* real number which is such that $b^n = a$; the symbol $\sqrt[n]{a}$ is used for the principal root of a.

If a is a *negative* real number and n is an *even* natural number, then a has *no real principal n^{th} root.*

We see now why we have restricted a to be a positive real number in the previous section on rational exponents. If a is a *negative* real number and q an *even* natural number, then $a^{\frac{p}{q}}$ does not exist in the field of the real numbers; we wish to avoid this difficulty. If a is a *positive* real number, then $a^{\frac{p}{q}}$ *always exists* in the field of the real numbers.

EXERCISE 3—5

Find the required principal roots of the following where such a root exists: (Use square and cube root tables for approximations when necessary.)

1. $\sqrt[3]{-64}$	2. $\sqrt[4]{81}$	3. $\sqrt{-36}$	4. $\sqrt[5]{32}$
5. $\sqrt[3]{47}$	6. $\sqrt{24}$	7. $\sqrt[3]{81}$	8. $\sqrt[4]{-625}$
9. $(49)^{\frac{1}{4}}$	10. $(-225)^{\frac{1}{6}}$	11. $(-21)^{\frac{1}{3}}$	12. $(-243)^{\frac{1}{5}}$
13. $(125)^{\frac{1}{3}}$	14. $(-512)^{\frac{1}{9}}$	15. $(256)^{\frac{1}{6}}$	16. $(-128)^{\frac{1}{7}}$

3 6 Scientific Notation

Scientists and engineers often write numbers, especially very large or very small numbers, in the form of a number, n, between 1 and 10 ($1 \leq n < 10$), multiplied by a power of ten.

Thus $3{,}765 = 3.765 \times 1000 = 3.765 \times 10^3$

And similarly $.0273 = 2.732 \div 100 = 2.732 \times \frac{1}{100} = 2.732 \times 10^{-2}$

$4{,}000{,}000{,}000{,}000 = 4 \times 10^{15}$

$.0000000000089 = 8.9 \times 10^{-12}$

Obviously, notation in this form is very useful for very large or very small numbers.

For example (to one digit accuracy),

the distance to one of the most distant galaxies is 10^{29} cm.,
the distance to α-Centauri, the nearest star to the sun, is 4×10^{18} cm.,
the distance of the farthest planet is 5×10^{14} cm.,
the radius of the earth is 6×10^{8} cm.,
the radius of a blood cell is 10^{-3} cm.,
the radius of an atom is about 5×10^{-8} cm.,
the radius of the nucleus of an atom is about 3×10^{-13} cm.

As the following examples show, we can perform all the usual operations of addition, subtraction, multiplication, division, and exponentiation in scientific notation.

Example 1

Calculate $2.731\times10^4+5.624\times10^5+7.258\times10^{-2}$ to four significant figures.

Solution:
$$\begin{aligned}&2.731\times10^4+5.624\times10^5+7.258\times10^2\\&=.2731\times10^5+5.624\times10^5+.007258\times10^5\\&=5.904\times10^5 \text{ to four significant figures}\end{aligned}$$

$$\begin{array}{r}5.624\\.2731\\.007258\\\hline5.904358\end{array}$$

Example 2

Calculate $(2.7\times10^4)\times(5.1\times10^7)\times(6\times10^{-8})$.

Solution:
$$\begin{aligned}&(2.7\times10^4)\times(5.1\times10^7)\times(6\times10^{-8})\\&=2.7\times5.1\times6\times(10^4\times10^7\times10^{-8})\\&=82.62\times10^3\\&=8.262\times10^4\end{aligned}$$

$$\begin{array}{r}2.7\\5.1\\\hline.27\\13.5\\\hline13.77\end{array}\qquad\begin{array}{r}13.77\\6\\\hline82.62\end{array}$$

Example 3

Find $(1.7\times10^3)\div(4\times10^{-5})$.

Solution:
$$\begin{aligned}&(1.7\times10^3)\div(4\times10^{-5})\\&=(1.7\div4)\times(10^3\div10^{-5})\\&=.425\times(10^3\times10^5)\\&=.425\times10^8\\&=4.25\times10^7\end{aligned}$$

Example 4

Evaluate $(1.2\times10^4)^3$.

Solution:
$$\begin{aligned}&(1.2\times10^4)^3\\&=(1.2)^3\times(10^4)^3\\&=1.728\times10^{12}\end{aligned}$$

Computer Notation

A more recent notation has been developed by scientists and engineers for use with computers. In computer notation a number is usually expressed as a decimal fraction n between 0.1 and 1.0 ($0.1\leq n<1.0$) multiplied by a power of 10 (for many computers the power of 10 is restricted to lie between 10^{-49} and 10^{+49}). Thus,

$2{,}731=.2731\times10^4$ (written as .2731 E4)

$4.86=.486\times10$ (written as .486 E1)

$0.00612=.612\times10^{-2}$ (written as .612 E−2)

All the usual arithmetical operations can be performed in this notation either by human beings or by a computer. Arithmetic performed in this notation by a computer is called *floating point arithmetic* because the computer does the calculations with the significant digits and the exponents separately, and only when it prints out the result does it recognize the true decimal nature of the result. The instructions to multiply 7354.8961 and 246.76943 would appear as follows:

.73548961E4 * .24676943E3,

and the computer would print out the result as

.1814963518306223E7.

In the more usual decimal notation, this is

1,814,963.518306223.

(Note that 3. * 4. in computer format means 3×4; similarly, 312./57. means $312 \div 57$.)

EXERCISE 3—6

Express the following in (i) scientific notation and (ii) computer notation.

1. (a) 235.1 (b) 7.568 (c) .4371
2. (a) 561,400 (b) 35.36 (c) .001736
3. (a) 3,468 (b) 27.41 (c) 1.693
4. (a) 5.463 (b) 128.4 (c) 1,234,678
5. (a) 9,876.54321 (b) .00003 (c) .07892

Perform the following calculations in the notation given, and state which notation it is. Rewrite the answers in ordinary decimal notation.

6. $2.345 \times 10^2 + 6.364 \times 10^2$
7. $5.32 \times 10^5 - 7.4 \times 10^4$
8. $2.25 \times 10^2 \times 3.1 \times 10^3$
9. $5.12 \times 10^3 \div 3.2 \times 10^2$
10. $2.88 \times 10^{-2} \div 9 \times 10^{-3}$
11. $6.25 \times 10^{-1} + 1.5 \times 10$
12. $.7892 \times 10^{-2} \times .3 \times 10^4$
13. $.375 \times 10^5 \div .15 \times 10^2$
14. $.72 \times 10^{-1} + .8 \times 10^{-2}$
15. $.569 \times 10^{13} - .321 \times 10^{12}$
16. $1.37 \times 10^{23} \div 1.5 \times 10^{11}$
17. $6.4 \times 10^{-24} \times 1.2 \times 10^8$
18. .362E9/.6E7
19. .735E0 * .6E4
20. (.9685E2+.3682E2) * .25E(−4)/.4E3.

3.7 Positive Real Bases and Real Exponents

The extension of the rules for exponents to the case where the exponent is a real number may seem difficult because all our previous definitions have involved whole numbers. For example,

$$a^n = a \times a \times a \ldots (n \text{ factors})$$

was the definition for positive integral exponents.

$a^{-n} = \frac{1}{a^n}$, where n was a positive integer, was used in the definition for negative integral exponents, and $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$, where q was a positive integer, was used in the definition for rational number exponents. We know that some real numbers, namely the irrational numbers, are not rational numbers. How then can we define a number involving real exponents?

We remember that an irrational number can always be approximated as closely as we wish by rational numbers. Thus

$$1.414 < \sqrt{2} < 1.415$$
$$1.4142 < \sqrt{2} < 1.4143 \qquad \text{and so on.}$$

We can define a^r for any rational number r; that is, we can define

$$a^{1.41421} \text{ and } a^{1.41422},$$

and as $\sqrt{2}$ lies between 1.41421 and 1.41422, so $a^{\sqrt{2}}$ lies between $a^{1.41421}$ and $a^{1.41422}$.

If we represent the two sets of values of a^r

$$a^1,\ a^{1.4},\ a^{1.41} \ldots \text{ and } a^2, a^{1.5}, a^{1.42} \ldots$$
$$\text{by } b_1, b_2, b_3 \ldots b_n \ldots \text{ and } c_1, c_2, c_3 \ldots c_n \ldots,$$

then the value of $a^{\sqrt{2}}$ always lies between the values of b_n and c_n, each of which gradually approaches a unique irrational number as n increases. This irrational number is the value of $a^{\sqrt{2}}$.

In this way we can define a^m when a is a positive real number and m is any real number.

For example, the value of $2.71828182459^{\sqrt{2}}$ correct to four significant figures is 4.113, to six significant figures is 4.11325, and to eight significant figures is 4.1132504.

We assume that the rules for exponents are obeyed by such real exponents; that is

$$\text{(i) } a^x a^y = a^{x+y} \qquad \text{(ii) } (a^x)^y = a^{xy}$$
$$\text{(iii) } a^x b^x = (ab)^x \qquad \text{(iv) } a^x \div b^x = \left(\frac{a}{b}\right)^x$$

for all positive real numbers a and $b (b \neq 0)$ and all real numbers x and y.

Note that $a \in \mathrm{Re}^+$, and therefore $a^m \in \mathrm{Re}^+$ for all $m \in \mathrm{Re}$; this is a consequence of the fact that a^m is determined by a sequence of values of a^r, where $r \in \mathrm{Ra}$, and a^r is always positive when $a \in \mathrm{Re}^+$.

3·8 The Exponential Functions

In Chapter 2 (pages 31, 32) we drew the graphs of such functions as $y=6-x$, $y=x^2$, by making a table of some of the elements of the set of ordered pairs determined by the function. We shall now do this for functions defined by equations such as $y=2^x$, $y=3^x$, $y=10^x$.

DEFINITION: A function defined by the rule $y=a^x$ is called an *exponential function*; it determines the set of ordered pairs

$$\{(x, a^x) | a \in \mathrm{Re}^+, x \in \mathrm{Re}\}.$$

It will be seen from the definition of a^x that there is a unique value of a^x for each value of x and that $a^x > 0$. Thus we confirm that the rule $y=a^x$ does in fact define a function, and that the domain is the set of real numbers. Also, the range, the set of values of a^x, is the set of *positive* real numbers; that is, $a^x \in \mathrm{Re}^+$.

Example 1

Sketch the graph of the function defined by $y=2^x$.

Solution: We can easily calculate a table of values for $y=2^x$.

x	$y=2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16
5	32

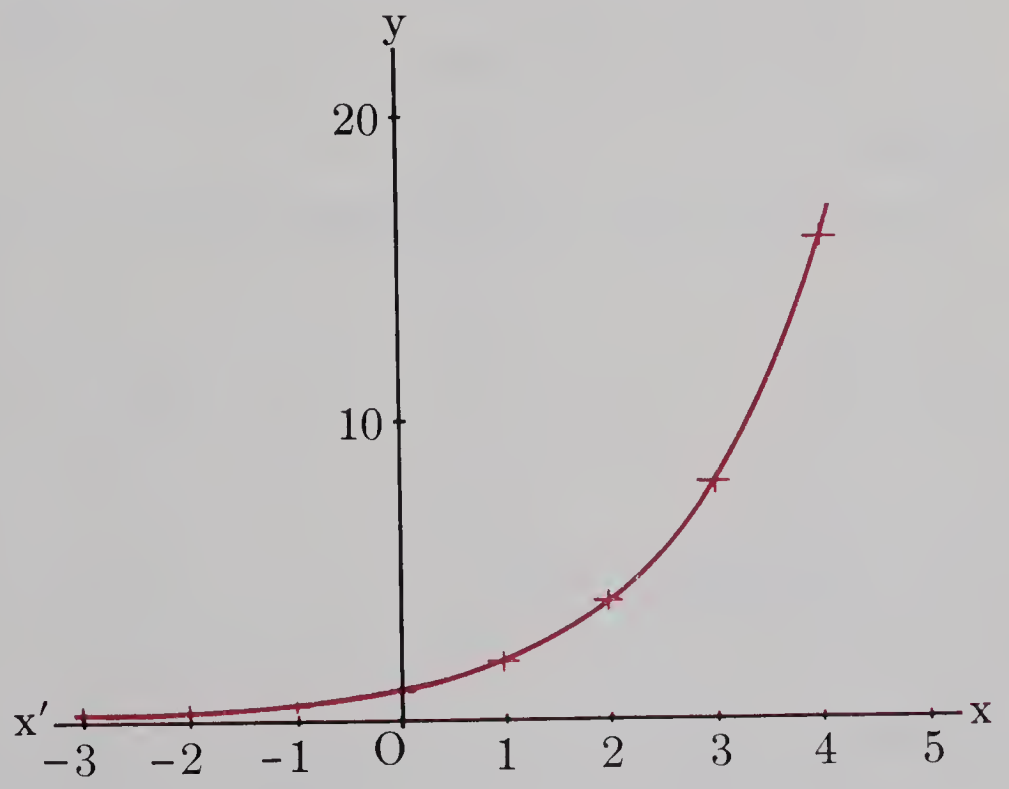


Figure 3-1 $\{(x, y) | y=2^x, x \in \mathrm{Re}\}$

From the table of values and the graph, we see that

(i) $2^x > 0$,

(ii) $2^x > 1$ if $x > 0$,

(iii) $2^x < 1$ if $x < 0$,

(iv) $2^{x_1} < 2^{x_2}$ if $x_1 < x_2$.

Example 2

Sketch the graph of the set of ordered pairs $\{(x, y)|y=(1.5)^{-x},\ x\in \mathrm{Re}\}$.

Solution:

x	$y=(1.5)^{-x}$
-3	3.375
-2	2.25
-1	1.5
0	1
1	.66
2	.44
3	.296

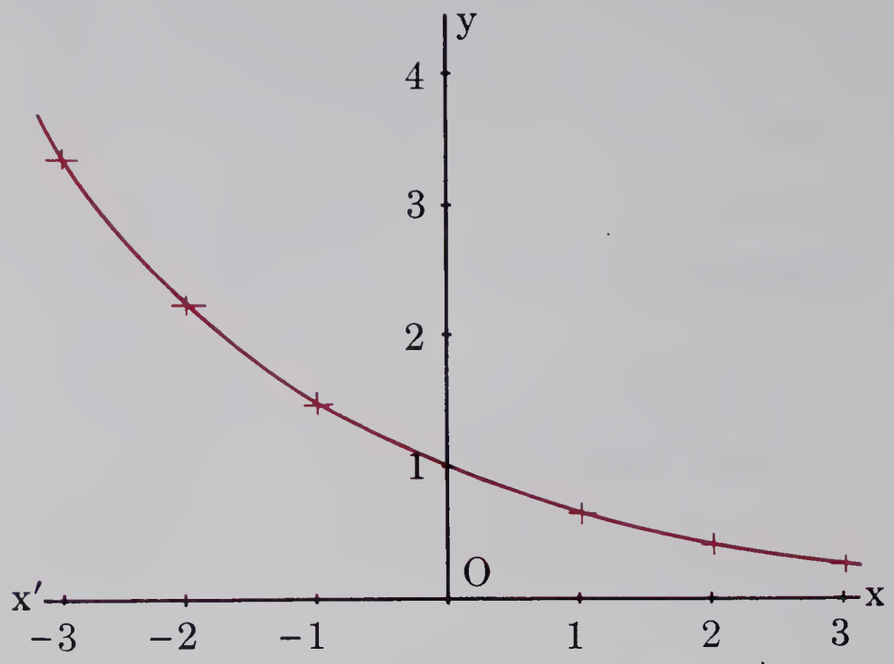


Figure 3-2 $\{(x, y)|y=(1.5)^{-x}, x\in \mathrm{Re}\}$

From the graph we see that

(i) $(1.5)^{-x}>0$

(ii) $(1.5)^{-x}<1$ if $x>0$,

(iii) $(1.5)^{-x}>1$ if $x<0$,

(iv) $(1.5)^{-x_1}>(1.5)^{-x_2}$ if $x_1<x_2$.

Example 3

Sketch the graph of the function defined by $y=(\frac{1}{2})^x$.

Solution:

x	$y=(\frac{1}{2})^x$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

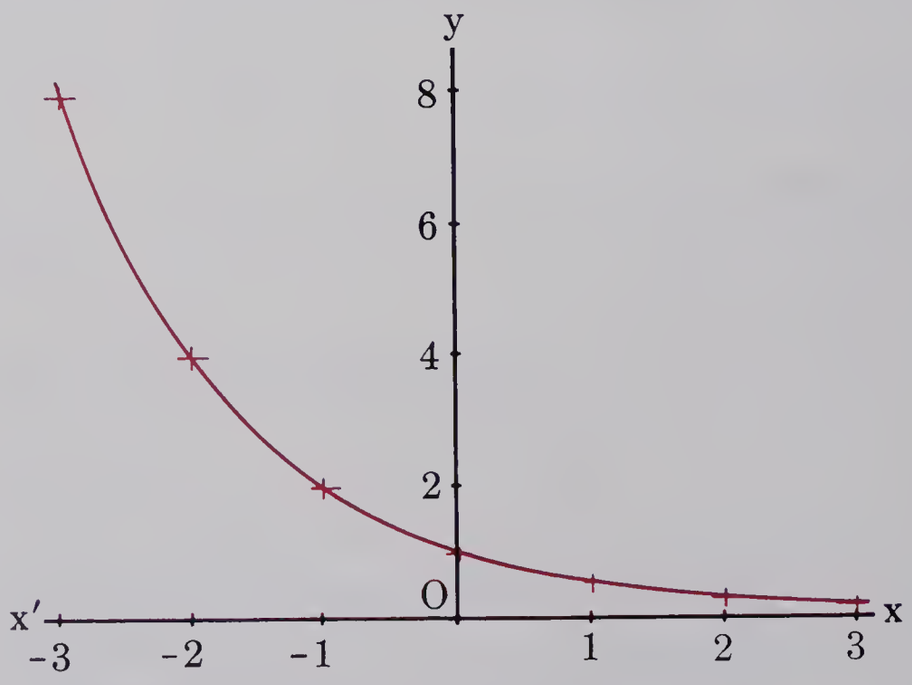


Figure 3-3 $\{(x, y)|y=(\frac{1}{2})^x, x\in \mathrm{Re}\}$

EXERCISE 3—7

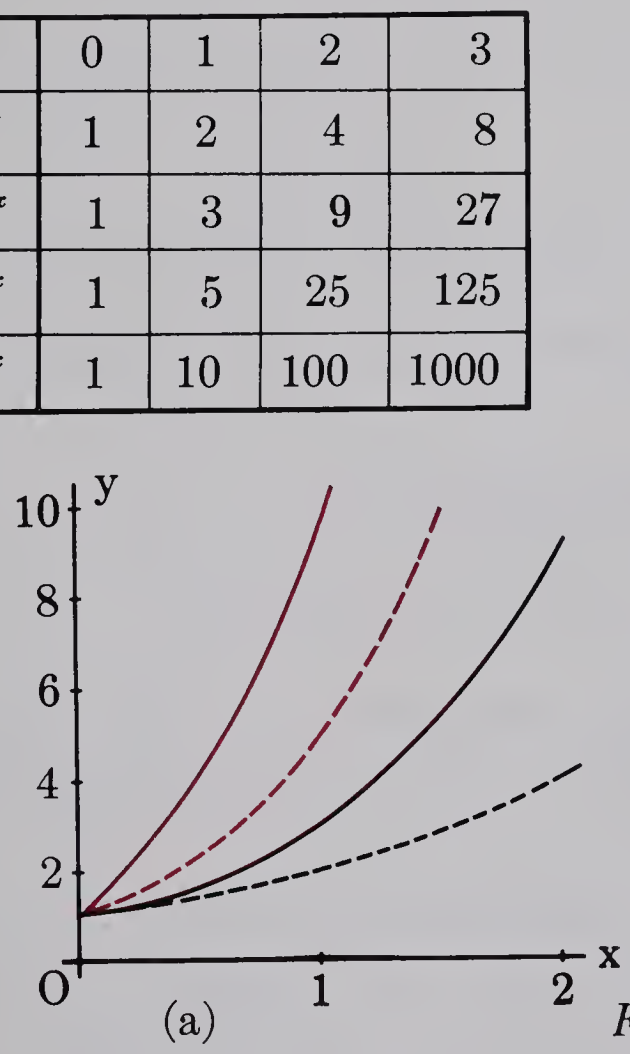
1. $1\frac{2}{3} < \sqrt{3} < 1\frac{3}{4}$ Use this fact to show that $1024 < 64^{\sqrt{3}} < 1448$. (Use tables of square and cube roots if necessary.)

2. Sketch the graphs of:
   (a) $\{(x, 3^x) | -2 \leq x \leq 3, x \in \text{Re}\}$; (b) $\{(x, y) | y = 2^{-x}, -4 \leq x \leq 2, x \in \text{Re}\}$;
   (c) the function defined by $y = (1.2)^x$ in the domain $-3 \leq x \leq 3$, $x \in \text{Re}$;
   (d) the function f in the domain $-2 \leq x \leq 2$, $x \in \text{Re}$, where $f(x) = (1.5)^{-x}$.
   (Evaluate (x, a^x) for each integral value of x in the given domain, and sketch a smooth curve through these points.)

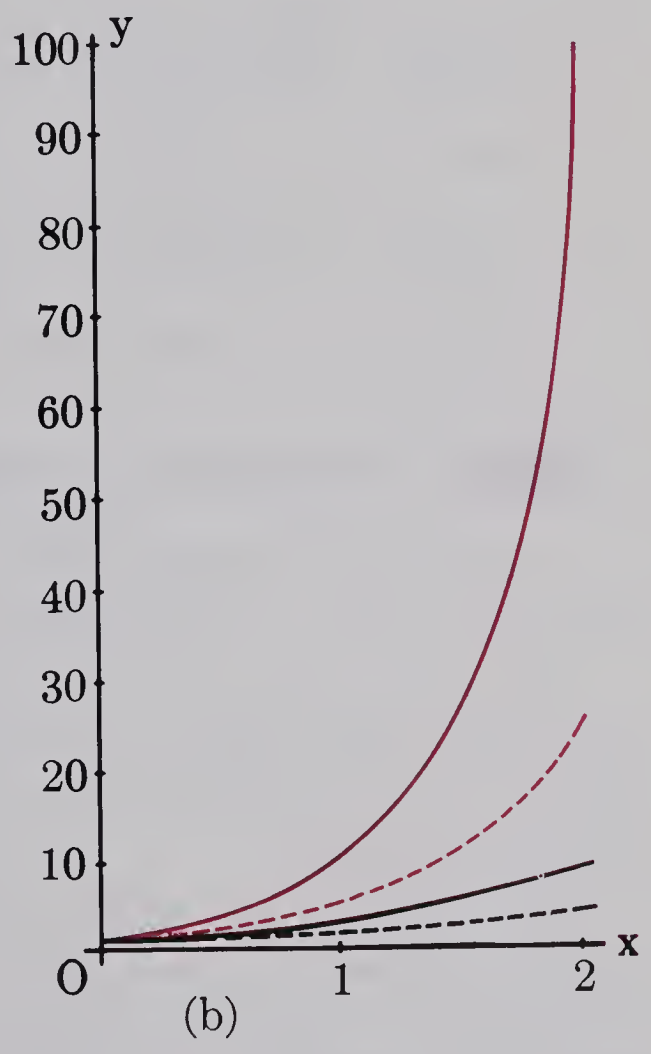
3 9 Growth Rates of the Exponential Functions

We shall now sketch the graphs of the functions defined by $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ on the same coördinate grid for $x > 0$.

x	0	1	2	3
2^x	1	2	4	8
3^x	1	3	9	27
5^x	1	5	25	125
10^x	1	10	100	1000



(a)



(b)

Figure 3-4

We see immediately that for any given number $a > 0$, $2^a < 3^a < 5^a < 10^a$; this set of inequalities orders the values of the function in the domain $x > 0$ for each value of x. Indeed we see that 10^x increases in size much more rapidly than 5^x as x increases. Similarly 5^x increases more rapidly than 3^x, and 3^x more rapidly than 2^x.

Note that $1^x = 1$ for all real x and therefore does not increase at all; 1^x is constant for all real x.

EXERCISE 3—8

1. Sketch the graphs of the functions whose values are given by $f(x)=2^x$, 3^x, 5^x, and 10^x in the domain $-2\leq x\leq 0$. (Use the table of reciprocals and the table of values in this section.)
   Verify that $2^b>3^b>5^b>10^b$ when $b<0$.

2. On the same diagram, sketch the graphs of the functions defined by
   (a) $y=(1.2)^{-x}$ (b) $y=(1.5)^{-x}$ (c) $y=2^{-x}$
   (d) $y=3^{-x}$ in the domain $-2\leq x\leq 1$. (See Exercise 3-7, (2).)

3. On the same diagram, sketch the graphs of the following sets of ordered pairs in the domain $-1\leq x\leq +1$:
   (a) $\{(x, (\frac{1}{2})^x)|x\in \mathrm{Re}\}$ (b) $\{(x, (\frac{1}{3})^x)|x\in \mathrm{Re}\}$ (c) $\{(x, 1^x)|x\in \mathrm{Re}\}$
   (d) $\{(x, 2^x)|x\in \mathrm{Re}\}$ (e) $\{(x, 3^x)|x\in \mathrm{Re}\}$ (f) $\{(x, 5^x)|x\in \mathrm{Re}\}$

4. In (2) which function decreases most rapidly with x? which least rapidly? Order the functions in the domains $-2\leq x<0$ and $0\leq x<1$.

5. Order the functions in (3) in the domains $x<0$ and $x>0$.

6. Compare the graphs in (2) and (3). Are any the same? If so why?

7. What do you notice about the graphs of $y=(\frac{1}{2})^x$, $y=2^{-x}$, and $y=2^x$?

3 10 Computations Using Graphs of $y=a^x$

(Large scale graphs of the functions defined by $y=2^x$, $y=3^x$, $y=5^x$, and $y=10^x$ for $0\leq y\leq 100$ will be found at the end of the book.)

We have always considered addition and subtraction to be simpler operations than multiplication and division, and we find the operations of multiplication and division to be easier than the operations of finding powers or roots of a number.

Our work in the earlier sections of this chapter has shown that, to *multiply powers* of a number, we *add exponents*, and similarly to *divide powers* of a number, we *subtract exponents* (rule i). To find *powers*, we *multiply exponents*, and to find *roots*, we *divide exponents* (rule ii). We shall now use these results to perform various computations using large scale graphs of $y=a^x$.

Note: We often use the abbreviated clause "$y=a^x$" for the formal clause, "the function defined by $y=a^x$," or for the more complex symbolism, "$f=\{(x, y)|y=a^x, x\in \mathrm{Re}\}$". However, we should not do this in formal proofs unless we so specify.

Example 1

(i) Calculate 5.4×7.7 by using the graphs of

(a) $y = 2^x$ (b) $y = 10^x$.

(ii) By using the same graphs, calculate 54×77.

(Use the large scale graphs at the back of the book.)

Solution: (i) (a) From the graph of $y = 2^x$, we see that

when $y = 5.4$, $x \simeq 2.38$;

when $y = 7.7$ $x \simeq 2.96$.

$\therefore 5.4 \simeq 2^{2.38}$, and $7.7 \simeq 2^{2.96}$.

$$(5.4)(7.7) \simeq 2^{2.38+2.96}$$
$$= 2^{5.34}$$
$$\simeq 41.6 \qquad \text{(from the graph)}$$

(b) $5.4 \simeq 10^{.73}$ and $7.7 \simeq 10^{.89}$

$$\therefore (5.4)(7.7) \simeq 10^{.73+.89}$$
$$= 10^{1.62} \quad \text{or} \quad = 10^{.62} \times 10^1$$
$$\simeq 42. \qquad \simeq 4.2 \times 10$$
$$= 42$$

(ii) (a) From the graph of $y = 2^x$, we see that

$54 \simeq 2^{5.73}$ and $77 \simeq 2^{6.24}$

$$\therefore 54 \times 77 \simeq 2^{5.73+6.24}$$
$$= 2^{11.97}.$$

This is not on the graph; therefore we proceed as follows:

$54 = 5.4 \times 10$ and $77 = 7.7 \times 10$

$$\therefore 54 \times 77 \simeq 5.4 \times 7.7 \times 10 \times 10$$
$$\simeq 2^{2.38} \times 2^{2.96} \times 10^2$$
$$= 2^{5.34} \times 10^2$$
$$\simeq 41.6 \times 10^2$$
$$= 4160.$$

(b) $54 \simeq 10^{1.73}$ and $77 \simeq 10^{1.89}$

$$\therefore 54 \times 77 \simeq 10^{1.73+1.89}$$
$$= 10^{3.62}$$
$$= 10^{0.62} \times 10^3$$
$$\simeq 42 \times 10^3$$
$$= 4200.$$

Although using $y = 2^x$ gives us more accurate results with graphs to a large scale, notice that the calculations are more complicated when we use $y = 2^x$ rather than $y = 10^x$ when numbers are changed by multiples of 10. This is seen to be natural when we remember that our *decimal notation is itself to the base 10.* (Consider, for example, our work on scientific notation.)

Example 2

Find (i) $78.3 \div 3.7$ (ii) $263 \div 179$ by using the graphs of

(a) $y = 5^x$ (b) $y = 10^x$.

Solution: (i) (a) $78.3 \simeq 5^{2.72}$ and $3.7 \simeq 5^{.80}$

$\therefore 78.3 \div 3.7 \simeq 5^{2.72-.80}$

$= 5^{1.92}$

$\simeq 21.0.$

(b) $78.3 \simeq 10^{1.90}$ and $3.7 \simeq 10^{.57}$

$\therefore 78.3 \div 3.7 \simeq 10^{1.90-.57}$

$= 10^{1.33}$

$\simeq 21.0.$

(ii) (a) $263 \div 179 = 26.3 \div 17.9$

$26.3 \simeq 5^{2.03}$ and $17.9 \simeq 5^{1.80}$

$\therefore 26.3 \div 17.9 \simeq 5^{2.03-1.80}$

$= 5^{.23}$

$\simeq 1.5.$

(b) $26.3 \simeq 10^{1.41}$ and $17.9 \simeq 10^{1.24}$

$263 \simeq 10^{2.41}$ and $179 \simeq 10^{2.24}$

$\therefore 263 \div 179 \simeq 10^{2.41-2.24}$

$= 10^{.17}$

$\simeq 1.5.$

Example 3

Find (i) $\sqrt{47}$ (ii) $\sqrt{4.7}$ by using the graphs of

(a) $y = 3^x$, (b) $y = 10^x$.

Solution: (i) (a) $47 \simeq 3^{3.5}$

$\sqrt{47} \simeq (3^{3.5})^{\frac{1}{2}}$

$= 3^{3.5 \times \frac{1}{2}}$

$= 3^{1.75}$

$\simeq 6.9$

(b) $47 \simeq 10^{1.68}$

$\sqrt{47} \simeq (10^{1.68})^{\frac{1}{2}}$

$= 10^{1.68 \times \frac{1}{2}}$

$= 10^{.84}$

$\simeq 6.9$

(ii) (a) $4.7 \simeq 3^{1.39}$

$\sqrt{4.7} \simeq (3^{1.39})^{\frac{1}{2}}$

$= 3^{1.39 \times \frac{1}{2}}$

$= 3^{.695}$

$\simeq 2.2$

(b) $4.7 \simeq 10^{.67}$

$\sqrt{4.7} \simeq (10^{.67})^{\frac{1}{2}}$

$= 10^{.67 \times \frac{1}{2}}$

$= 10^{.34}$

$\simeq 2.2$

Again we note that, because $47 \simeq 10^{1.68}$ and $4.7 \simeq 10^{.68}$, the two calculations to the base 10 are very similar. In fact, if we work with $y = 10^x$, we need only the graph for x in the domain $0 \leq x \leq 1$, and therefore y in the range $1 \leq y \leq 10$. With this restricted part of the function we can calculate any product, quotient, power, or root involving numbers of any size. The number of significant figures that we can use in the result will depend on how accurately we can read the graph or table of values, not on the absolute size of the numbers involved. As we shall see in the next chapter, this simplifies the table of values that has to be prepared for use in computations; therefore, in that chapter, we shall work only in the base 10.

EXERCISE 3—9

Using the large scale graphs, evaluate:

1. $(9.7)(6.8)$ using $y = 2^x$ and $y = 10^x$
2. $(8.3)(7.9)$ using $y = 3^x$ and $y = 10^x$
3. $(17.2)(5.7)$ using $y = 5^x$ and $y = 10^x$
4. $(81.6) \div (12.3)$ using $y = 5^x$ and $y = 10^x$
5. $(93.4) \div (8.9)$ using $y = 2^x$ and $y = 10^x$
6. $(71.6) \div (9.3)$ using $y = 3^x$ and $y = 10^x$
7. $\sqrt{87.3}$ using $y = 5^x$ and $y = 10^x$
8. $\sqrt[3]{65.2}$ using $y = 2^x$ and $y = 10^x$
9. $\sqrt[5]{76.3}$ using $y = 3^x$ and $y = 10^x$
10. $\sqrt[4]{38.6}$ using $y = 5^x$ and $y = 10^x$

Use $y = 10^x$ to evaluate:

11. $23.7 \times 36.8 \div 12.7$
12. $\sqrt{\dfrac{67.4 \times 83.6}{41.7}}$
13. $\left(\dfrac{38.7}{6.9 \times 9.3}\right)^3$
14. $\left(\dfrac{29.4 \times 13.1}{5.63}\right)^2$
15. $\sqrt[3]{14.7 \times 31.4 \times 5.68}$
16. $\sqrt[5]{2.39 \times 7.62 \times 4.72}$
17. $\dfrac{3.62 \times 5.98}{12.7 \times 4.16}$
18. $\dfrac{4.72 \times 8.91}{1.37 \times 6.38}$
19. $\sqrt[3]{\dfrac{12.3}{4.78 \times 7.64}}$
20. $\sqrt[7]{\dfrac{15.9 \times 34.7}{4.82 \times 8.76}}$

3 11 Chapter Summary

The Rules for Exponents

(i) $a^x a^y = a^{x+y}$ (ii) $(a^x)^y = a^{xy}$

(iii) $a^x b^x = (ab)^x$ (iv) $a^x \div b^x = \left(\frac{a}{b}\right)^x$

hold for $a \in \mathrm{Re}^+$, $b \in \mathrm{Re}^+$, $x \in \mathrm{Re}$, and $y \in \mathrm{Re}$, and for $a \in \mathrm{Re}$, $b \in \mathrm{Re}$, $x \in \mathrm{I}$, and $y \in \mathrm{I}$.

$a^n \in \mathrm{Re}^+$ when $a \in \mathrm{Re}^+$ and $n \in \mathrm{Re}$.

The Principal Root of a Number (Supplementary)

(i) If $a \in \mathrm{Re}^+$ and $n \in \mathrm{N}$, then $\sqrt[n]{a} = a^{\frac{1}{n}} = b$ is the principal root of a where $b \in \mathrm{Re}^+$ and $b^n = a$.

(ii) If $a \in \mathrm{Re}^-$ and $n \in \mathrm{N}$ and n is odd, then $\sqrt[n]{a} = a^{\frac{1}{n}} = b$ where $b \in \mathrm{Re}^-$ and $b^n = a$.

(iii) If $a \in \mathrm{Re}^-$ and $n \in \mathrm{N}$ and n is even, then a has no principal *real* root.

Scientific Notation

$\mathrm{N} = n \times 10^v$ where $\mathrm{N} \in \mathrm{Re}$, $n \in \mathrm{Re}$ $(1 \leq n < 10)$, $v \in \mathrm{I}$.

Computer Notation

$\mathrm{M} = m \times 10^u$ where $\mathrm{M} \in \mathrm{Re}$, $m \in \mathrm{Re}$ $(.1 \leq m < 1)$, $u \in \mathrm{I}$ $(-49 < u < +49)$.
This is written $m\mathrm{E}u$.

An Exponential Function is defined by $y = a^x$ and determines the set of ordered pairs $\{(x, a^x) | a \in \mathrm{Re}^+, x \in \mathrm{Re}\}$.
If p is any given *positive* real number and $a, b \in \mathrm{Re}^+$, $b > a$, then $0 < a^p < b^p$.
If q is any given *negative* real number and $a, b \in \mathrm{Re}^+$, $b > a$, then $0 < b^q < a^q$.

REVIEW EXERCISE 3—10

1. Simplify:

(a) $\dfrac{a^3 a^5}{a^4 a^2}$ (b) $\left(\dfrac{p^{-2} q^3}{r^4}\right)^3$

(c) $(p^2 q)^{\frac{1}{3}} \times (p^3 q^4)^{\frac{1}{6}}$ (d) $x^{m-2} x^{3+n} x^{5+n-2m}$

(e) $\left(\dfrac{\sin\theta}{\cos\theta}\right)^{-1} . (\sin\theta)^2$ (f) $\dfrac{16^2 \times 4^{3p}}{8^{p+4}}$

(g) $\sqrt{p^6 q^4}$ (h) $\sqrt[3]{x^6 \div y^9}$

(i) $\dfrac{x^{1-r}}{x^{1-s}}$

(j) $\dfrac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^2y^3y^{-1}}$

(k) $(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})$

(l) $(a^{\frac{2}{3}}+b^{\frac{2}{3}})a^{\frac{1}{3}}b^{\frac{1}{3}}$

(m) $\sqrt[6]{8x^9y^3z^3}$

(n) $27^{\frac{1}{3}}-16^{\frac{1}{4}}+25^{\frac{1}{2}}$

(o) $\sqrt[4]{\dfrac{36r^2s^2}{81t^2}}$

(p) $\dfrac{a-b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$

(q) $\dfrac{(p^2qr^3)^4}{p^3r^5}$

(r) $\dfrac{(64xy^2)^{\frac{1}{6}}(25y^4)^{\frac{1}{2}}}{(27xy^4)^{\frac{1}{3}}}$

(s) $(9x^3)^{\frac{1}{2}}\times(32y^3)^{\frac{1}{5}}$

(t) $(3xy^2)^3(x^3y^6)^{\frac{1}{2}}$

2. Find the principal roots (supplementary):

(a) $\sqrt{+49}$

(b) $\sqrt[3]{216}$

(c) $\sqrt[5]{-243}$

(d) $\sqrt[4]{-81}$

(e) $\sqrt[3]{28}$

(f) $\sqrt{43}$

(g) $\sqrt[6]{64}$

(h) $\sqrt[4]{64}$

(i) $\sqrt[3]{-27}$

3. Write the following numbers in (i) scientific notation and (ii) computer notation:

(a) 231.56

(b) 579,654.

(c) 8.9361

(d) .005137

(e) 51.67

(f) 5176.83

(g) .000008124

(h) 4,731.85

(i) 39.79

(j) .023189

(k) $\sqrt{2.89}$

(l) $\sqrt{256}$

(m) $\dfrac{.12}{2.50}$

(n) $\dfrac{3.1}{8.7}$

(o) $\sqrt{16{,}000{,}000}$

(p) $\sqrt[3]{27{,}000}$

(q) $\sqrt[5]{3{,}200{,}000}$

(r) $\sqrt{.000016}$

(s) $\frac{514}{25}$

(t) $\frac{5}{3}$

(u) $\frac{15}{400}$

(v) $\sqrt{.000169}$

(w) $\sqrt[3]{125{,}000}$

(x) $\sqrt[4]{810{,}000}$

4. Sketch the graphs of the functions defined by

(a) $y=2.5^{-x}$

(b) $y=1.4^{x}$

(c) $y=.5^{x}$

in the domain $-2\leq x<+2$.

Order these functions in the domains $x<0$ and $x>0$.

5. Use the large scale graphs to calculate:

(a) 8.4×6.8 using $y = 5^x$

(b) $67.4 \div 7.3$ using $y = 3^x$

(c) $\dfrac{9.1 \times 7.4}{5.6}$ using $y = 2^x$

(d) $\sqrt{83.6}$ using $y = 10^x$

(e) $\sqrt[3]{92.8}$ using $y = 3^x$

(f) $\dfrac{27.3 \times 31.6}{47.2}$ using $y = 5^x$

(g) $\dfrac{471}{18.2 \times 26.8}$ using $y = 10^x$

(h) $\sqrt[5]{538}$ using $y = 10^x$

(i) $\dfrac{8.6 \times 6.4 \div 12.4 \times 5.8}{4.6 \times 18.2}$ using $y = 2^x$

(j) $\sqrt[4]{92.4 \times 31.8 \times 53.6}$ using $y = 3^x$

6. Use $y = 10^x$ to calculate:

(a) $\dfrac{7.55 \times 1.84}{5.60}$

(b) $\dfrac{9.46 \times 6.43}{19}$

(c) $6.81 \times 4.68 \times 5.29$

(d) $8.76 \times 3.15 \times 2.84$

(e) $\dfrac{12.8}{4.73 \times 1.92}$

(f) $\dfrac{8.98}{1.75 \times 3.92}$

(g) $\sqrt{3.67 \times 5.34}$

(h) $\sqrt[3]{6.41 \times 12.3}$

(i) $\sqrt[5]{82.9 \times 21.7}$

(j) $\sqrt[4]{46.7 \times 68.4}$

Chapter 4

Logarithmic Functions

4 1 Calculations Using the Graph of $y = 10^x$

Using scientific notation, we can express any number as a product of a number a $(1 \leq a < 10)$ and an integral power of 10. For example,

$1{,}367 = 1.367 \times 10^3$, $26.57 = 2.657 \times 10^1$, $.08147 = 8.147 \times 10^{-2}$.

Also, as we have seen in Section 3-6, we can multiply and divide and find powers and roots using numbers expressed in scientific notation.

Using the laws of exponents, we can multiply by adding exponents, divide by subtracting exponents, form powers by multiplying exponents, and extract roots by dividing exponents, as in Section 3-9. We also recall that, because the base of our number system is 10, the exponential function to base 10 is the most convenient to use for such computations.

Let us now combine these observations about scientific notation and exponents and use them to develop a method to perform calculations with numbers of any magnitude.

[Use the large scale graph of $y = 10^x$ at the end of the book, but in the domain $0 \leq x \leq 1$ and range $1 \leq y \leq 10$ only.]

Example 1

Evaluate (a) $2{,}100 \times .43$ (b) $.078 \div .23$.

Solution: (a) $2{,}100 = 2.1 \times 10^3$ and $.43 = 4.3 \times 10^{-1}$

$\simeq 10^{.3222} \times 10^3$ $\simeq 10^{.6335} \times 10^{-1}$ (from the graph)

$\therefore 2{,}100 \times .43 \simeq 10^{.3222} \times 10^3 \times 10^{.6335} \times 10^{-1}$

$= 10^{.9557} \times 10^2$

$\simeq 9.03 \times 10^2$ (from the graph)

$= 903.$

(b) $.078 = 7.8 \times 10^{-2}$ and $.23 = 2.3 \times 10^{-1}$

$\simeq 10^{.89} \times 10^{-2}$ $\simeq 10^{.36} \times 10^{-1}$

$$.078 \div .23 \simeq 10^{.89} \times 10^{-2} \div (10^{.36} \times 10^{-1})$$
$$= 10^{.89} \times 10^{-2} \times 10^{-.36} \times 10^{1}$$
$$= 10^{.53} \times 10^{-1}$$
$$\simeq 3.4 \times 10^{-1}$$
$$= .34$$

Example 2

Evaluate (a) $(56)^4$ (b) $(.00063)^{\frac{1}{4}}$.

Solution: (a)
$$56 = 5.6 \times 10^{1}$$
$$\simeq 10^{.75} \times 10^{1}$$
$$\therefore (56)^4 \simeq (10^{.75} \times 10^{1})^4$$
$$= (10^{.75})^4 \times (10^{1})^4$$
$$= 10^{3} \times 10^{4}$$
$$= 10^{7}$$
$$= 10{,}000{,}000.$$

(b)
$$.00063 = 6.3 \times 10^{-4}$$
$$\simeq 10^{.80} \times 10^{-4}$$
$$(.00063)^{\frac{1}{4}} \simeq (10^{.80} \times 10^{-4})^{\frac{1}{4}}$$
$$= (10^{.80})^{\frac{1}{4}} \times (10^{-4})^{\frac{1}{4}}$$
$$= 10^{.20} \times 10^{-1}$$
$$\simeq 1.6 \times 10^{-1}$$
$$= .16$$

EXERCISE 4—1

Using the large scale graph (at the end of the book) of $y = 10^x$, in the domain $0 \leq x \leq 1$, evaluate:

1. $230 \times .34$	2. $.043 \times 1.9$	3. $7.9 \div .026$
4. $.083 \div 41$	5. $\sqrt{8900}$	6. $\sqrt[3]{.97}$
7. $\sqrt{.67}$	8. $\sqrt[5]{83000}$	9. $\sqrt[3]{270}$
10. $\sqrt[4]{.73}$	11. $.37 \times 6.1$	12. $.0071 \times 870$
13. $82 \div 2.9$	14. $910 \div .31$	15. $672 \div 423$
16. $\dfrac{47 \times 6.9}{.34}$	17. $\dfrac{.027}{37 \times 230}$	18. $\dfrac{71.4}{9.81 \times 3.24}$
19. $(58)^{\frac{2}{3}}$	20. $(380)^{\frac{5}{4}}$	21. $(.76)^{-\frac{1}{2}}$
22. $(8800)^{-\frac{2}{3}}$	23. $(35.6)^{-\frac{3}{2}}$	24. $(7.36)^{\frac{1}{2}}$

4 2 The Graph of $\{(x,y)|y=10^x, 0\leq x\leq 1, x\in \mathrm{Re}\}$

We have used the graph of $y=10^x$ both in the last chapter and in this chapter; however we have not drawn the graph ourselves. How would we obtain a table of values from which to draw the graph reasonably accurately in the domain $0\leq x\leq 1$? Notice that in the table on page 69 we obtained only two points $\{(0, 1), (1, 10)\}$ with x values in this domain, and the range is $1\leq y\leq 10$. We cannot draw the graph using only these two endpoints! To obtain a reasonable graph from which we might read y to 2 significant figures, we would need a number of points, fairly evenly spread over the domain and the range. We can find approximations to the ordered pairs, represented by such points by using simple powers of 2, 3, and 7 and the laws for exponents.

Before we do this, we note the following results:

$1=10^0$, $9=3^2$, $4=2^2$, $10=10^1$, $8=2^3$, $5\times 2=10^1$.

If we form powers of 2, we obtain the following table

n	1	2	3	4	5	6	7	8	9	10
2^n	2	4	8	16	32	64	128	256	512	1024

and we see that

$$2^{10}\simeq 10^3.$$
$$\therefore 2 \simeq (10^3)^{\frac{1}{10}}$$
$$= 10^{.30}.$$

In this way we have obtained the ordered pair (.30, 2) in the domain and range required and this ordered pair approximately satisfies the equation $y=10^x$.

Also $4=2^2\simeq(10^{.30})^2=10^{.60}$

gives the ordered pair (.60, 4)

and $8=2^3\simeq(10^{.30})^3=10^{.90}$

gives the ordered pair (.90, 8).

The equation $5\times 2=10^1$ gives

$$5\times 10^{.30}\simeq 10^1.$$
$$\therefore 5\simeq 10^1\div 10^{.30}=10^{1-.30}=10^{.70},$$

and so we have the ordered pair (.70, 5).

Now also $3^4=81\simeq 80=8\times 10^1.$

$$\therefore 3^4\simeq 10^{.90}\times 10^1=10^{1.9}.$$
$$\therefore 3=(10^{1.9})^{\frac{1}{4}}\simeq 10^{.48},$$

which gives the ordered pair (.48, 3),

$6=3\times 2\simeq 10^{.48}\times 10^{.30}=10^{.78}$

gives the ordered pair (.78, 6),

and $9=3^2\simeq(10^{.48})^2=10^{.96}$

gives the ordered pair (.96, 9).

Finally $7^2 = 49 \simeq 50 = 5 \times 10^1$.

$\therefore 7^2 \simeq 10^{.70} \times 10^1 = 10^{1.70}$.

$\therefore 7 \simeq (10^{1.70})^{\frac{1}{2}} = 10^{.85}$

and so we have the ordered pair (.85, 7).

A table of approximate values from which we can draw the graph of $y = 10^x$ can now be formed. (The values of y for $x = .50$, .25, and .15 are found from the square root tables; that is,
$\sqrt{10} = 10^{\frac{1}{2}} = 10^{.50}$, $(\sqrt{10})^{\frac{1}{2}} = (10^{.50})^{\frac{1}{2}} = 10^{.25}$, and $\sqrt{2} = 2^{.50} = 2^{\frac{1}{2}} = (10^{.30})^{\frac{1}{2}} = 10^{.15}$.)

Table of values for $y = 10^x$

x	0	.15	.25	.30	.48	.50	.60	.70	.78	.85	.90	.96	1
y	1	1.4	1.8	2	3	3.2	4	5	6	7	8	9	10

We can use this table of values to find approximate values of x when we are given a value of y that has prime factors in the set $\{2, 3, 5, 7\}$.

Example

Find x such that $10^x = 18$.

Solution: $18 = 2 \times 3 \times 3$

$2 \simeq 10^{.30}$ and $3 \simeq 10^{.48}$ (from the table)

$\therefore 18 \simeq 10^{.30} \times 10^{.48} \times 10^{.48}$

$= 10^{1.26}$.

$\therefore x = 1.26$.

EXERCISE 4—2

1. Using the above table of values, draw the graph of $y = 10^x$, $0 \leq x \leq 1$.
2. Use the table of values to find approximations to x for the following values of y:

   (a) 12 (b) 15 (c) 16 (d) 21

   (e) 24 (f) 27 (g) 32 (h) 35
3. Use the following approximate equalities to evaluate ordered pairs which satisfy $y = 10^x$ for $y = 11$, 13, 17, 19, 23. Values of x should be given to two decimal places.

   (a) $11^2 = 121 \simeq 120 = 3 \times 4 \times 10$ (b) $7 \times 13 = 91 \simeq 90 = 3^2 \times 10$

   (c) $3 \times 17 = 51 \simeq 50 = 5 \times 10$ (d) $3 \times 19 = 57 \simeq 56 = 7 \times 8$

   (e) $4 \times 19 = 76 \simeq 75 = 5^2 \times 3$ (f) $3 \times 23 = 69 \simeq 70 = 7 \times 10$

4 3 $y=10^x$ and $x=10^y$ The Logarithmic Function

In Exercise 4-2 (1) we drew the graph of $\{(x,y)|y=10^x;\ 0\leq x\leq 1\}$. If we consider the equation $x=10^y$, we see that the roles of x and y have been reversed from their roles in $y=10^x$. Thus a table of values for $x=10^y$, $0\leq y\leq 1$, would be as follows: (See the table opposite in Section 4-2.)

Table of values for $x=10^y$

x	1	1.4	1.8	2	3	3.2	4	5	6	7	8	9	10
y	0	.15	.25	.30	.48	.50	.60	.70	.78	.85	.90	.96	1

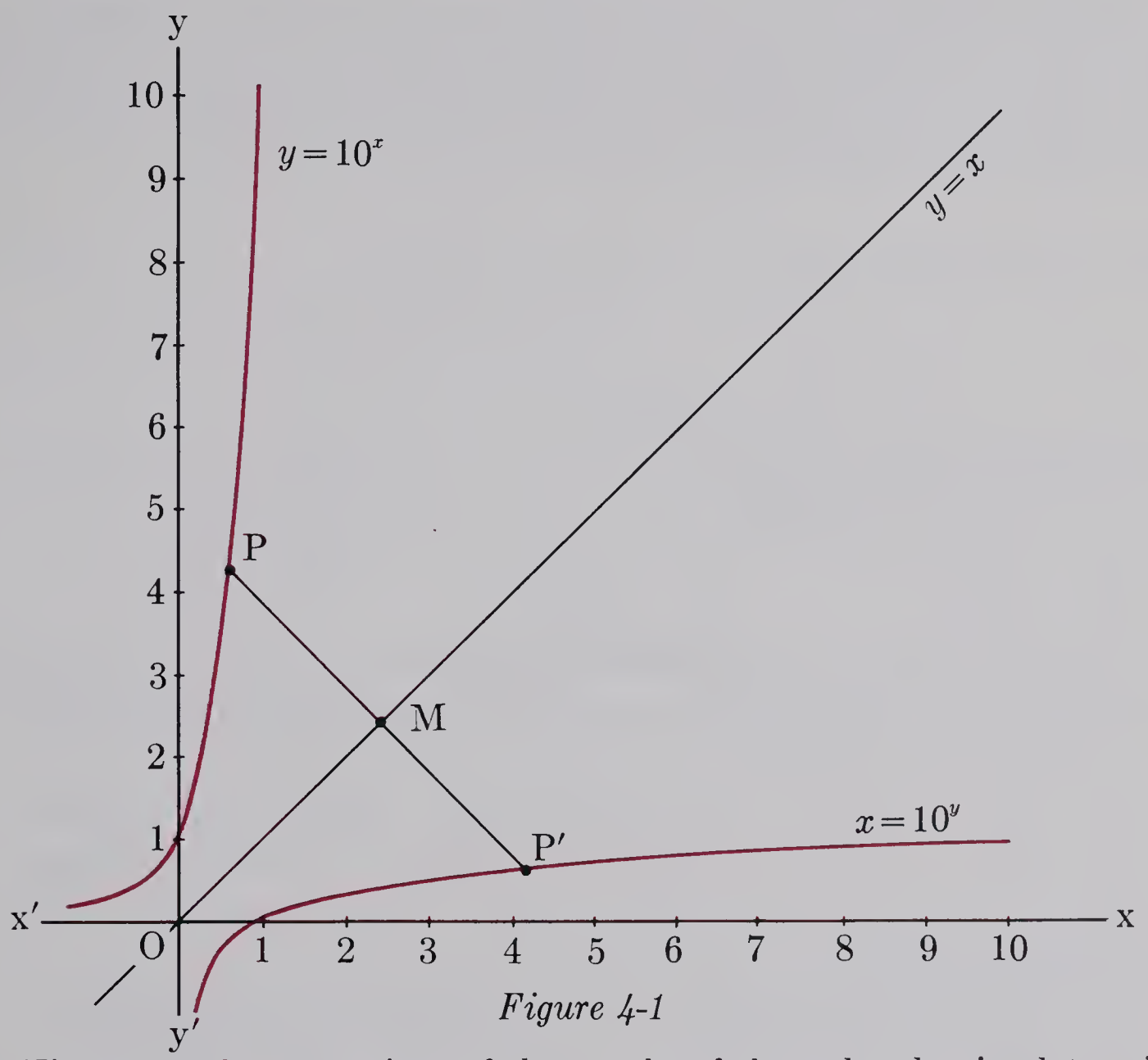


Figure 4-1

Figure 4-1 shows portions of the graphs of the ordered pairs determined by $y=10^x$ and $x=10^y$.

We also show the line $y=x$ in Figure 4-1, and we see that if we draw a line PMP′ which is perpendicular to the line $y=x$ at M and cuts curve $y=10^x$ at P and curve $x=10^y$ at P′, then PM = MP′. Thus if we placed a mirror along the line $y=x$, the reflection of the graph of $y=10^x$ would be the graph of $x=10^y$, and similarly the reflection of the curve $x=10^y$ would be $y=10^x$.

We know if $x = 10^y$ that to each value of $y \in \text{Re}$ corresponds one and only one value of $x \in \text{Re}^+$; that is, $x = 10^y$ defines x as a function of y. The graph and table of values would indicate also that to each value of $x \in \text{Re}^+$ corresponds one and only one value of $y \in \text{Re}$. Thus there is a one-to-one correspondence between x and y; so $x = 10^y$ also defines y as a function of x.

This function is so important in mathematics that it has been given a special name and symbol.

DEFINITION: If $x = a^y$ $(a \in \text{Re}^+)$, then $y = \log_a x$ $(a, x \in \text{Re}^+)$, and the function defined by $y = \log_a x$ is called the logarithmic function to base a. $\text{Log}_a x$ is read 'log of x to base a.'

In our particular case, $y = \log_{10} x$ defines the logarithmic function of x to the base ten, and the graph of the ordered pairs $\{(x, y) | y = \log_{10} x; x \in \text{Re}^+\}$ is the graph of the logarithmic function to base 10. Therefore, the graph of $y = \log_{10} x$ is by definition the graph of $x = 10^y$. In fact,

$$x = 10^y \quad \text{iff} \quad y = \log_{10} x.$$

These two equations give two important identities:

$$\text{(i)}\ x = 10^{\log_{10} x} \quad \text{and} \quad \text{(ii)}\ y = \log_{10}(10^y).$$

EXERCISE 4—3

1. Using the large scale graph (at the end of the book) of $y = 10^x$, $0 \leq x \leq 2$, as the graph of $x = \log_{10} y$, find (as accurately as possible)
   (a) $\log_{10} 18$ (b) $\log_{10} 20$ (c) $\log_{10} 24$
   (d) $\log_{10} 28$ (e) $\log_{10} 35$ (f) $\log_{10} 12$.
2. Using the values for the logarithms to the base 10 found in (1), and the values of the logarithms of 1 to 10 from the table of values of $y = 10^x$, evaluate the following:
   (a) $\log_{10} 3 + \log_{10} 4$ (b) $\log_{10} 9 + \log_{10} 2$ (c) $\log_{10} 4 + \log_{10} 5$
   (d) $\log_{10} 7 + \log_{10} 4$ (e) $\log_{10} 24 - \log_{10} 6$ (f) $\log_{10} 35 - \log_{10} 5$
3. Use the graph and the results of (2) to verify that (to the order of accuracy possible)
   (a) $\log_{10} 3 + \log_{10} 4 = \log_{10} 12$ (b) $\log_{10} 9 + \log_{10} 2 = \log_{10} 18$
   (c) $\log_{10} 4 + \log_{10} 5 = \log_{10} 20$ (d) $\log_{10} 7 + \log_{10} 4 = \log_{10} 28$
   (e) $\log_{10} 24 - \log_{10} 6 = \log_{10} 4$ (f) $\log_{10} 35 - \log_{10} 5 = \log_{10} 7$.
4. Verify to the approximation possible from the graphs that
   (a) $2 \log_{10} 5 = \log_{10} 25$ (b) $3 \log_{10} 3 = \log_{10} 27$ (c) $5 \log_{10} 2 = \log_{10} 32$
   (d) $\frac{1}{2} \log_{10} 36 = \log_{10} 6$ (e) $\frac{1}{4} \log_{10} 81 = \log_{10} 3$ (f) $\frac{1}{2} \log_{10} 64 = \log_{10} 8$.
5. (a) Verify identity (i) for $x = 2, 3, 5, 8, 10$.
   (b) Verify identity (ii) for $y = 1, 4, 6, 7, 9$.

4 4 Rules for Logarithms

The value of $\log_{10}x$ for a given value of x is usually referred to as the logarithm of x (to base 10).

The results of (3) and (4) in Exercise 4-3 suggest the following:

Rules for Logarithms

(i) $\log_{10}s+\log_{10}t=\log_{10}(st)$

(ii) $\log_{10}s-\log_{10}t=\log_{10}\left(\frac{s}{t}\right)$

(iii) $t\ \log_{10}s=\log_{10}(s)^t$

(iv) $\frac{1}{t}\log_{10}s=\log_{10}\sqrt[t]{s}$

In fact, these equations are identities, and they give the four basic rules for calculation with logarithms. With these rules, we can use logarithms to multiply or divide by adding or subtracting logarithms, and to find powers or roots by multiplying or dividing logarithms by integers.

These rules can be proved by using the definition of a logarithm and the laws of exponents.

Example

Prove that $\log_{10}s+\log_{10}t=\log_{10}(st)$.

Solution: Let $p=\log_{10}s$ and $q=\log_{10}t$.

Then $s=10^p$ and $t=10^q$; (definition)

$\therefore st=10^p\times10^q$

$=10^{p+q}$. (product law of exponents)

$\therefore p+q=\log_{10}(st)$; (definition)

$\therefore \log_{10}s+\log_{10}t=\log_{10}(st)$.

EXERCISE 4—4

1. Verify the Rules (i) and (ii) for Logarithms using the following values for s and t: (Use the table in Section 4-3, page 81.)

   (a) $s=100,\ t=1000$ (b) $s=.001,\ t=10$

   (c) $s=2,\ t=3$ (d) $s=3,\ t=3$

   (e) $s=8,\ t=4$ (f) $s=6,\ t=3$

2. Verify the Rules (iii) and (iv) for Logarithms using the following values for s and t:

   (a) $s=100,\ t=2$ (b) $s=100000,\ t=5$

   (c) $s=.001,\ t=3$ (d) $s=2,\ t=3$ (rule iii)

   (e) $s=9,\ t=2$ (rule iv) (f) $s=1,\ t=5$ (rules iii and iv)

3. (a) Prove $\log_{10}s-\log_{10}t=\log_{10}\left(\frac{s}{t}\right)$ (b) Prove $t\ \log_{10}s=\log_{10}\ (s)^t$

   (c) Prove $\frac{1}{t}\log_{10}s=\log_{10}\sqrt[t]{s}$.

4.5 The Tables of the Logarithmic and Exponential Functions to Base 10

The tables of values for the exponential and logarithmic functions, which we calculated in the previous sections, are obviously not accurate enough to use in any computations with numbers that are given to three or four significant digits. However, more detailed and accurate tables have been calculated by mathematicians so that computations can be performed using logarithms and exponentials. Four- and five-figure tables are most frequently used, but tables given to seven, eight, and even more significant digits have been published.

The tables of the logarithmic and exponential functions given on page 396 contain four-figure numbers, and the results of computations performed with them are usually correct to three significant figures, sometimes to four. The tables are often called *logarithms* and *antilogarithms* in many books and sets of tables.

We shall now show that these tables are used in much the same way as the trigonometric tables which you have used already.

The following is a row from the table of the exponential function to the base 10:

	0	1	2	3	4	5	6	7	8	9
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748

The first two digits of the exponent $[0 \leq x \leq 1]$ are given in the left-hand column and the third along the top of the table.

Notice that the decimal point has been inserted in the numbers in the left column, but not anywhere else in the table. In the body of the table, *the decimal point is understood to follow the first digit.*

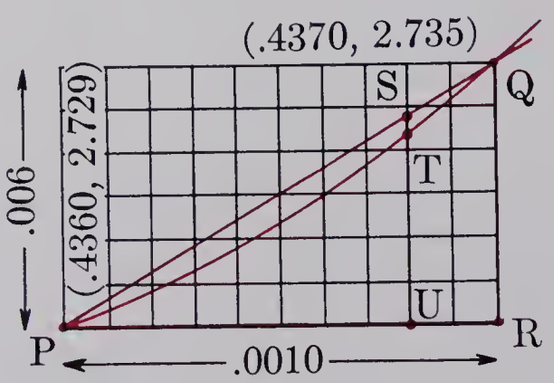
For example,

$$10^{.436} = 2.729 \text{ and } 10^{.437} = 2.735.$$

To find the value with a four-digit exponent, we have to use a process called *linear interpolation*. We shall consider an example.

To find the value of $10^{.4368}$.

We consider the two points on the graph of $y = 10^x$ which are given by (.4360, 2.729) and (.4370, 2.735) from the table. Remember that the ordered pairs found from the table correspond to points on the graph. In the diagram at the right, the curve PTQ represents an enlargement of the graph of the exponential function to base 10 near



these points. We wish to find the value of y corresponding to x at U, that is, the value of y at T; in our example, the value of x at U is .4368. We cannot find this value of y at T by using the table because there is no tabulated value for $x=.4368$. The value of y at S on the secant PQ is the best simple approximation that we can calculate to the value of y at T.

$$\text{The slope of the line PQ} = \frac{\text{RQ}}{\text{PR}}$$

$$= \frac{2.735-2.729}{.4370-.4360}$$

$$= \frac{.006}{.0010}.$$

But the slope of the line PQ is also given by $\frac{\text{US}}{\text{PU}}$; therefore in our example $\frac{\text{US}}{\text{PU}} = \frac{.006}{.0010}$.

Now the length of PU, call it $\triangle x$, is the difference between the value of x at U and the value of x at P. In our example,

$$\triangle x = .4368 - .4360 = .0008.$$

Similarly, the length of US is the difference between the value of y at T and the value of y at U; in our example the value of y at U $=2.729$ and y_T, the value of y at T, is what we wish to find.

In our example, therefore,

$$\frac{y_T - 2.729}{.0008} = \frac{.006}{.0010}.$$

$$y_T = 2.729 + .006 \times \frac{.0008}{.0010}$$

$$= 2.729 + .006 \times \frac{8}{10}$$

$$= 2.729 + .0048$$

$$\simeq 2.734 \quad \text{(to four significant digits)}.$$

This value of y_T is the approximation we use for the value of y at S on the graph of $y=10^x$.

Thus the ordered pair

$$(.4368,\ 2.734)$$

gives an approximation to the point S on the graph of $y=10^x$.

Example 1

Find the value of 10^x for $x = .6734$.

Solution: From the tables,

for $x = .6740$, $10^x = 4.721$;

for $x = .6730$, $10^x = 4.710$.

$\therefore$ the differences for x and 10^x are .0010 and .011 respectively.

For $x = .6734$ and $x = .6730$, $\triangle x = .0004$.

$$\therefore 10^x \simeq 4.710 + .011 \times \frac{.0004}{.0010}$$

$$= 4.710 + .011 \times \frac{4}{10}$$

$$\simeq 4.710 + .004$$

$$= 4.714.$$

Example 2

Find the value of 10^x if $x = 2.6734$.

Solution: The tables are calculated only for $0 \leq x \leq 1$ and so the values of 10^x are such that $1 \leq 10^x \leq 10$.

Therefore we write

$$10^{2.6734} = 10^{2+.6734} = 10^2 \cdot 10^{.6734}.$$

Note that the fractional (decimal) exponent used in the tables must always be a positive fraction less than 1.

Now $10^{.6734} \simeq 4.714$ (Example 1)

$$\therefore 10^{2.6734} \simeq 10^2 \times 4.714$$

$$= 471.4.$$

We gradually abbreviate the interpolation calculation as we gain confidence with the use of the tables, until finally we perform the interpolation mentally.

Example 3

Find the value of $y = 10^x$ when $x = -.2644$.

Solution: Remember that the proper fractional exponent *must be positive* in order to use the tables.

Therefore we write

$$x = -.2644 = -1 + .7356$$

$$\text{and } 10^x = 10^{-1+.7356} = 10^{-1} \times 10^{.7356}.$$

$10^{.7360} = 5.445$

$10^{.7350} = 5.433$

Diff. $= .012$

$$10^{-1+.7356} = 10^{-1}\left(5.433 + .012 \times \frac{6}{10}\right)$$

$$= 10^{-1}\,(5.440)$$

$$= .5440.$$

We use the tables of the logarithmic function to the base 10 in the same way. Note that no decimal points are shown anywhere in the table of the logarithmic function.

Example 4

Find the value of $\log_{10}x$ for $x=5.837$.

Solution: From the tables,

for $x=5.840$, $\log_{10}x=.7664$;

for $x=5.830$, $\log_{10}x=.7657$.

$\therefore$ the differences for x and $\log_{10}x$ are .010 and .0007 respectively.

For $x=5.837$ and $x=5.830$, $\triangle x=.007$.

$$\therefore \log_{10}x \simeq .7657+.0007\times\frac{.007}{.010}$$

$$=.7657+.0007\times\tfrac{7}{10}$$

$$=.7657+.00049$$

$$\simeq .7662.$$

Example 5

Find the value of $\log_{10}x$ for $x=237.4$.

Solution: $237.4=2.374\times10^{2}$

$\therefore \log_{10}237.4=\log_{10}2.374+2.$ (log rule (i))

For $x=2.380$, $\log_{10}x=.3766$.

For $x=2.370$, $\log_{10}x=.3747$.

$\therefore$ differences are .010 and .0019 respectively.

For $x=2.374$, and $x=2.370$, $\triangle x=.004$.

$$\therefore \log_{10}2.374 \simeq .3747+.0019\times\tfrac{4}{10}$$

$$=.3747+.00076$$

$$\simeq .3755;$$

$$\therefore \log_{10}237.4 \simeq +2+.3755.$$

Note that the fractional (decimal) part of the logarithm found from the tables is a positive fraction less than 1. The integral part of the logarithm is the exponent of 10 when the number is written in scientific notation.

Example 6

Find the value of $\log_{10}(.007582)$.

Solution: $.007582=7.582\times10^{-3}$

$$\therefore \log_{10}(.007582)=\log_{10}(10^{-3}\times7.582)=-3+\log_{10}(7.582)$$

$\log_{10}7.590=.8802$

$\log_{10}7.580=\underline{.8797}$

Diff. $=.0005$

$$\therefore \log_{10}(.007582)=-3+.8797+.0005\times\tfrac{2}{10}$$

$$=-3+.8798.$$

Note that it is usual to leave the logarithm in this form so that the proper fractional part is always positive as in the tables.

EXERCISE 4—5

1. Find the values of 10^x for the following values of x:

(a) $+.1837$	(b) $+.6223$	(c) $+.8397$
(d) $+2+.8821$	(e) $+3+.0513$	(f) $-2+.1174$
(g) $+1+.1174$	(h) $-3+.2693$	(i) $+4+.8144$
(j) $-1+.3393$	(k) $-2+.0875$	(l) $+1+.3279$
(m) $-2+.7331$	(n) $-5+.6470$	(o) $+8+.3822$
(p) $+1+.7624$	(q) $-2+.8695$	(r) $-1+.9348$
(s) $+.3641$	(t) $+1.7628$	(u) $+3.4681$
(v) $-.5743$	(w) -2.4295	(x) -4.1896

2. Find the values of $\log_{10}x$ for the following values of x:

(a) 1.738	(b) 3.426	(c) 5.649
(d) 9.372	(e) 7.523	(f) 2.917
(g) 6.157	(h) 8.373	(i) 12.84
(j) .05637	(k) 3675.	(l) 869.5
(m) .002303	(n) 67.44	(o) .08872
(p) 599300	(q) .9708	(r) 1343
(s) 15.69	(t) .06784	(u) 4.395
(v) .005763	(w) 713.8	(x) 218700

4 6 Multiplication and Division Using Logarithms

We shall now use the table of the logarithmic function and the table of the exponential function to perform computations involving multiplication and division. We use the logarithmic function table to find the exponent x, when we are given a number $y=10^x$, from the equation $x=\log_{10}y$. We then perform multiplication of numbers by adding exponents and division of numbers by subtracting exponents, according to the rules for exponents. When we have calculated the resultant exponent, we then use the table of the exponential function to find the number $y=10^x$ which is the result of our computation.

1. *To multiply two numbers s and t.*

We use rule (i):

$$\log_{10}(st)=\log_{10}s+\log_{10}t.$$

Find $p=\log_{10}s$ and $q=\log_{10}t$ from the table of the logarithmic function.
Then $p+q=\log_{10}st$. (rule (i) for logarithms)
Find $r=10^{p+q}$ from the table of the exponential function.
Then $r=st$.

In computational work it is customary to omit the suffix 10 on the logarithm; thus we write log a instead of $\log_{10}a$ in the examples.

Remember that, when we are using the tables, a logarithm is always written in the form of an integer (positive or negative) and a positive proper decimal fraction. The value of the integer is always found by writing the number in scientific notation: it is the exponent of 10 in that notation.

Example 1

Evaluate $3.716\times18.24\times.2841$.

Solution: If $x=3.716\times18.24\times.2841$,

then $\log x=\log 3.716+\log 18.24+\log .2841$.

$\log 3.716 = \quad 0+.5701 \qquad .5694+.0011\times\frac{6}{10}$

$\log 18.24 = +1+.2611 \qquad .2601+.0024\times\frac{4}{10}$

$\log .2841 = -1+.4535 \qquad .4533+.0015\times\frac{1}{10}$

$\therefore \log x = +1+.2847.$

$x=1.927\times10^{1} \qquad 1.923+.005\times\frac{7}{10}$

$\therefore 3.716\times18.24\times.2841=19.27.$

2. *To divide two numbers s and t.*

We use rule (ii):

$$\log_{10}\left(\frac{s}{t}\right)=\log_{10}s-\log_{10}t.$$

Find $p=\log_{10}s$ and $q=\log_{10}t$ from the table of the logarithm function.
Then $p-q=\log_{10}(s\div t)$. (rule (ii) for logarithms)
Find $r=10^{p-q}$ from the table of the exponential function.
Then $r=s\div t$.

Example 2

Divide 32.75 by 4.863.

Solution: If $x=32.75\div4.863$,

then $\log x=\log 32.75-\log 4.863$.

$\log 32.75 = +1+.5152 \qquad .5145+.0014\times\frac{5}{10}$

$\log 4.863 = \quad 0+.6869 \qquad .6866+.0009\times\frac{3}{10}$

$\log x = \quad 0+.8283$

$\therefore\ x=6.735. \qquad 6.730+.015\times\frac{3}{10}$

$\therefore\ 32.75\div4.863=6.735.$

Example 3

Evaluate $63.58 \times 47.56 \div 781.9$.

Solution: If $x = 63.58 \times 47.56 \div 781.9$,

then $\log x = \log 63.58 + \log 47.56 - \log 781.9$.

$$\begin{aligned}
\log 63.58 &= +1+.8034 && .8028+.0007\times\tfrac{8}{10}\\
\log 47.56 &= \underline{+1+.6772} && .6767+.0009\times\tfrac{6}{10}\\
&\quad +3+.4806\\
\log 781.9 &= +2+.8932 && .8927+.0005\times\tfrac{9}{10}\\
\therefore \log x &= \quad 0+.5874\\
x &= 3.868\times 10^{0} && 3.864+.009\times\tfrac{4}{10}
\end{aligned}$$

$\therefore 63.58 \times 47.56 \div 781.9 = 3.868.$

We have written the interpolation calculations at the right. Most of these can be performed mentally when the processes and the use of the tables become more familiar.

Example 4

Evaluate $\dfrac{2.341 \times 36.24 \times 5.476}{81.72 \times .9631 \times 46.37}$.

Solution: If $x = \dfrac{2.341 \times 36.24 \times 5.476}{81.72 \times .9631 \times 46.37}$,

then $\log x = \log (2.341 \times 36.24 \times 5.476) - \log (81.72 \times .9631 \times 46.37)$

$= [\log 2.341 + \log 36.24 + \log 5.476] - [\log 81.72 + \log .9631 + \log 46.37]$.

$$\begin{aligned}
\log 2.341 &= \quad .3694 && \log 81.72 = +1+.9123\\
\log 36.24 &= +1+.5592 && \log .9631 = -1+.9837\\
\underline{\log 5.476} &= \underline{\quad .7385} && \underline{\log 46.37} = \underline{+1+.6662}\\
&\quad +2+.6671 && +3+.5622\\
&\quad \underline{+3+.5622}\\
\log x &= -1+.1049\\
x &= 1.274\times 10^{-1}\\
\therefore x &= .1274.
\end{aligned}$$

Note: We have used equal signs ($=$) in all the above computations instead of approximately equal ($\simeq$); this is the usual practice, but we must remember that the various numbers involved are correct only to four significant digits at most.

We should always check the magnitude of our answer by an approximate calculation. Thus, in Example 4, the answer is given approximately by

$$\frac{2\times40\times5}{80\times1\times50}=\frac{400}{4000}\simeq.1.$$

EXERCISE 4—6

Evaluate:

1. 7.819×6.081
2. 8.24×10.74
3. 3.758×98.24
4. 78.57×547.9
5. 97.77×959
6. 9.435×69.84
7. 95500×19.21
8. $.6322\times.08105$
9. $8.209\div4.956$
10. $15.84\div7.854$
11. $61.66\div.7279$
12. $.3259\div610.2$
13. $.9391\div44.29$
14. $.001351\div801.3$
15. $7.37\div.005221$
16. $539.1\div.0986$
17. $8893\div5.737$
18. $4.108\div772.5$
19. $3.661\times46.22\times.4849$
20. $19.49\times7.209\times.09258$
21. $4.108\times77.25\times614.9$
22. $561.3\times.6495\times9.049$
23. $7920.\times5.341\div.00218$
24. $.00064\times4.054\div954.8$
25. $32.25\times1800\times.2792$
26. $300.8\times2406\times47010$
27. $\dfrac{89.47\times4.180}{4.087\times.8089}$
28. $\dfrac{9.714\times.2860\times998.2}{9.03}$
29. $\dfrac{7.385\times365.4}{46.89\times.3339}$
30. $\dfrac{.4728\times.4625}{39.44\times862.1}$
31. $\dfrac{87.80\times751.\times.05871}{665.8}$
32. $\dfrac{1022.\times.09492}{8.241\times.001733}$
33. $\dfrac{137.4\times5713}{1917\times8097}$
34. $\dfrac{12.17\times.7647}{893.8\times.06154}$
35. $\dfrac{146.8\times59.45\times.05187}{5608.\times.0908}$
36. $\dfrac{66.6\times1.591\times8.767}{87.3\times.6242}$
37. $\dfrac{3673000}{1.574\times61.06}$
38. $\dfrac{.005830}{718.5\times2.884}$
39. $\dfrac{95090}{.0006935}$
40. $\dfrac{.000005466}{740200}$

47 The Computation of Roots and Powers

While the use of the tables of the logarithmic and exponential functions simplifies computations involving multiplication and division of numbers, it is in the calculation of roots that the great power of logarithms becomes apparent. The difficult process of finding roots (and any process for finding roots other than square roots is very complicated) is replaced by the simple process of division by a natural number.

3. *To find a root of a number.*

We use rule (iv):

$$\log_{10}\sqrt[t]{s}=\frac{1}{t}\log_{10}s.$$

Find $p=\log_{10}s$ from the table of the logarithmic function.

Then $p\div t=\frac{1}{t}\log_{10}s=\log_{10}\sqrt[t]{s}.$

Find $r=10^{\frac{p}{t}}$ from the table of the exponential function.

Then $r=\sqrt[t]{s}.$

Example 1

Find (a) $\sqrt{37.46}$ (b) $\sqrt[3]{128.4}$

(c) $\sqrt[3]{3876}$ (d) $\sqrt[3]{.0007364}.$

Solution: (a) $\log 37.46=1+.5736$ $\quad .5729+.0011\times\frac{6}{10}$

$\frac{1}{2}\log 37.46=+.7868$

$\therefore\sqrt{37.46}=6.121$ $\quad 6.109+.015\times\frac{8}{10}$

(b) $\log 128.4=2+.1086$ $\quad .1072+.0034\times\frac{4}{10}$

$\frac{1}{3}\log 128.4=.7029$

$\therefore\sqrt[3]{128.4}=5.046$ $\quad 5.035+.012\times\frac{9}{10}$

(c) $\log 3876=3+.5884$ $\quad .5877+.0011\times\frac{6}{10}$

$\frac{1}{3}\log 3876=1+.1961$

$\sqrt[3]{3876}=1.570\times 10$ $\quad 1.570+.004\times\frac{1}{10}$

$=15.70$

(d) $\log(.0007364) = -4+.8671$ $\qquad .8669+.0006\times\frac{4}{10}$

$\frac{1}{3}\log(.0007364) = \frac{1}{3}(-4+.8671)$

$= \frac{1}{3}(-6+2+.8671)$

$= -2+.9557$

$\sqrt[3]{.0007364} = 9.030\times10^{-2}$ $\qquad 9.016+.020\times\frac{7}{10}$

$= .09030$

Remember that the decimal part of the logarithm must always be in the range 0 to 1. Hence in this last example the logarithm of $.0007364 = -4+.8671$ has to be written in the form $-6+2.8671$ *before* it is divided by 3. This is necessary so that the final logarithm, which is used as an exponent, has *a negative part that is an integer* and *a positive decimal part in the range 0 to 1*.

4. *To find a power of a number.*

We use rule (iii):

$$\log_{10} s^t = t \log_{10} s.$$

Find $p = \log_{10} s$ from the table of the logarithmic function.
Then $tp = t \log_{10} s = \log_{10} s^t$.
Find $r = 10^{tp}$ from the table of the exponential function.
Then $r = s^t$.

Example 2

Evaluate (a) $(4673)^{\frac{2}{5}}$ (b) $(.8732)^{\frac{4}{3}}$ (c) $(23.47)^5$.

Solution: (a) $\log 4673 = 3+.6696$ $\qquad .6693+.0009\times\frac{3}{10}$

$\frac{2}{5}\log 4673 = \frac{2}{5}\times(3+.6696)$

$= \frac{1}{5}\times(7+.3392)$

$= 1+.4678$

$\therefore (4673)^{\frac{2}{5}} = 29.37.$ $\qquad 2.931+.007\times\frac{8}{10}$

(b) $\log .8732 = -1+.9411$ $\qquad .9410+.0005\times\frac{2}{10}$

$4\log .8732 = -4+3.7644$

$\frac{4}{3}\log .8732 = \frac{1}{3}(-1+.7644)$

$= \frac{1}{3}(-3+2.7644)$

$= -1+.9215$

$(.8732)^{\frac{4}{3}} = .8347$ $\qquad 8.337+.019\times\frac{5}{10}$

(c) $\log 23.47 = 1+.3705$ $\qquad .3692+.0019\times\frac{7}{10}$

$5\log 23.47 = 5+1.8525$

$= 6+.8525$

$\therefore (23.47)^5 \simeq 7{,}121{,}000.$ $\qquad 7.112+.017\times\frac{5}{10}$

EXERCISE 4—7

Evaluate:

1. $\sqrt{84.45}$
2. $\sqrt{.7193}$
3. $\sqrt{215.3}$
4. $\sqrt[3]{102.7}$
5. $\sqrt{1.583}$
6. $\sqrt[4]{5.802}$
7. $(.8425)^{\frac{2}{3}}$
8. $(57.42)^{\frac{4}{5}}$
9. $(9.129)^{\frac{3}{4}}$
10. $(41.28)^{-\frac{1}{3}}$
11. $(25.38)^2$
12. $(81.89)^4$
13. $(8.737)^3$
14. $(2.421)^5$
15. $(7.011)^{-2}$
16. $(35.87)^2$
17. $(.9657)^3$
18. $(477.2)^{-3}$
19. $(.01794)^3$
20. $\sqrt{41.91\times 8.084}$
21. $\sqrt{474.8\times .1613}$
22. $(3.230\times 71.41)^{\frac{1}{2}}$
23. $\sqrt{\dfrac{59.92}{1.745}}$
24. $\dfrac{\sqrt{4.594}}{\sqrt[3]{599.2}}$
25. $\sqrt[3]{\dfrac{7.406\times 2.856}{3.697\times 3.757}}$
26. $\sqrt[5]{\dfrac{36.22\times 5.187}{4.915\times .2124}}$
27. $(65.09\times 3.426)^{\frac{1}{3}}$
28. $(80.81\times 19.41)^{\frac{1}{5}}$
29. $\sqrt{2.811\div 31.76}$
30. $\sqrt[3]{4.931\div 72.87}$

4.8 Equations Involving Logarithmic and Exponential Functions (Supplementary)

Equations involving logarithmic and exponential functions arise very frequently from problems in physics, chemistry, biology, and engineering. These equations can be solved exactly in some cases but generally we have to use a table of values in their solution.

Example 1

Solve $2(3^x)+32=86$.

Solution:

$$\begin{aligned} 2(3^x)+32&=86\\ 2(3^x)&=86-32\\ 2(3^x)&=54\\ 3^x&=27\\ x&=3 \qquad \text{(by inspection)}\end{aligned}$$

Example 2

Solve $5 \log_{10} x + 12 = 2 \log_{10} x + 21$.

Solution:

$$\begin{aligned} 5 \log_{10} x + 12 &= 2 \log_{10} x + 21 \\ 3 \log_{10} x &= 9 \\ \log_{10} x &= 3 \\ x &= 10^3 \\ x &= 1000 \end{aligned}$$

Example 3

Solve $4 \times 10^x = 25$.

Solution:

$$\begin{aligned} 4 \times 10^x &= 25 \\ 10^x &= 6.25 \\ x &= \log_{10} 6.25 \\ &\simeq .7959 \end{aligned}$$

Example 4

Solve $2 \times 6^x = 5 \times 7^{2x}$.

Solution:

$$2 \times 6^x = 5 \times 7^{2x}$$

Take logarithms to the base 10 of both sides of the equation.

$$\begin{aligned} \log_{10} 2 + \log_{10}(6^x) &= \log_{10} 5 + \log(7^{2x}) \\ \log_{10} 2 + x \log_{10} 6 &= \log_{10} 5 + 2x \log_{10} 7 \\ .3010 + .7782x &\simeq .6990 + 2 \times .8451x \\ .3010 + .7782x &\simeq .6990 + 1.6902x \\ .3010 - .6990 &\simeq (1.6902 - .7782)x \\ -.3980 &\simeq .9120x \\ x &\simeq -\frac{.3980}{.9120} \\ &\simeq -\frac{398}{912} \simeq -.4364 \end{aligned}$$

Example 5

Solve $3 \log_{10} x - 2 \log_{10} 5x = 3.2$.

Solution:

$$\begin{aligned} 3 \log_{10} x - 2 \log_{10} 5x &= 3.2 \\ 3 \log_{10} x - 2 \log_{10} x - 2 \log_{10} 5 &= 3.2 \\ \log_{10} x &\simeq 3.2 + 2 \times .6990 \\ &= 3.2 + 1.3980 \\ &= 4.5980 \\ x &\simeq 39630 \end{aligned}$$

EXERCISE 4—8

Solve the following equations:

1. $10^x = 1000$
2. $10^x = .00001$
3. $5^x = 625$
4. $3^x = 27$
5. $2^x = 64$
6. $7^x = 49$
7. $11^x = 1331$
8. $13^x = 169$
9. $10^x = 69$
10. $10^{2x} = 495$
11. $10^{3x} = .4739$
12. $10^{-x} = 29.32$
13. $7^x = 35.72$
14. $(.6)^x = .7931$
15. $(5.764)^x = 69.85$
16. $(3.482)^{-x} = .08942$
17. $(12)^{2x-3} = 144$
18. $7^{x+3} = 49$
19. $5^{3x+4} = 25$
20. $4^{x+2} = 8$
21. $2 \times 3^x = 7 \times 5^x$
22. $(15)^x = 4 \times (8)^{2x}$
23. $10 \times (2.736)^x = (5.431)^{2x}$
24. $2.678(7.384)^x = (9.362)^{2x+1}$
25. $5 \log_{10} x + 3 \log_{10} x = 12$
26. $2.3 \log_{10} x - 4 = 5.8 \log_{10} x + 1$
27. $15.34 \log_{10} x = 6.841$
28. $5.438 \log_{10} x^2 = 4.312 \log_{10} x + 7.218$
29. $3.4 \log_{10} x = 2.1 \log_{10} x + 5.673$
30. $2.457 \log_{10} x = 7.895 \log_{10} x + 6.237$
31. $\log_{10} x^{1.567} = 3.794 \log_{10} x$
32. $\log_{10} x^{6.374} = 5.148 \log_{10} x - 3.751$
33. $4.831 \log_{10} x = -8.137$
34. $\log_{10} x^3 = 4.732 \log_{10} x + 1.731$
35. $10^{3.16x} - 4(10)^{1.58x} + 3 = 0$
36. $(\log_{10} x)^2 - 5 \log_{10} x + 6 = 0$

4 9 Logarithms to Bases Other Than Ten (Supplementary)

Just as it was possible to have exponential functions to bases other than 10, so we can have logarithmic functions to bases other than 10. We have studied base 10 in earlier sections because it is the simplest for computation in our decimal notation. For some mathematical problems base 2 is more useful, and for most problems in science and engineering the base is an irrational number called e ($\simeq 2.718\ldots$). This number e is an irrational number like π and is discussed as the result of a special type of compound interest in Section 5-5.

DEFINITION: $y = \log_a x$ if and only if $x = a^y$

These logarithms to base a obey all the rules we established for logarithms to base 10.

If we prepare a table of values for $y=\log_2 x$, we obtain

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-2	-1	0	1	2	3	4

We recall that the table of values for $y=2^x$, the exponential to base 2, is

x	-2	-1	0	1	2	3	4
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

These two tables merely confirm what is obvious from the equations $y=2^x$ and $x=2^y$ (which is equivalent to $y=\log_2 x$); that is, the roles of y and x are interchanged. Mathematicians say that the logarithmic and exponential functions are *inverse functions.* This is also true for the exponential and logarithmic functions to the base 10 as we saw on pages 81 and 82.

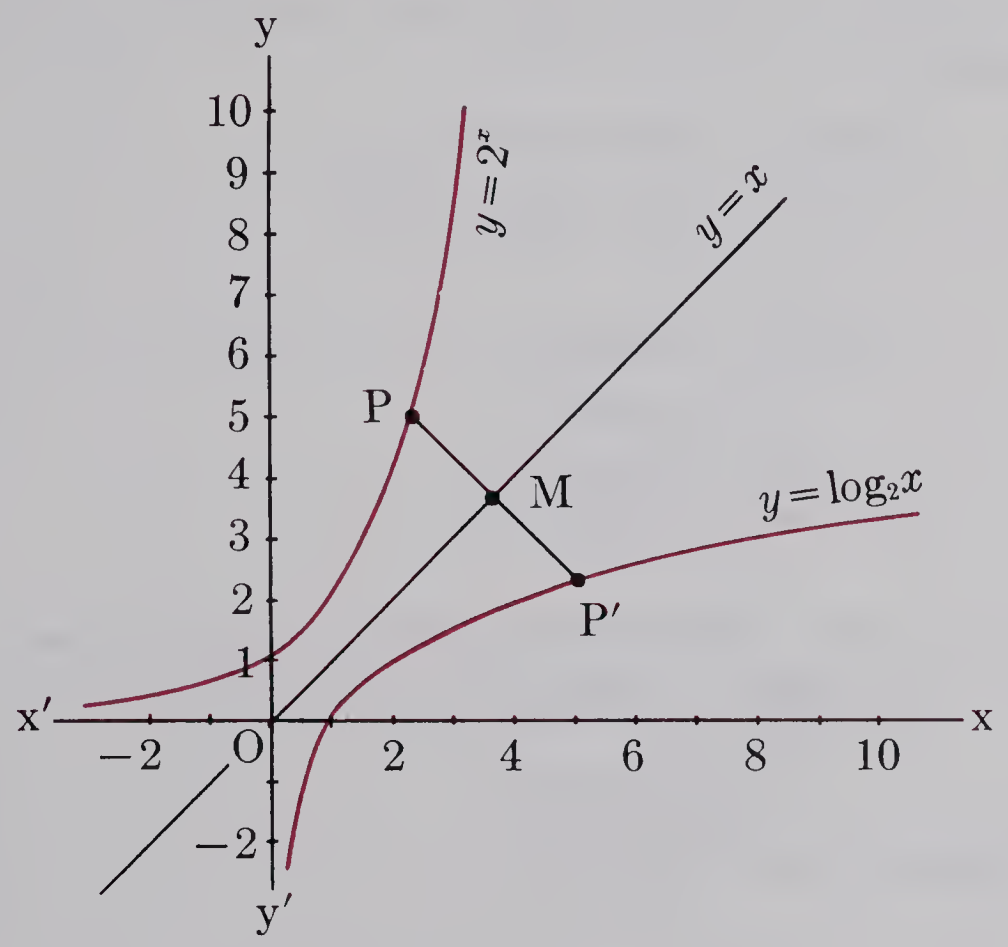


Figure 4-2

Figure 4-2 shows the graphs of both functions together with that of the line $y=x$. We see that if we draw a line like PMP′, which is perpendicular to $y=x$ at M and meets $y=2^x$ and $x=2^y$ at P and P′ respectively, then PM = MP′. This property is characteristic of inverse functions. The graph of $x=2^y$ is said to be the reflection of the graph of $y=2^x$ in the line $y=x$.

Example 1

Simplify $5\log_a x+2\log_a x^3+3\log_a x^{-1}$.

Solution:
$$\begin{aligned}&5\log_a x+2\log_a x^3+3\log_a x^{-1}\\&=5\log_a x+6\log_a x-3\log_a x\\&=8\log_a x\end{aligned}$$

Example 2

Evaluate the logarithm of $\dfrac{3^5\times\sqrt[5]{27}}{9^{1.2}\times(\frac{1}{3})^2}$ to base 3.

Solution:
$$\begin{aligned}\log_3\left(\frac{3^5\times\sqrt[5]{27}}{9^{1.2}\times(\frac{1}{3})^2}\right)&=5\log_3 3+\tfrac{1}{5}\log_3 27-(1.2\log_3 9+2\log_3\tfrac{1}{3})\\&=5\times1+\tfrac{1}{5}\times\log_3 3^3-(1.2\log_3 3^2+2\log_3 1-2\log_3 3)\\&=5+\tfrac{1}{5}\times3-1.2\times2-0+2\\&=5+\tfrac{3}{5}-2.4+2\\&=5.6-.4\\&=5.2\end{aligned}$$

It is possible to convert the logarithm to base a of a number to a logarithm to base b.

Let n be a given number.
Then if $x=\log_a n$ and $y=\log_b n$,
$$n=a^x \quad \text{and} \quad n=b^y.$$
$$\therefore\ a^x=b^y.$$
$$\therefore\ \log_a(a^x)=\log_a(b^y).$$
$$x=y\log_a b$$
$$\therefore\ \log_a n=\log_b n\cdot\log_a b$$
and $\log_b n=\log_a n/\log_a b$.

If we now consider the special case when $n=a$, we obtain
$$\log_a a=\log_b a\cdot\log_a b.$$
But $\log_a a=1$.
$$\therefore\ 1=\log_b a\cdot\log_a b.$$
Thus $\log_b a$ and $\log_a b$ are reciprocals.

Example 3

Find $\log_7 3.576$, using the table of logarithms to base 10.

Solution:
$$\log_7 3.567=\log_{10}3.567/\log_{10}7$$
$$\simeq\frac{.5523}{.8451}=\frac{5.523}{8.451}$$
$$\therefore\log_7 3.567\simeq.6533$$

$\log_{10}5.523=.7421$
$\log_{10}8.451=.9270$
$$\log_{10}\frac{5.523}{8.451}=-1+.8151$$

Example 4

Solve $5 \log_3 x + 12 = 2 \log_3 x + 21$.

Solution:

$$5 \log_3 x + 12 = 2 \log_3 x + 21$$
$$3 \log_3 x = 9$$
$$\log_3 x = 3$$
$$x = 3^3$$
$$x = 27$$

EXERCISE 4—9

Express each of the following as a single logarithm:

1. $\log_2 5 + \log_2 7 + \log_2 6$
2. $\log_5 4 + \log_5 6 - \log_5 3$
3. $2 \log_4 5 + \log_4 7$
4. $3 \log_7 3 + 2 \log_7 5$
5. $2 \log_3 7 - (\log_3 14 + \log_3 35)$
6. $\log_2 5^2 - \log_2 15 + \log_2 9$
7. $\frac{1}{2} \log_e 4 + \frac{1}{3} \log_e 27$
8. $\frac{1}{5} \log_b 32 + \log_b 3 - \log_b 6$
9. $.25 \log_e (.25) + .5 \log_e 4$
10. $2 \log_u (\frac{1}{4}) + \frac{1}{2} \log_u 16$
11. $a \log_e b + 2a \log_e c$
12. $6 \log_{10} 3 - 2 \log_{10} 27$
13. $x \log_a a + y \log_b b$
14. $\log_a b^v - w \log_a c$
15. $\log_a x^b + \dfrac{c}{\log_x a}$
16. $w \log_x p - \dfrac{\log_q p}{\log_q x}$

Evaluate each of the following without the use of tables:

17. $\log_2 (4\sqrt{4})$
18. $\log_3 (81\sqrt{3})$
19. $\log_{\sqrt{2}} (8)$
20. $\log_\pi (\pi^{\frac{2}{3}})$
21. $\log_5 625$
22. $\log_{1.5} 2.25$
23. $\log_2 (8)^{\frac{1}{5}}$
24. $\log_3 (27)^{-\frac{1}{2}}$

Evaluate each of the following using logarithms to base 10:

25. $\log_3 2.436$
26. $\log_5 6.527$
27. $\log_8 5.347$
28. $\log_{12} 21.73$
29. $\log_{2.718} 10$
30. $\log_\pi 2$
31. $\log_2 4$
32. $\log_{10} 1.732$

Solve the following:

33. $4 \log_5 x + 7 = 2 \log_5 x + 11$
34. $3 \log_4 x + 5 = \log_4 x + 1$
35. $2 \log_2 x + 3 = 4 \log_2 x + 1$
36. $2 \log_4 x + 1 = 3 - \log_4 x$
37. $5 \log_\pi x + 2 = \log_\pi x^2 - 1$
38. $3 - 2 \log_{10} x^2 = 3 \log_{10} x - 2$
39. $\log_3 x = 2 - \log_3 9$
40. $\log_3 3^5 = \log_3 x^2$

4 10 Logarithms on the Number Line (Supplementary)

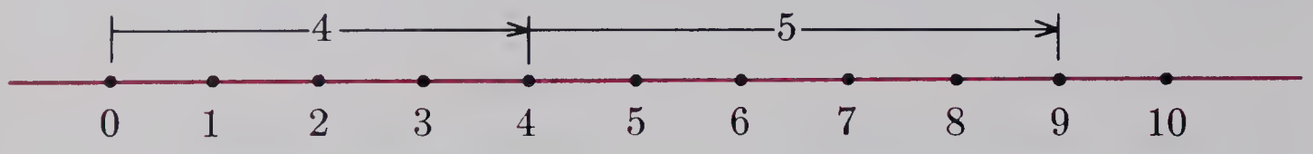
We have seen that, by the use of logarithms to the base 10,

(i) multiplication of numbers is replaced by addition of logarithms,

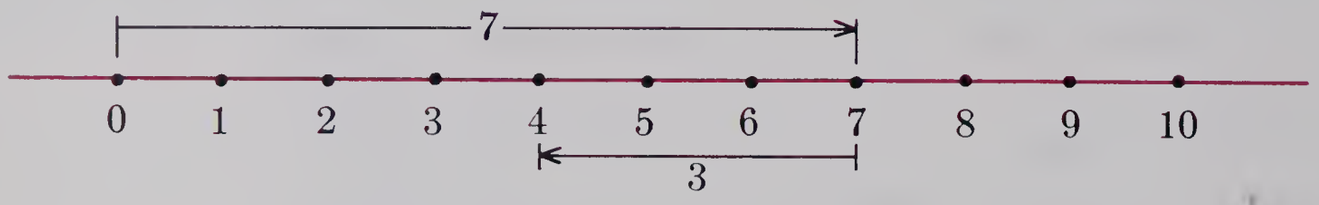
(ii) division of numbers is replaced by subtraction of logarithms.

We have also studied the number line in previous grades and know that simple addition and subtraction calculations can be performed approximately by using the number line.

Thus $4+5=9$ is represented by

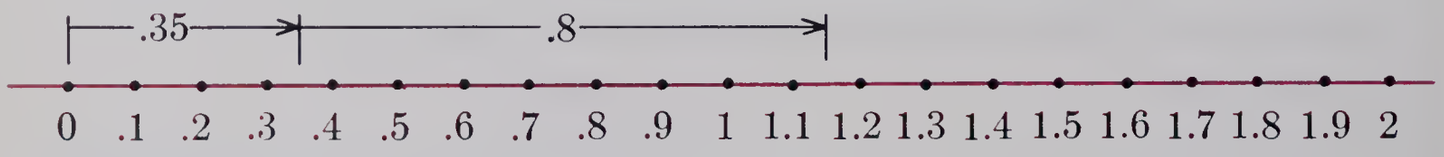


and $7-3=4$ is represented by

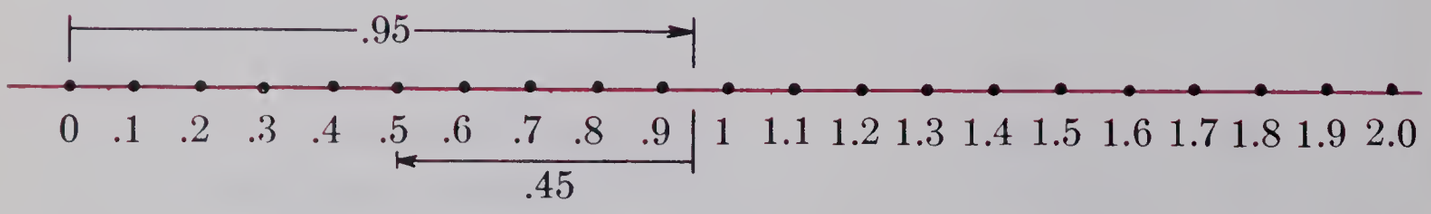


Decimal fractions can also be represented on the number line and addition and subtraction performed in the same way.

$$.35+.8=1.15$$



$$.95-.45=.5$$



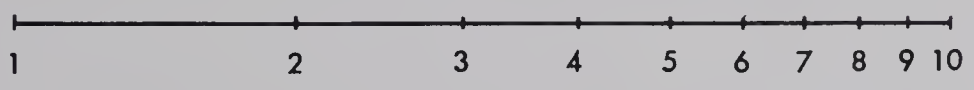
Instead of marking the equidistant points, which we usually label .1, .2 . . . , we can mark the points whose distances from the origin are proportional to the numbers

.30, .48, .60, .70, .78, .85, .90, .95, and 1.

These numbers are of course approximately the logarithms of

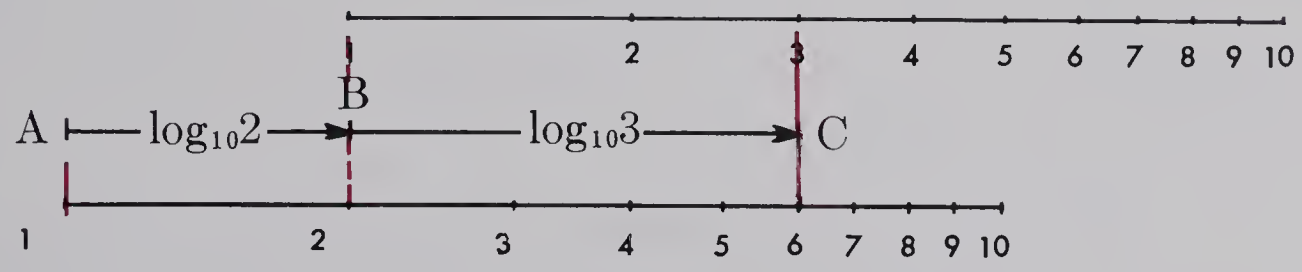
2, 3, 4, 5, 6, 7, 8, 9, and 10, respectively.

If we give the origin on the line the label 1, instead of our usual label of zero, and then label the other points 2, 3, . . . 10 in order, we obtain what is called a logarithmic scale.



In other words, the length of the line segment from the origin point, 1, to any point n $(2 \leq n \leq 10)$ is proportional to $\log_{10} n$.

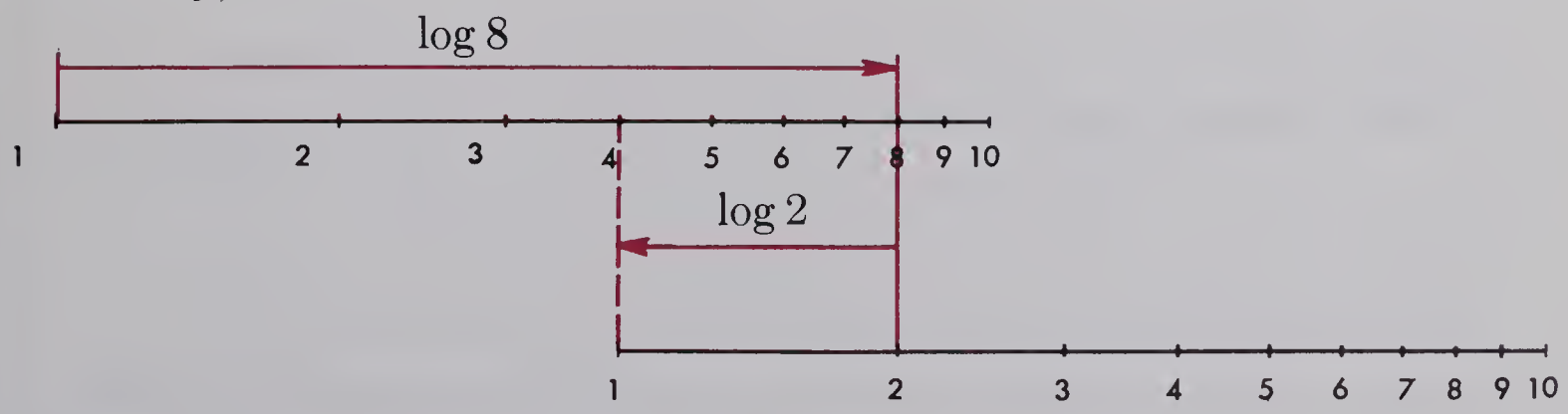
We can now represent multiplication of two numbers by the addition of logarithmic line segments. Thus to calculate 2×3:



The length of the line segment AB = .30,
the length of the line segment BC = .48,
and the length of the line segment AC = .78;

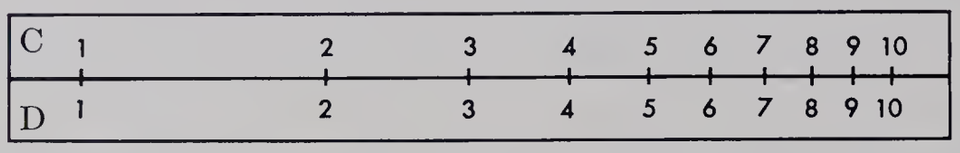
that is, the lengths AB, BC, and AC correspond to the logarithms of 2, 3, and 6, and therefore represent $\log_{10} 2 + \log_{10} 3 = \log_{10} 6$.

Similarly, for $8 \div 2$:



EXERCISE 4—10

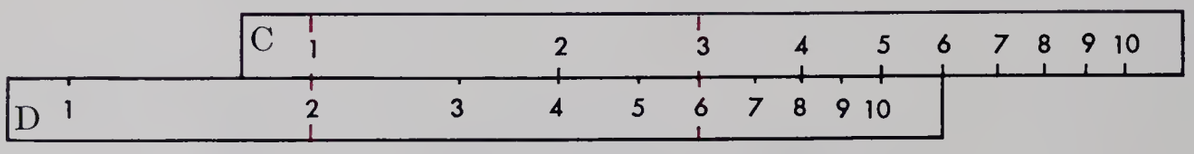
1. On a piece of Bristol board or stiff paper about 5″ long and 2″ wide, draw the centre line. On this line, mark off points corresponding to the logarithms to the base 10 of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, using a scale 10 cm. = distance from point 1 to 10. (For example, the distance from 1 to 2 is 30 mms.; see diagram.) Cut the card along the centre line and label one half C and the other D as shown.



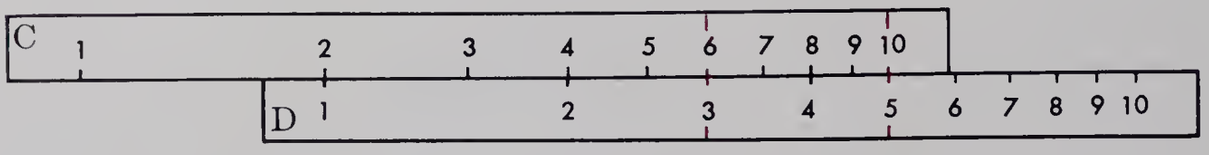
2. Using scale C placed against scale D, perform the following multiplications:

   (a) 2×4 (b) 8×5 (c) 7×8

   Examples: $2\times3=6$



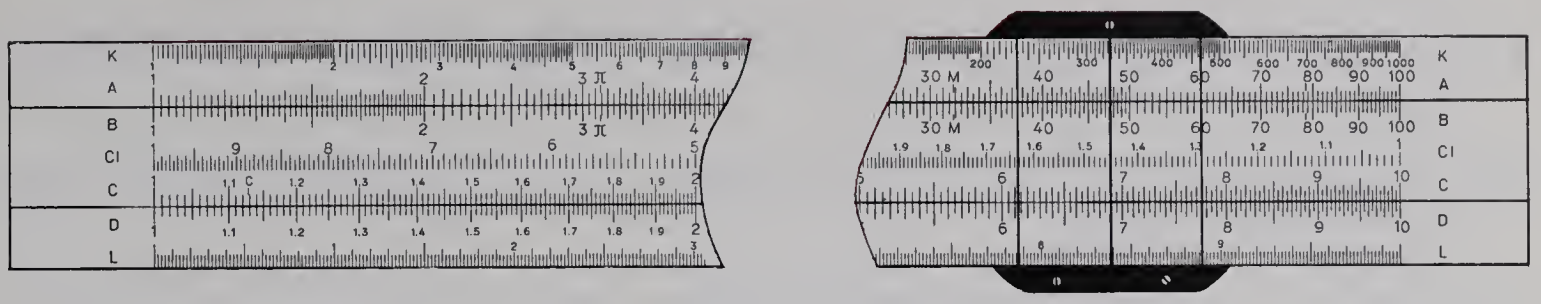
$5\times6=30$



3. Using a similar method to that in (2), perform the following divisions:

   (a) $9\div3$ (b) $20\div5$ (c) $75\div25$

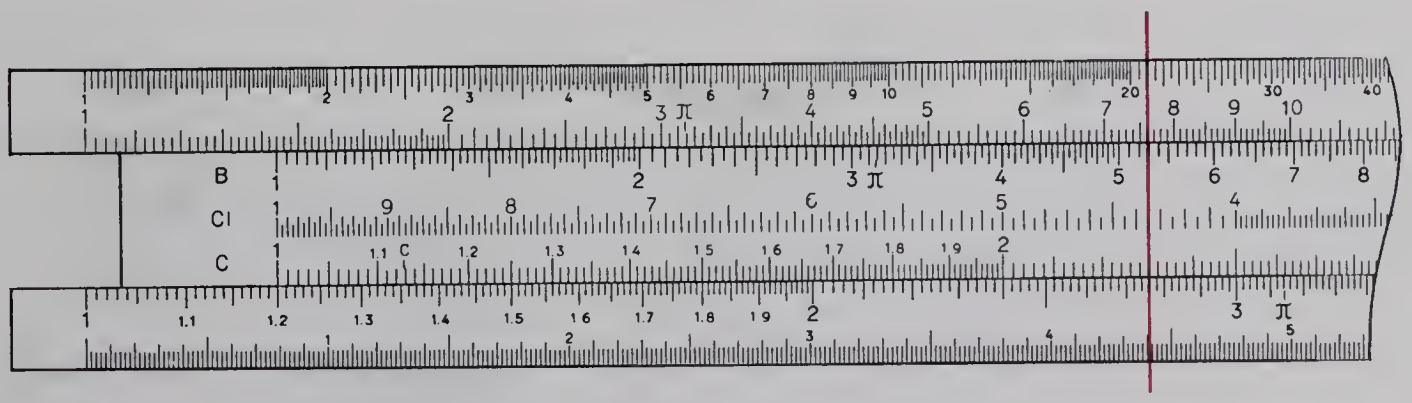
4 11 The Slide Rule (Supplementary)



The slide rule is a more elaborate version of the calculating device you made in the previous exercise. Some rules have a number of scales that are used for special purposes. The C and D scales are normally used for multiplication and division, and once the operation of these two scales is mastered, it is not difficult to learn how to use the other scales for their special purposes.

The C scale is on the moving part of the rule, which is often called the *slide*; the D scale is on the fixed part, which is called the *body* of the rule. The glass runner is called the *indicator* or *cursor*, and the fine line on the indicator is called the *hairline.* The mark asociated with the number 1 on any scale is called the *index.* Marks on different scales are *aligned* when they lie on the hairline at the same position of the slide.

Operation of the slide rule consists of aligning marks on two scales using the hairline, and then moving the indicator hairline to a new position that indicates the result of the mathematical operation involved. We shall study the operations of multiplication and division and also squaring and finding square roots. The diagram shows the position of the slide rule for the calculation $1.2 \times 2.3 = 2.76$.



Multiplication

To multiply two numbers a and b:

(i) Identify the mark corresponding to a on the D scale, and place the hairline on it.

(ii) Move the slide so that *an* index of the C scale coincides with the hairline.

(iii) Move the indicator so that the hairline lies on the mark corresponding to b on the C scale.

(iv) Read the number corresponding to the mark on the D scale, which is under the hairline.

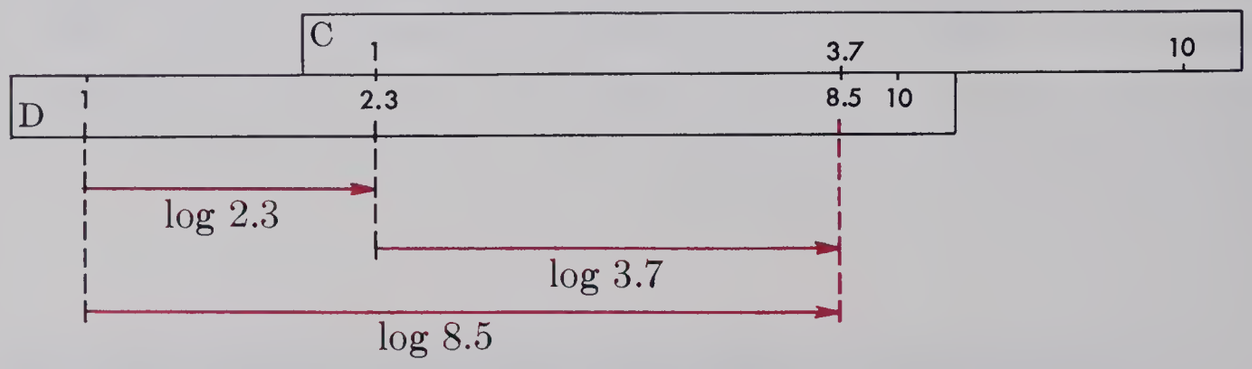
This is the result $a \times b$.

Notice that in (ii) we have said *an* index of the C scale; there are two, one at the left end of the scale and the other at the right. If the product $a \times b$ (considered as numbers in the range $1 < a < 10$, $1 < b < 10$) is less than 10, then we use the left index; if the product is greater than 10, we use the right index. It will be noticed that only in one case *or* the other does the mark for b on the C scale actually come into alignment with a mark on the D scale.

Example 1

Multiply 2.3×3.7.

Solution: The diagram shows the arrangement of the scales; the red drawing of the indicator shows the answer 8.5 (actually a little more than 8.5 but certainly less than 8.55).



Note that we have put in the decimal points. *There are no decimals on the slide rule.* We have to estimate the position of the decimal point ourselves. The example we have just worked could have been

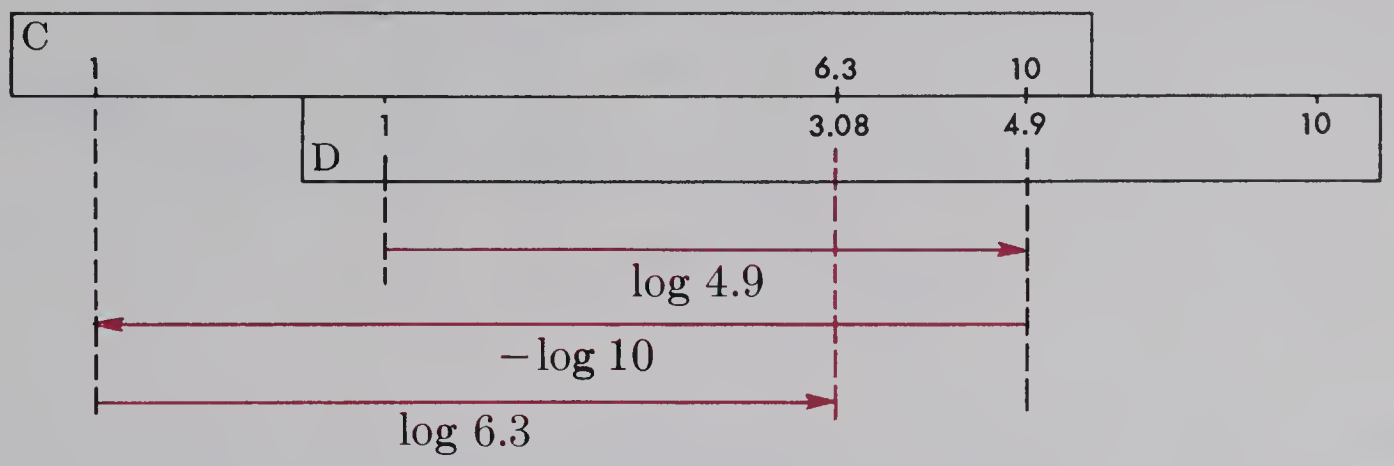
$$\begin{aligned} & \log 2.3 + \log 3.7 = \log 8.5 \\ \text{or} \quad & \log 2300 + \log 3700 = \log 8500000 \\ \text{or} \quad & \log .00023 + \log .00037 = \log .000000085. \end{aligned}$$

Note also that we had to use the *left* index of the C scale.

Example 2

Multiply 4.9×6.3.

Solution: The diagram shows the arrangement of the scales and the result 30.8.



Mathematically, $\log 4.9 - 1 + \log 6.3 = \log 3.08$.

$\therefore \log 4.9 + \log 6.3 = \log 3.08 + 1$.

Note that we had to use the *right* index of the C scale and therefore the term -1 in the above question is the result of the automatic subtraction of log 10 as shown in the diagram.

Division

To divide a number a by a number b:

(i) Identify the mark corresponding to b on the D scale and place the hairline on it.

(ii) Move the slide so that the mark corresponding to a on the C scale is aligned with the hairline.

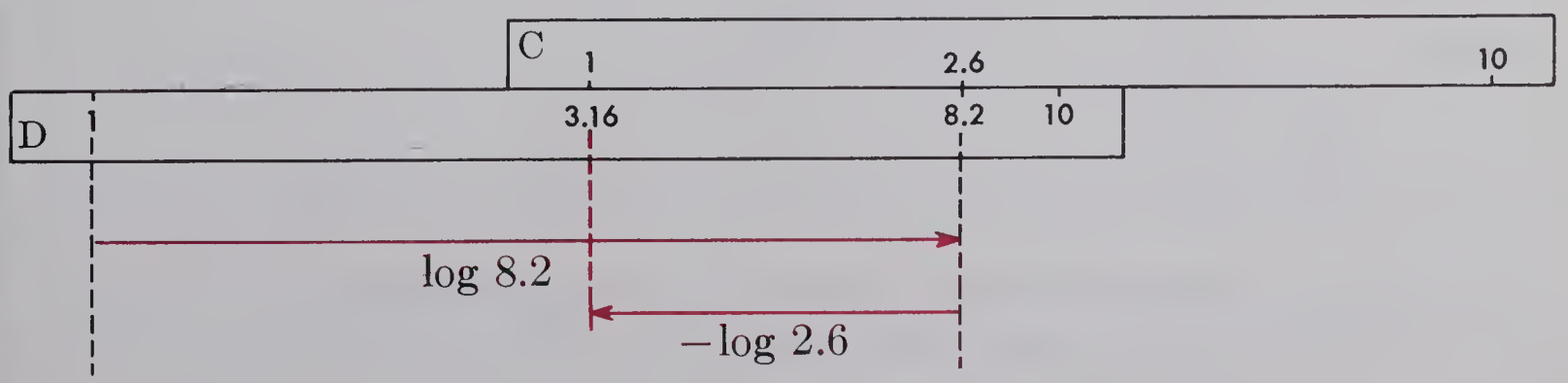
(iii) Move the indicator so that the hairline is on *an* index of the C scale.

(iv) Read the number corresponding to the mark on the D scale, which is under the hairline.

Note that again there are two cases.

Example 3

Divide 8.2 by 2.6.

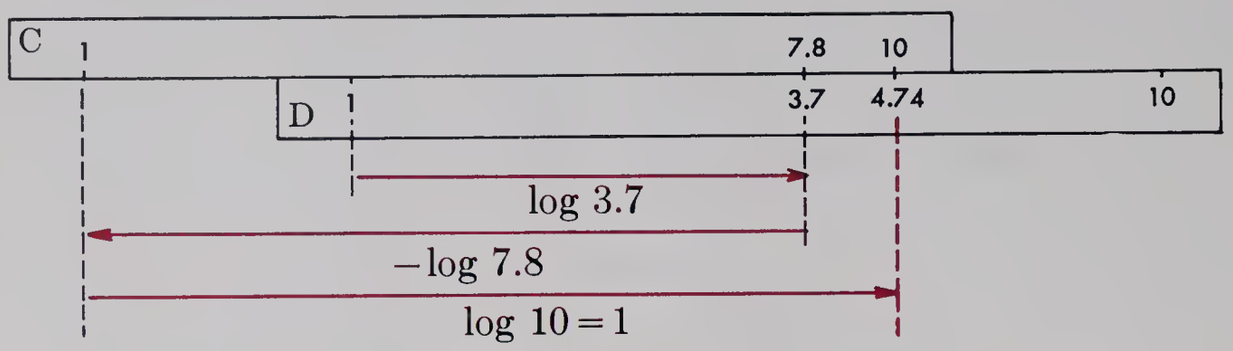


Example 4

Divide 3.7 by 7.8.

Solution: $\log 3.7 - \log 7.8 + 1 = \log 4.74$

$\therefore \log 3.7 - \log 7.8 = -1 + \log 4.74.$



Again in the above cases we must estimate the position of the decimal point. Successive operations of multiplication and division are carried out by continuing in the same way.

Square roots and squares

On most slide rules finding square roots and squares can be done without any movement of the slide. Two other scales A and B are used. It will be noted that the A and B scales are identical and each consists of two successive logarithmic scales, each half the length of either the C or D scales.

If we label the *left* index of the D scale as 1.0, then the *right* index corresponds to the number 10.0. Similarly on the A scale, if the *left* index is labelled 1.0, then the *centre* index corresponds to 10.0 and the right index to 100.0. Thus 10.0 on the D scale corresponds to 100.0 on the A scale. Similarly 5.0 on the D scale corresponds to 25.0 on the A scale and so on.

To find the square of a number:

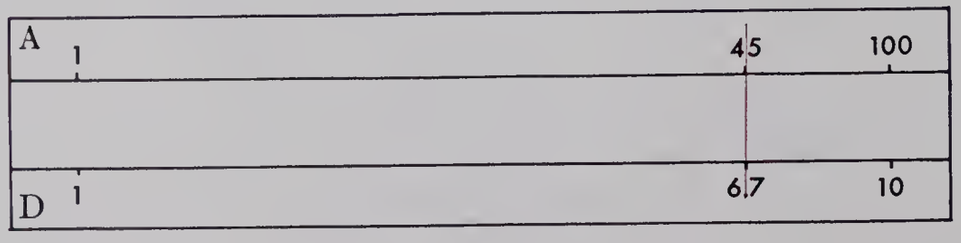
(i) Align the hairline of the indicator with the mark corresponding to the number x on the D scale.

(ii) Read the number corresponding to the mark under the hairline on the A scale; this number is x^2.

Example 5

Find $(6.7)^2$.

Solution: $\log 45 = \log(6.7)^2$



To find the square root of a number:

(i) Align the hairline with the mark corresponding to the number x on the A scale.

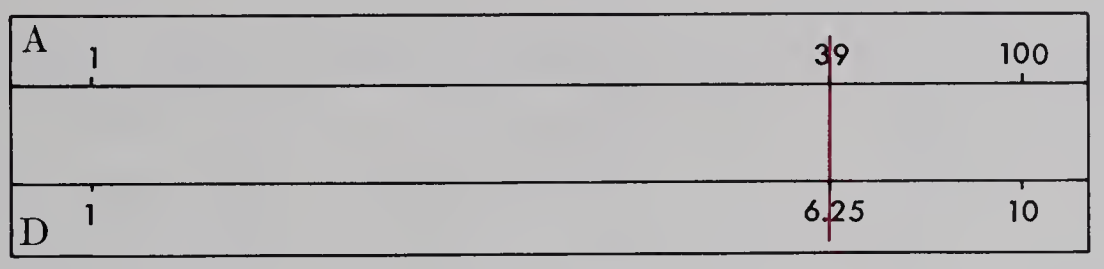
(ii) Read the number corresponding to the mark under the hairline on the D scale; this number is $\sqrt{x}$.

Example 6

Find the square root of 39.

Solution:

$$\log \sqrt{39} = \log 6.25$$



EXERCISE 4—11

Calculate the following using a slide rule:

1. 2.9×3.6	2. 6.9×8.8	3. 9.3×4.6	4. 69×1.7
5. 50×42	6. $.47\times71$	7. $6.3\div2.3$	8. $1.36\div8.2$
9. $86\div73$	10. $9.7\div81$	11. $.19\div7.8$	12. $22\div.58$
13. 2.9^2	14. 45^2	15. $.32^4$	16. $\sqrt{7.8}$
17. $\sqrt{538}$	18. $\sqrt{.85}$	19. $\sqrt{13.8}$	20. $\sqrt{.0236}$

21. $4.3\times7.8\div5.8$ 22. $28.4\div(5.6\times8.4)$ 23. $\sqrt{41.2\times3.6\times1.28}$

4 12 Chapter Summary

Calculations with graphs of $y=10^x$. The logarithmic function to base 10.

Tables of $y=10^x$ and $y=\log_{10}x$. Interpolation in the tables.

Multiplication and division using logarithmic and exponential function tables.

Formation of powers and roots using logarithmic and exponential function tables.

Equations involving logarithmic and exponential functions. Logarithms to bases other than 10. The slide rule (supplementary)

REVIEW EXERCISE 4—12

1. Sketch the graphs of the following for $0 \leq x \leq 2$ on the same diagram:

   (a) $y = 7^x$ (b) $y = 10^x$ (c) $y = \log_{10} x$ (d) $y = \log_5 x$

2. Use the graphs of $y = 10^x$ and $y = \log_{10} x$ to evaluate:

   (a) 7.8×1.9 (b) 60×8.1 (c) $.82 \times 4.1$

   (d) $82 \div 49$ (e) $5.8 \div 14$ (f) $.78 \div 54$

   (g) $\sqrt{93}$ (h) $\sqrt{3.5}$ (i) $\sqrt[3]{73}$

   (j) $\dfrac{36 \times 4.6}{6.1}$ (k) $\dfrac{5.2 \times .91}{49 \times 7.2}$ (l) $\dfrac{79 \times 4.1}{5.3 \times 2.8}$

3. Use the logarithmic and exponential function tables to evaluate:

   (a) 9.187×1.806 (b) $.428 \times 470.1$ (c) $73.79 \div 9.591$

   (d) $.4851 \div 45.87$ (e) 193.9×92.44 (f) $1.531 \div 310.8$

   (g) $.07387 \times 1.225$ (h) $68.91 \div 19.35$ (i) 94.91×4.504

   (j) $\dfrac{1.663 \times 22.46}{48.49 \times .0297}$ (k) $\dfrac{.00643 \times 1.257}{47.35 \times 8.459}$ (l) $\dfrac{689.5 \times .02837}{9.027 \times 18.24}$

   (m) $\sqrt{82.67}$ (n) $\sqrt[3]{3.917}$ (o) $\sqrt{.7021}$

   (p) $(72.38)^2$ (q) $(5.372)^5$ (r) $(2.457)^{-3}$

   (s) $(637.9)^{-2}$ (t) $(4.732)^{\frac{3}{4}}$ (u) $(82.53)^{\frac{2}{5}}$

   (v) $\sqrt{32.58 \times 46.71}$ (w) $\sqrt[3]{7.832 \times 6.157}$ (x) $\sqrt{4.189 \div 8.469}$

   (y) $\sqrt{\dfrac{24.78 \times 5.167}{14.79 \times 2.368}}$ (z) $\sqrt[3]{\dfrac{5.962 \times 94.37 \times 41.68}{7.463 \times 3.859 \times 127.4}}$

4. Solve:

   (a) $5^{2x} = 625$ (b) $(2.478)^x = 756.9$

   (c) $(42.96)^{x+1} = (8.743)^{2x-1}$ (d) $(549.3)^{3x-1} = (37.28)^{2-x}$

   (e) $3.418 \log_{10} x - 2.547 = 1.672$ (f) $4 \log_{10} x - 3.674 = 6 \log_{10} x$

5. Calculate the value of the following expressions to the best accuracy possible with a slide rule:

   (a) $39.2 \div 6.56$ (b) 2.39×19.6

   (c) 7.77×21.6 (d) $96.8 \div 3.16$

   (e) $31.66 \times 9.84 \times 21.4$ (f) $32.7 \times 1.98 \div 6.42$

   (g) $\sqrt{6.48}$ (h) $\sqrt{12.8}$

   (i) $(23.4)^2$ (j) $\sqrt{3.68 \times 61.8}$

Chapter 5

Simple and Compound Interest

5 1 Simple Interest

If we purchase a \$500, 5% government bond, we receive 5% of \$500 or \$25 each year as an interest payment for the use of our money. At the end of the period for which the bond is issued, our \$500 would be returned as the particular bond issue was redeemed. Usually with a government bond, although not with other bonds, we may redeem the bond at any time during its life.

In the example given, the \$500 originally lent is known as the *principal*, the 5% per annum is known as the *rate*, and the \$25 per annum received for the use of the money is the annual *interest*. In such a case, since the annual interest is always calculated on the principal, we are receiving *simple interest*. For our bond, if held for 10 years, the total amount received over this period would be the return of our \$500 at the end of the period plus 10 interest payments of \$25 each, a total of \$750. This sum is known as the *amount*.

If \$1 is invested for one year at a rate of 5% per annum simple interest, the interest earned will be \$.05, and the amount at the end of the year will be \$(1+.05). Therefore, if \$500 is invested for one year at a rate of 5% per annum simple interest, the amount at the end of that year will be \$500(1+.05). If the \$1 is invested for two years, the interest earned will be \$2(.05) and the amount will be \$[1+2(.05)]. Therefore, on \$500, the amount will be \$500[1+2(.05)]. In the same way, the amount of \$500 in n years at 5% per annum will be \$500 [1+n(.05)].

In this example, \$500 is the principal and .05 is the rate expressed in decimal fraction form. In a general case, therefore, the amount of \$P in n years at 100i% simple interest is \$P(1+ni); that is

$$A = P(1+ni)$$

where A is the amount,
P is the principal,
n is the number of years, and
i is the rate expressed in decimal fraction form.

Example 1

Find the amount of \$250 in 12 years at $4\frac{1}{2}\%$ per annum simple interest.

Solution: $P=250,\quad i=.045,\quad n=12$

$$\begin{aligned} A &= P(1+ni) \\ &= 250(1+.54) \\ &= 250\ (1.54) \\ &= 385 \end{aligned}$$

$\therefore$ Amount is \$385.

Example 2

If \$400 amounts to \$560 in 10 years, find the rate of simple interest.

Solution: $A=560,\quad P=400,\quad n=10$

$$A = P\ (1+ni)$$

$$\therefore\ 560 = 400\ (1+10i)$$

$$\therefore\ \ 7 = 5\ (1+10i).$$

$$\therefore\ \ 7 = 5+50i.$$

$$\therefore\ 50i = 2.$$

$$i = \frac{2}{50}$$

$$= .04$$

$\therefore$ Annual rate is 4% or .04.

EXERCISE 5—1

1. Find the amount in each of the following at simple interest:

	Principal	Time	Rate per annum
(a)	\$100	6 years	$4\frac{1}{2}\%$
(b)	\$320	8 years	$3\frac{1}{4}\%$
(c)	\$450	$7\frac{1}{2}$ years	6%
(d)	\$385	15 years	$2\frac{1}{4}\%$
(e)	\$216.25	5 years	3%
(f)	\$1865.50	18 years	$4\frac{1}{4}\%$

2. Find the annual rate of simple interest in each of the following:

	Principal	Time	Amount
(a)	\$450	7 years	\$544.50
(b)	\$250	12 years	\$400
(c)	\$620	8 years	\$793.60

(d)	\$84.50	6 years	\$95.91
(e)	\$170	$3\frac{1}{2}$ years	\$187.85
(f)	\$42.50	5 years	\$49.41

3. Find the time in years necessary for each of the following principals to accumulate to the given amount at the given annual rate of simple interest:

	Principal	Amount	Rate per annum
(a)	\$300	\$390	5%
(b)	\$440	\$704	$7\frac{1}{2}$%
(c)	\$650	\$981.50	$4\frac{1}{4}$%
(d)	\$1830	\$2166.26	$5\frac{1}{4}$%
(e)	\$77.50	\$89.90	4%
(f)	\$166.40	\$215.90	$3\frac{1}{2}$%

5.2 Compound Interest

The calculation of an amount at simple interest, as in the previous section, is now very rarely used. With a government bond or company share, the interest is usually paid annually, semi-annually, or quarterly. Only the principal remains to be returned at the end of the loan period.

When the interest is not withdrawn at the end of each period but is left to accumulate, it will be added to the principal giving a new figure on which the interest will be calculated for the next period. This is now the most common method of calculating an amount or accumulated value, and we say that the amount is calculated at *compound interest.*

Let us consider a bank deposit of \$100 when the bank pays interest at the rate of 3% compounded annually. For each \$1 on deposit at the beginning of a year and left on deposit during the year, the amount due at the end of the year will be \$1.03.

For the principal at the beginning of any year, the accumulated value at the end of the year will be that principal multiplied by 1.03.

The following table shows the amount due at the end of various years:

Year	Amount at start of year	Amount at end of year
1	\$100	\$100 (1.03)
2	\$100 (1.03)	\$100 (1.03) (1.03)
3	\$100 $(1.03)^2$	\$100 $(1.03)^2$ (1.03)
4	\$100 $(1.03)^3$	\$100 $(1.03)^3$ (1.03)
.	.	.
.	.	.
.	.	.
n	\$100 $(1.03)^{n-1}$	\$100 $(1.03)^n$

For example the amount or accumulated value at the end of 8 years will be

$$\$100(1.03)^8.$$

From this, we see that for a principal of \$P and a rate of interest of $100i\%$ compounded annually, the amount at the end of n years is given by the formula

$$A = P(1+i)^n.$$

Unless it is definitely stated otherwise, we always consider that the interest is calculated annually or compounded annually. However, in many cases the interest is compounded semi-annually, quarterly, or monthly. In such cases, the interest rate given is the *nominal* annual rate. If we are told that the rate of interest is 6% compounded semi-annually, for 10 years, then

$$A = P(1.03)^{20}.$$

The semi-annual rate is 3%, and during the 10 years the number of interest periods is 20.

The calculation of the amount by ordinary arithmetical methods would obviously be tedious and time-consuming. One improvement would be to make use of our tables of logarithms. For instance,

$$\text{if} \qquad A = 240\ (1.03)^{20},$$
$$\text{then} \quad \log A = \log 240 + 20 \log 1.03.$$

However, even this would take time and therefore, to simplify the calculations, tables of values for $(1+i)^n$ have been calculated for various values of i and n (see table, page 400). To find the value of $(1.03)^{20}$, we look in the 3% column and 20 year row. The value is 1.80611. This, of course, is accurate only to 5 decimal places; consequently our solution cannot be guaranteed beyond the fifth significant digit and even the fifth figure may not be correct. No result obtained by using these tables should be given to more than 5 significant digits.

Example 1

Calculate the amount of \$240 after 10 years at 6% compounded semi-annually.

Solution: $P = 240, \quad n = 20, \quad i = .03$

$$\begin{aligned} A &= P(1+i)^n \\ &= 240(1.03)^{20} \\ &\simeq 240(1.80611) \\ &= 433.4664 \end{aligned}$$

$\therefore$ Amount is \$433.47.

Example 2

Calculate the interest received on \$1250 at the end of 8 years at 4% if the interest is compounded

(a) annually, (b) quarterly.

Solution: (a) Annually

$$P = 1250, \quad n = 8, \quad i = .04$$
$$A = 1250\ (1.04)^8$$
$$\simeq 1250\ (1.36857)$$
$$\simeq 1710.71$$

$\therefore$ Amount is \$1710.70.
$\therefore$ Interest is \$460.70.

(b) Quarterly

$$P = 1250, \quad n = 32, \quad i = .01$$
$$A = 1250\ (1.01)^{32}$$
$$\simeq 1250\ (1.37494)$$
$$\simeq 1718.68$$

$\therefore$ Amount is \$1718.70.
$\therefore$ Interest is \$468.70.

Example 3

Calculate the equivalent annual rate (or effective rate) if the rate is 12% compounded monthly.

Solution: For a principal of \$1 for 1 year

$$A = (1.01)^{12}$$
$$\simeq 1.12683.$$

$\therefore$ equivalent annual rate is 12.683%.

EXERCISE 5—2

1. Complete each row in the following table:

Principal	Rate	Compounded	Time	Amount
\$200	4%	Annually	12 years	
\$350	5%			$350\ (1.025)^{18}$
\$480		Quarterly		$480\ (1.015)^{28}$
	8%			$1200\ (1.04)^{20}$
\$640	6%	Monthly	3 years	
	3%			$680\left(1+\frac{.03}{12}\right)^{15}$
		Semi-annually		$900\ (1.02)^{16}$
\$P	12%			$P\ (1.03)^{40}$
		Semi-annually		$200\ (1.025)^{2n}$
			16 years	$P\ (1+i)^{16}$
			n years	$P\ (1+i)^{4n}$

2. Find the accumulated value of each of the following:
   (a) \$200 in 12 years at 4%,
   (b) \$350 in 9 years at 5% compounded semi-annually,
   (c) \$480 in 7 years at 6%,
   (d) \$480 in 7 years at 6% compounded quarterly,
   (e) \$1200 in 9 years at 4% compounded semi-annually,
   (f) \$675 in 2 years at 12% compounded monthly,
   (g) \$1000 in $6\frac{1}{2}$ years at 3% compounded semi-annually,
   (h) \$750 in $6\frac{1}{2}$ years at 4% compounded quarterly,
   (i) \$500 in $3\frac{3}{4}$ years at 8% compounded semi-annually,
   (j) \$800 in 5 years 1 month at 12% compounded quarterly.

3. Find the equivalent annual rate in each of the following:
   (a) 5% compounded semi-annually,
   (b) 8% compounded quarterly,
   (c) 12% compounded monthly,
   (d) 6% compounded quarterly,
   (e) 6% compounded semi-annually,
   (f) 12% compounded quarterly,
   (g) 12% compounded semi-annually.

4. If \$100 is deposited in a savings account at the end of each year for 3 years, find the accumulated value 10 years after the first deposit, if the rate of interest is 3% compounded semi-annually.

5 3 Graphs of $A=P(1+ni)$ and $A=P(1+i)^n$

Let us consider the formula $A=P(1+ni)$, when $P=100$ and $i=.04$.

$$\text{Then } A=100(1+.04n)$$
$$=100+4n.$$

For each value of n, there corresponds one and only one value of A. We also see that A and n appear only to the first power so that $A=100+4n$ defines a linear function and its graph will be a straight line. The domain will be the set of nonnegative real numbers since n represents a number of interest periods. The range will be the set of positive real numbers. From a practical point of view we shall restrict ourselves, at the moment, to the set of nonnegative rational numbers for domain and range.

For the formula $A=P(1+i)^n$, when $P=100$ and $i=.04$,

$$A=100(1.04)^n.$$

Again this formula defines a function but in this case we see that we have an

exponential function. For any given principal and rate we can draw a graph of amount against time. This graph will be a straight line for simple interest and an exponential curve for compound interest. The graphs of $A=100+4n$ and $A=100(1.04)^n$ are drawn together in Figure 5-1 for the domain $0 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$100+4n$	104	108	112	116	120	124	128	132	136	140
$100(1.04)^n$	104	108.16	112.49	116.99	121.67	126.53	131.59	136.86	142.33	148.02

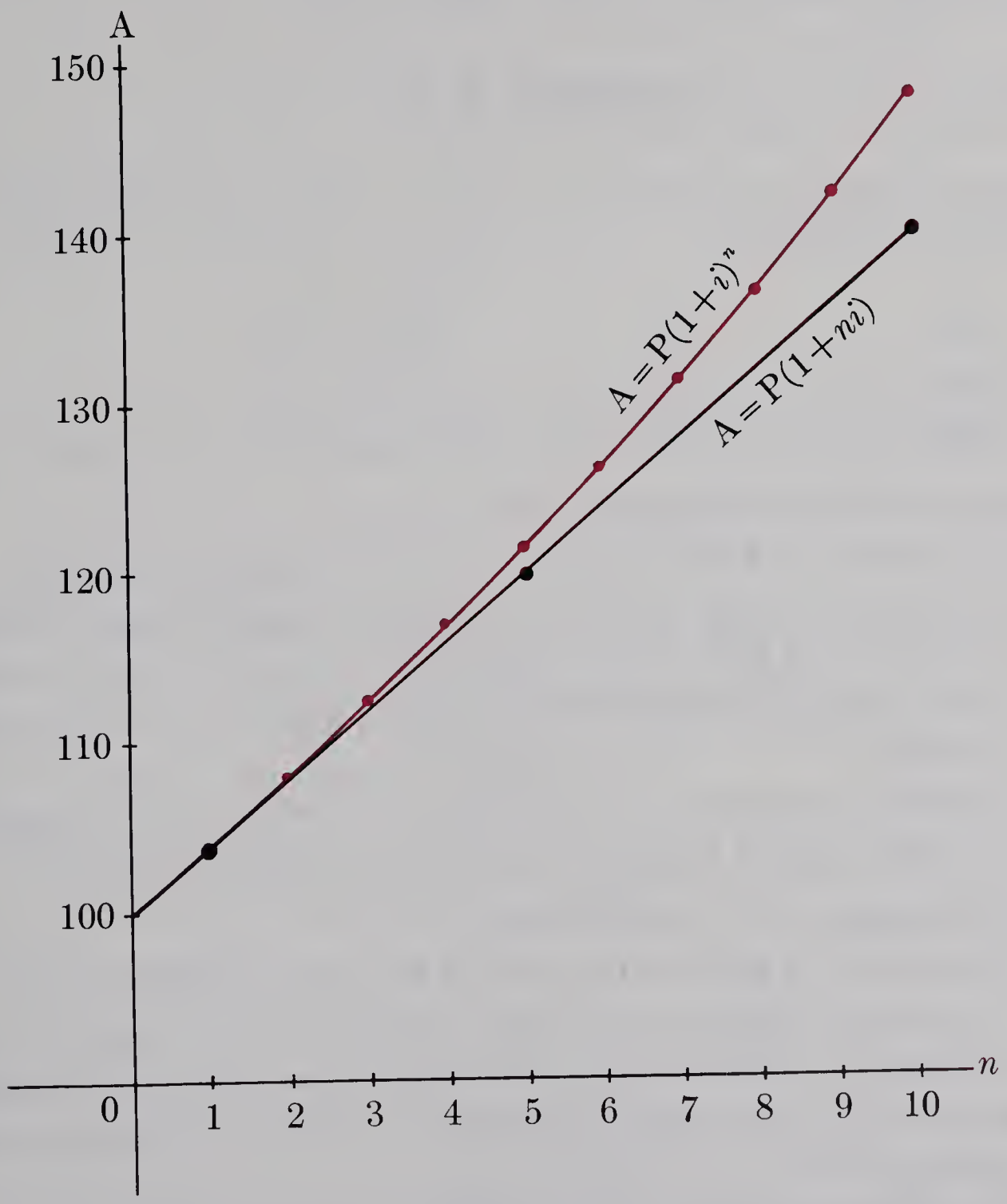


Figure 5-1

For the graph of $A=P(1+ni)$, the horizontal intercept or, in this case the n-intercept if it exists, will be given by $0=P(1+ni)$.

$$\therefore\ n=-\frac{1}{i}.$$

Since our domain is the set of nonnegative real numbers and $i>0$, there will be no n-intercept.

The vertical intercept will be the value of A when $n=0$, so that the A-intercept is P(100 in Figure 5-1).

For the graph of $A=P(1+i)^n$, the vertical intercept (or A-intercept) is again P but there is obviously no horizontal intercept (or n-intercept). For this to exist, A must be zero and therefore, since $P\neq 0$, $(1+i)^n$ must be zero. There is no value of n for which this is true.

EXERCISE 5—3

1. Draw the graphs of $A=P(1+ni)$ and $A=P(1+i)^n$ for the following values of P, i, and n:

	P	i	n
(a)	\$250	.04	$0\leq n\leq 10$
(b)	\$500	.03	$0\leq n\leq 12$
(c)	\$400	.02	$0\leq n\leq 10$

2. Using the graph of (1a) above, state:
   (i) the amount of \$250 for $6\frac{1}{2}$ years at 4% compounded annually,
   (ii) the amount of \$250 for $6\frac{1}{2}$ years at 4% per annum simple interest,
   (iii) the amount of \$250 for $3\frac{1}{2}$ years at 8% compounded semi-annually,
   (iv) in how many years \$250 will amount to \$300 at 4% compounded annually,
   (v) the difference in the amount of \$250 for 6 years at 4% compounded annually, and for 6 years at 4% per annum simple interest.

3. Using the graph of (1b) above, state:
   (i) the amount of \$500 for $10\frac{1}{2}$ years at 3% compounded annually,
   (ii) the amount of \$500 for $10\frac{1}{2}$ years at 3% per annum simple interest,
   (iii) the amount of \$500 for $4\frac{1}{2}$ years at 6% compounded semi-annually,
   (iv) in how many years \$500 will amount to \$700 at 6% compounded semi-annually,
   (v) the difference in the amount of \$500 for 5 years at 6% compounded semi-annually, and for 10 years at 3% per annum simple interest.

4. Using the graph of (1c), state:
   (i) the amount of \$400 for $7\frac{1}{2}$ years at 2% compounded annually,
   (ii) the amount of \$400 for $7\frac{1}{2}$ years at 2% per annum simple interest,
   (iii) the amount of \$400 for 6 years at 8% compounded quarterly,
   (iv) in how many years \$400 will amount to \$500 at 4% compounded semi-annually,
   (v) the difference in the amount of \$400 for 4 years at 2% compounded annually, and 8 years at 2% per annum simple interest.

5 4 Present Value

We may wish to have a certain sum of money available at some future date and wish to calculate how much we should deposit now in order that the required sum will be available on that date. We assume that our deposit will be earning compound interest while it is on deposit.

If the rate of interest is 5% compounded annually, let us consider how much must be deposited now in order that \$1000 may be available at the end for 10 years.

Using the formula $A = P(1+i)^n$, with $A = 1000$, $i = .05$, $n = 10$,
we see that $1000 = P(1.05)^{10}$.

$$\therefore P = \frac{1000}{(1.05)^{10}}.$$

This value of P is the sum (in dollars) which must be deposited and is known as the *Present Value* (P.V.) of \$1000 in 10 years at 5% compounded annually.

In general, $\text{P.V.} = \frac{A}{(1+i)^n}$ is the present value of \$A in n years at $100i\%$ compounded annually.

To avoid the tedious process of division, we use tables of values of $\frac{1}{(1+i)^n}$ which have been calculated (See table, page 402), and are used in a similar manner to the tables of values $(1+i)^n$.

Example 1

What is the present value of \$1000 payable 10 years from now if the rate of interest is 5% compounded annually?

Solution: $A = 1000$, $i = .05$, $n = 10$

$$\text{P.V.} = \frac{A}{(1+i)^n}$$
$$= 1000 \times \frac{1}{(1.05)^{10}}$$
$$\simeq 1000 \ (.61391)$$
$$= 613.91$$

$\therefore$ present value is \$613.91.

Example 2

A father wishes to arrange that his son will have \$6000 available on his eighteenth birthday to provide for his future education. If the rate of interest is 4% compounded semi-annually, how much must the father deposit in a trust account on his son's sixth birthday?

Solution: $A = 6000, \quad i = .02, \quad n = 24$

$$\text{P.V.} = \frac{A}{(1+i)^n}$$
$$= \frac{6000}{(1.02)^{24}}$$
$$\simeq 6000\ (.62172)$$
$$= 3730.32$$

$\therefore$ Amount to be deposited is \$3730.30.

We must remember that the cents value here may be incorrect due to round-off errors, and a more precise statement would be that the amount is greater than \$3730.25 and less than \$3730.35.

EXERCISE 5—4

1. Find the present value of
   (i) \$650 due in 7 years at 4%,
   (ii) \$300 due in 6 years at 6% compounded semi-annually,
   (iii) \$300 due in 6 years at 6% compounded quarterly,
   (iv) \$750 due in 18 years at 5% compounded semi-annually,
   (v) \$25 due in 3 years at 12% compounded monthly,
   (vi) \$1380 due in $7\frac{1}{2}$ years at 3% compounded semi-annually,
   (vii) \$1380 due in $7\frac{1}{2}$ years at 3% compounded annually,
   (viii) \$250 due in $4\frac{1}{4}$ years at 4% compounded semi-annually.
2. A house is purchased by paying \$5000 now and \$4000 at the end of one year. If the rate of interest is 5% compounded semi-annually, what is the equivalent cash price of the house?
3. A trust fund is set up to provide a boy with \$10,000 on his twenty-first birthday. If it is agreed that he receive the money on his eighteenth birthday, and the interest rate is 4% compounded semi-annually, how much will he receive?
4. A car is purchased with a down payment of \$1200 and it is agreed to pay a further \$200 at the end of each month for four months. If the rate of interest is 24% compounded monthly, what is the equivalent cash price?

5 5 The Number e = 2.71828 . . . (Supplementary)

In Section 5-2 we studied the question of the effective rate of interest if the interest is compounded annually, semi-annually, quarterly, and monthly. We know that an amount of \$1 at 10% per annum interest compounded *annually* would be 2.59374 at the end of ten years; that is,

$$A(10) = \left(1+\frac{1}{10}\right)^{10} \simeq 2.59374.$$

If the interest were compounded half yearly at the same rate for 10 years,

$$A(20) = \left(1+\frac{1}{20}\right)^{20} \simeq 2.65330,$$

and if quarterly for 10 years,

$$A(40) = \left(1+\frac{1}{40}\right)^{40} \simeq 2.68506,$$

and if monthly,

$$A(120) = \left(1+\frac{1}{120}\right)^{120} \simeq 2.7140453.$$

We notice that the amount, A, is increasing steadily as the number of periods for interest increases, even though the rate *per period* goes down. However, the increase is not very rapid. What do you think would happen if the interest were compounded daily or hourly at the same annual rate? The amounts, A, would be

$$A(3650) = \left(1+\frac{1}{3650}\right)^{3650} \simeq 2.71824495,$$

$$\text{and } A(87600) = \left(1+\frac{1}{87600}\right)^{87600} \simeq 2.71826354.$$

Notice that the last two values agree in value only to five significant figures; so we have to go to extremely large numbers to get agreement to ten or more digits.

In fact, even if we compounded the interest every second or even every micro-second (a micro-second is one millionth of a second), the value of $A(n)$ would still be less than 2.718282. Notice that we now have a situation very like that which we obtained when we first tried to calculate $\sqrt{2}$. In this present case, it is possible to show that

$$2.71826354 < A(n) = \left(1+\frac{1}{n}\right)^{n} < 2.71828183 \text{ for all } n > 87{,}600.$$

This indicates that $\left(1+\frac{1}{n}\right)^{n}$ approaches a unique real number as n becomes indefinitely large; this number is called e, and to 21 significant figures

$$e = 2.71828182845904523536.$$

This irrational number is very important in science and engineering; almost every possible law of growth or decay depends on this number.

Returning to our study of this number in the context of compound interest, we see that, if the total length of time in our example is increased to twenty years (that is, doubled), then, if the interest rate is kept at 10% per annum, the amount at the end of twenty years,

$$\text{compounded yearly, is } \left(1+\frac{1}{10}\right)^{20} = [\mathrm{A}(10)]^2,$$

$$\text{compounded monthly, is } \left(1+\frac{1}{120}\right)^{240} = [\mathrm{A}(120)]^2.$$

Hence as n increases, the amount at the end of 20 years will approach e^2.

Similarly, for 30 years the ultimate amount would be e^3, 40 years, e^4, and, in fact, for $10x$ years it would be e^x.

Note that e^x is an exponential function.

Tables of the exponential function $y=e^x$ have been calculated, as have tables of $\log_e x$ which is called the *natural logarithmic function*. However, as we know from Section 4-10, we can find natural logarithms by converting from logarithms to the base 10. To do this, we need to know that

$$\log_{10} e = .4343 \qquad \text{or} \qquad \log_e 10 = 2.30258$$

and to use

$$\log_e x = \log_{10} x \cdot \log_e 10 \qquad \text{or} \qquad \log_{10} x = \log_e x \cdot \log_{10} e.$$

Example

The rate of decay of radium as a result of radioactive decay is proportional at any time to the amount of radium still left at that time. As a result the amount of radium, Q grams, at any time t is given by $Q = Q_0 e^{-kt_0}$ where Q_0 grams is the amount of radium present originally at $t=0$ and k is a constant. The time taken to reduce the amount of radium by 50%, that is to $\frac{1}{2}Q_0$, is called the *half-life*, t_0. If $k=.000435$ when t is measured in years, what is the half-life?

Solution: When $t=t_0$, $Q=\frac{1}{2}Q_0$,

$$\therefore \tfrac{1}{2}Q_0 = Q_0 e^{.000435t_0}$$

$$\tfrac{1}{2} = e^{.000435t_0}$$

$$e^{.000435t_0} = 2$$

$$.000435t_0 = \log_e 2$$

$$= \frac{\log_{10} 2}{\log_{10} e}$$

$$.000435t_0 \simeq \frac{.3010}{.4343}$$

$$t_0 \simeq \frac{.3010}{.4343} \div .000435$$

$$\simeq 1594$$

$$\log_{10}.3010 = -1+.4786$$

$$\log_{10}.4343 = -1+.6378$$

$$-1+.8408$$

$$\log_{10}.000435 = -4+.6385$$

$$3+.2023$$

The half-life of radium is 1594 years.

EXERCISE 5—5 (Supplementary)

1. Draw a graph of $y=e^x$, obtaining a table of values for $x=0, .5, 1, 1.5, 2$.

2. Calculate the logarithm to the base e of 2, 3, 4, 5, 6, 7, 8, and 9. Use a table of logarithms only in the calculations of $\log_e 2$, $\log_e 3$, and $\log_e 7$. Assume $\log_e 10 = 2.303$.

3. The temperature, T°C, of a body cooling from 100°C to 0°C is given by $T=100e^{-kt}$. If $k=.1$ when t is measured in minutes, find (a) the temperature after 20 minutes, (b) the time it takes to cool to 20°C.

4. The number N of bacteria in an experiment is given by the functional relation $N=N_0e^{kt}$, where N_0 is the original number of bacteria when $t=0$. Find k if t is measured in hours and the number N doubles every hour.

5 6 Chapter Summary

If A is the amount, P.V. the present value, P the principal, n the number of years (or interest periods), and i the rate of interest expressed in decimal fraction form,

then

at simple interest $A=P(1+ni)$,

at compound interest $A=P(1+i)^n$.

$$\text{P.V.}=\frac{A}{(1+i)^n}$$

REVIEW EXERCISE 5—6

1. Find the amount of \$850 for 10 years at
   (i) 4% simple interest, (ii) 4% compounded semi-annually.

2. Calculate the equivalent annual rate if the nominal rate is 8% compounded quarterly.

3. Find the present value of \$1200 due in 15 years if the rate of interest is 5% compounded semi-annually.

4. If \$200 is deposited in a bank at the end of each year for 5 years, and the accumulated value is withdrawn 12 years after the first deposit, what is the accumulated value if the rate of interest is 3% compounded annually?

5. Find the accumulated value at compound interest of each of the following:
   (a) \$250 in 6 years at 4%,
   (b) \$250 in 6 years at 4% compounded semi-annually,
   (c) \$850 in $12\frac{1}{2}$ years at 5% compounded semi-annually,
   (d) \$1000 in 3 years at 15% compounded monthly,
   (e) \$372.50 in 9 years at 6% compounded quarterly,
   (f) \$500 in $6\frac{1}{4}$ years at 8%.

6. Find the equivalent annual rate in each of the following:
   (a) 18% compounded monthly,
   (b) 8% compounded semi-annually,
   (c) 4% compounded quarterly,
   (d) 18% compounded semi-annually,
   (e) 18% compounded quarterly.

7. Find the present value of
   (a) \$2000 due in 8 years at 5%,
   (b) \$1500 due in 5 years at 4% compounded quarterly,
   (c) \$800 due in 2 years at 24% compounded monthly,
   (d) \$800 due in 2 years at 24% compounded quarterly,
   (e) \$450 due in $7\frac{1}{2}$ years at 7% compounded semi-annually.

8. If $A(n)=\left(1+\frac{1}{n}\right)^n$ tends to the value e as n becomes indefinitely large, and if $B(n)=\left(1+\frac{1}{n}\right)^{2n}$ tends to the value e^2, show that $C(n)=\left(1+\frac{2}{n}\right)^n$ will also tend to e^2 as n becomes indefinitely large.

9. If the lights are left on when a car is parked, the battery discharges and the voltage (V volts) of the battery is given at any time by $V=V_0e^{-kt}$ where $V_0=12$ volts. If $k=.01$, find to the nearest minute the time it takes to reduce the voltage to 9 volts.

Chapter 6

The Quadratic Function

6 1 Introduction

In Chapter 1 we studied the real linear function defined by

$$y=mx+b, \quad m, b\in \mathrm{Re}, \quad x\in \mathrm{Re}.$$

The domain of the function is the set of real numbers, and its graph is the set of points

$$\{(x, y)|y=mx+b; x\in \mathrm{Re}\}.$$

The graph is a line, and $y=mx+b$ is called the equation of the line. It was pointed out that all lines, except those parallel to the y-axis, have equations of this form (lines parallel to the y-axis are not graphs of functions; they have equations of the form $x=c$, $c\in \mathrm{Re}$). The general linear equation has the form

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0, \quad \mathrm{A}, \mathrm{B}, \mathrm{C}\in \mathrm{Re}, \quad \text{A and B not both 0.}$$

In this chapter we shall consider the graph and properties of the real quadratic function defined by

$$y=ax^2+bx+c, \quad a, b, c\in \mathrm{Re}, \quad a\neq 0,$$

over the domain of real numbers. This is the graph of the set of points

$$\{(x, y)|y=ax^2+bx+c; x, a, b, c\in \mathrm{Re}, a\neq 0\}.$$

6 2 The Function Defined by $y=ax^2$, $a \in \text{Re}$, $a \neq 0$

We saw the graph of the function defined by $y=x^2$ on page 32. Now let us consider the following example.

Example

Sketch the graph of the function defined by $y=2x^2$.

Solution: The graph is the infinite set of points (x, y) with $y=2x^2$. Obviously we cannot find and plot all these points. We must find a subset of these points that will indicate to us the shape of the curve. Such a subset is given by the following table.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	50	32	18	8	2	0	2	8	18	32	50

We plot this subset of points and draw the smoothest possible curve through them (Figure 6-1).

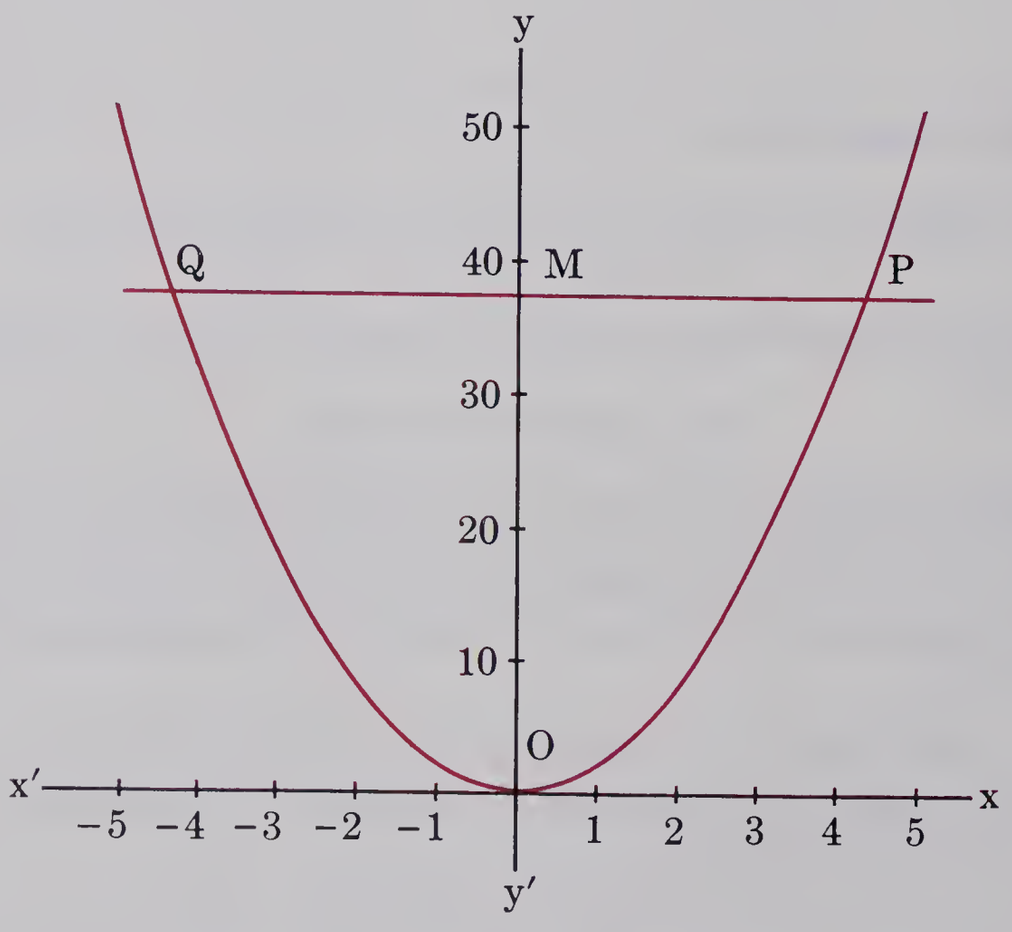


Figure 6-1 $y=2x^2$

Notes:

(1) It may be wondered why integral values in the set $-5 \leq x \leq 5$ were assigned to x to produce this table. The ability to select a suitable set of values to obtain a subset of points useful for sketching a curve will come as more examples are studied.

(2) In our example the numbers y increase numerically much faster than the corresponding numbers x, and therefore a larger scale is used on the y-axis. (Of course the scale on *each* axis is uniform.)

Observations from the graph:

(1) No point of the curve is below the x-axis. Why not?

(2) The curve opens upward. A curve that opens upward in a certain region is said to be concave upward in that region. A curve that opens downward in a certain region is said to be concave downward in that region.

(3) A minimum point of a curve is a point lower than (or at least no higher than) any of its immediately neighbouring points. The origin is the minimum point of the curve in this example.

A maximum point of a curve is a point higher than (or at least no lower than) any of its immediately neighbouring points. The curve in this example has no maximum point.

(4) The curve seems to be symmetric about the y-axis; that is, there is a branch of the curve to the left of the y-axis which is a mirror image of the branch to the right. (If we could fold the plane along the y-axis, the two branches would coincide.)

Recall the definition that a figure is symmetrical with respect to a line as an axis of symmetry if, for every point of the figure, there is another point of the figure such that the axis is the perpendicular bisector of the line segment joining the two points.

Then the symmetry can be established as follows:

Take any point M$(0, k)$ on the y-axis.

Construct QMP perpendicular to the y-axis cutting the graph of $y = 2x^2$ at points P and Q.

Then P is $\left(\sqrt{\frac{k}{2}}, k\right)$ and Q is $\left(-\sqrt{\frac{k}{2}}, k\right)$; hence $\text{MP} = \sqrt{\frac{k}{2}} = \text{QM}$,

and by definition the curve is symmetrical about the y-axis.

There is obviously a connection between the symmetry of this curve and the set of values chosen for x in the table of values. If we had known about this symmetry, we could have chosen $0 \leq x \leq 5$ as the set of values and still have been able to plot the graph for $-5 \leq x \leq 5$.

Now consider the graph of the function defined by $y = -3x^2$ (Figure 6-2) and answer the following questions.

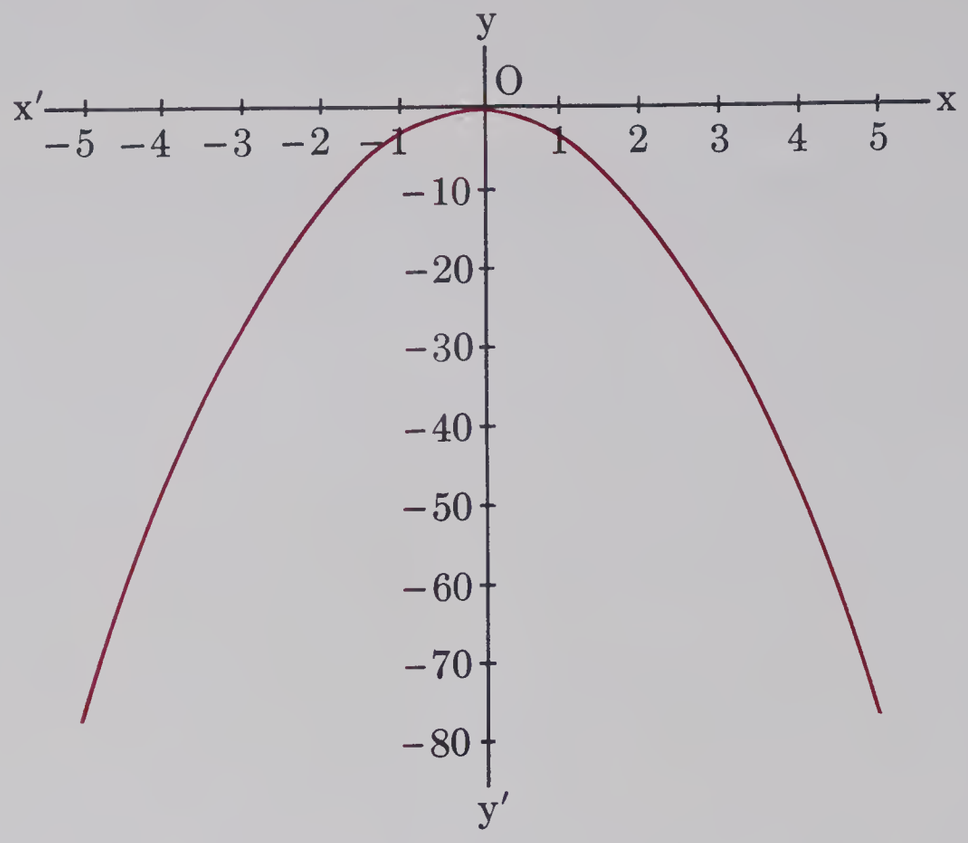


Figure 6-2 $y = -3x^2$

(1) What would be a suitable set of values of x for setting up a table of values to obtain points on this graph? Would $-100 \leq x \leq 100$ be a suitable set? Would $25 \leq x \leq 100$?

(2) Why is no part of the curve above the x-axis?

(3) Is the curve concave upward or concave downward?

(4) Does the curve have a minimum point? A maximum point?

(5) Does the curve have a line of symmetry? If so, what is the equation of the line of symmetry?

We have now studied graphs of the function defined by $y = ax^2$ for $a = 1$ (page 32), $a = 2$ (Figure 6-1), and $a = -3$ (Figure 6-2). These functions are special cases of the general quadratic function defined by $y = ax^2 + bx + c$, $a \neq 0$ (what special cases?).

The curves are called parabolas and $y = ax^2 + bx + c$ is called the equation of a parabola. Thus $y = x^2$, $y = 2x^2$, and $y = -3x^2$ are the equations of the parabolas we have drawn. In later sections, we shall consider parabolas with equations such as $y = 2x^2 - 3$ and $y = 6x^2 - x + 7$.

The line of symmetry of a parabola is called the axis of symmetry of the parabola. This line intersects the parabola at a point called the vertex of the parabola.

Examine the members of the family of parabolas with equations $y = ax^2$ shown in Figure 6-3, and verify the statements made in the following summary.

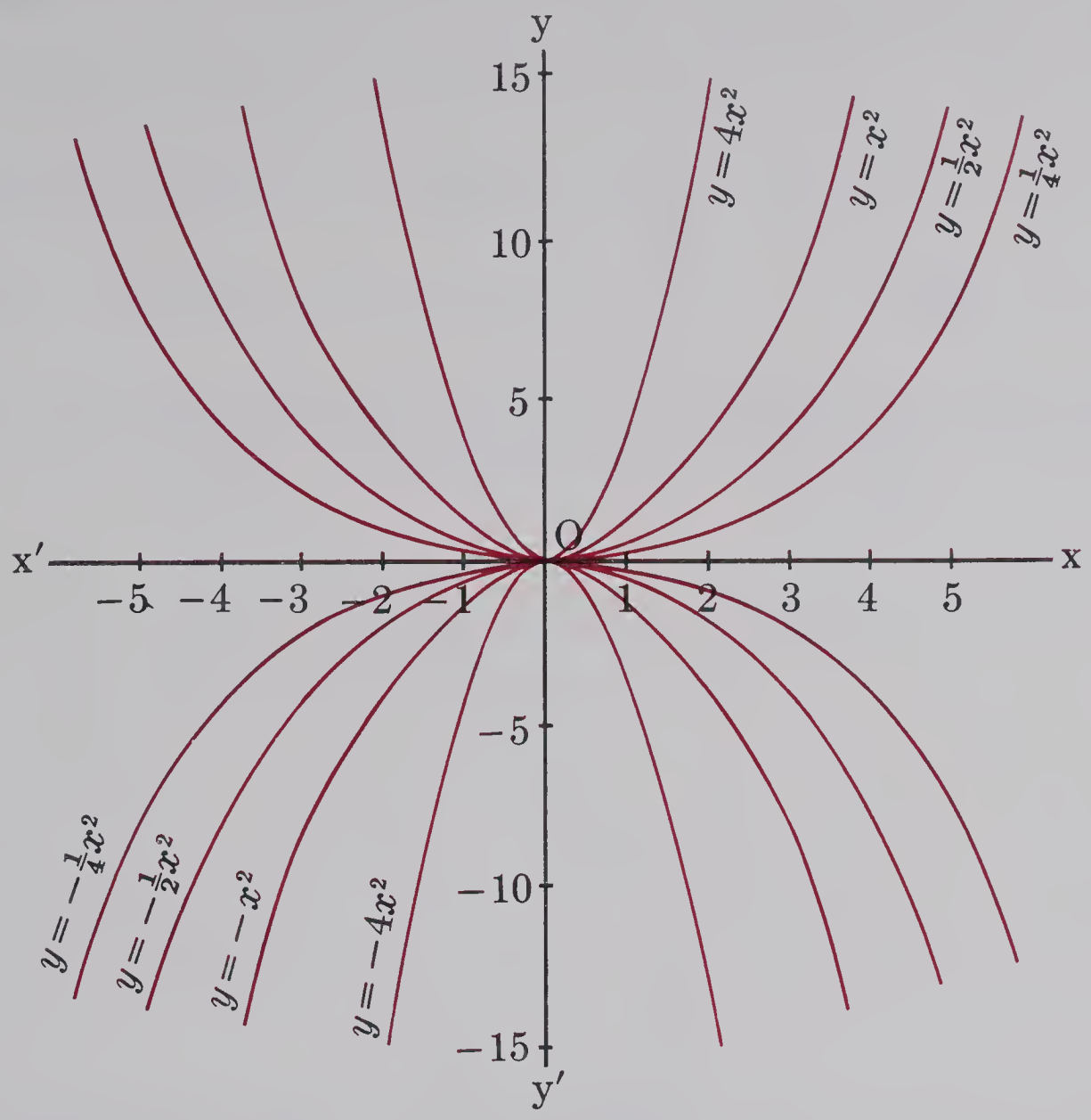


Figure 6-3 $y = ax^2$

Summary

The graph of the function defined by $y = ax^2$ $(a \neq 0)$ is a parabola:

(i) Its axis of symmetry is the y-axis, that is, the line whose equation is $x = 0$.

(ii) Its vertex is the origin, O, the point (0, 0).

(iii) If $a > 0$, the parabola is concave upward.

(iv) If $a < 0$, the parabola is concave downward.

(v) If $|a| < 1$, the parabola is "flatter" than the parabola whose equation is $y = x^2$.

(vi) If $|a| > 1$, the parabola is "sharper" than the parabola whose equation is $y = x^2$.

EXERCISE 6—1

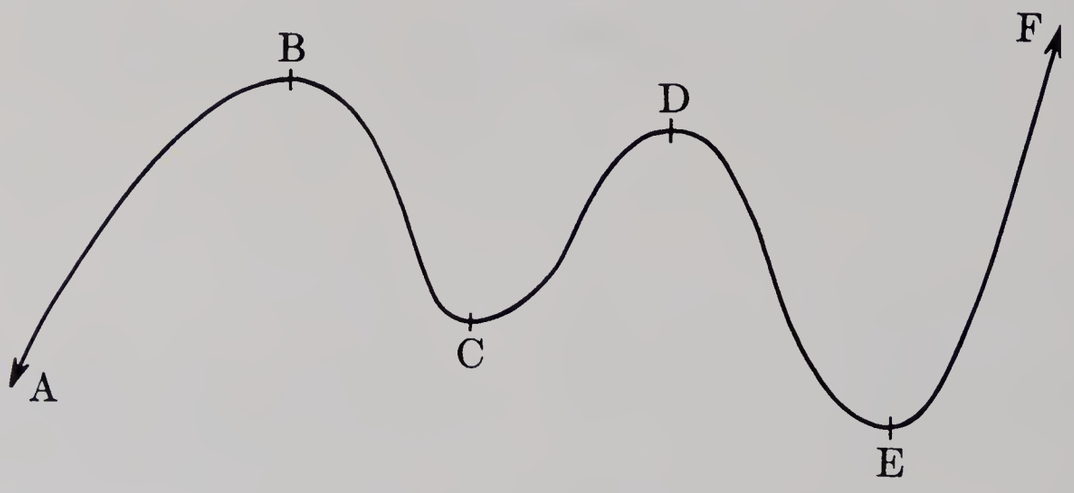
1. State orally whether the parabolas with the following equations are concave upward or concave downward and whether they are sharper or flatter than the parabola $y=x^2$.

   (a) $y=\frac{1}{9}x^2$ (b) $y=-9x^2$ (c) $y=64x^2$

   (d) $y=-0.0003x^2$ (e) $y=-\dfrac{1}{987}x^2$ (f) $y=(14)^3x^2$

   (g) $y=\dfrac{-1}{\sqrt{6295}}x^2$ (h) $y=\pi x^2$ (i) $y=-10\sqrt{5}x^2$

2. State where the following curve is concave upward and where concave downward. Name any maximum and minimum points on the curve.



3. Draw rough sketches to illustrate the relative shape and position of the pairs of parabolas with the following equations.

   (a) $y=5x^2$, $y=x^2$ (b) $y=-10x^2$, $y=-x^2$

   (c) $y=\frac{1}{3}x^2$, $y=x^2$ (d) $y=-\frac{1}{10}x^2$, $y=-x^2$

   (e) $y+\frac{1}{7}x^2=0$, $y+x^2=0$ (f) $y-\frac{1}{10}x^2=0$, $y-x^2=0$

   (g) $y=\pi x^2$, $y=x^2$ (h) $y=-10\sqrt{5}x^2$, $y=-x^2$.

4. Prove algebraically, as in the Example (page 125), that the y-axis is an axis of symmetry of the parabola whose equation is $y=ax^2$.

5. Show that the graph of the function f is symmetric with respect to the y-axis if $f(x)=f(-x)$. Apply this criterion in (4).

6. Why is the linear equation $Ax+By+C=0$ more general than the linear equation $y=mx+b$ (all coefficients are real numbers)?

7. Why are lines parallel to the y-axis not graphs of functions?

6 3 The Families of Parabolas with Equations $y=ax^2+d$ and $y=a(x-m)^2$

Example 1

Draw the curves with equations

(i) $y=3x^2+2$ (ii) $y=3x^2-1$.

Solution: (i) The equation $y=3x^2+2$ may be written

$$y-2=3x^2.$$

This is in the form

$$Y=3X^2$$

where $Y \equiv y-2$ (read "Y is $y-2$") and $X \equiv x$. But this is the equation of a parabola that is concave upward (why?), symmetric about the line $X=0$ (i.e., $x=0$, the y-axis), and that has its vertex at the point $X=0$, $Y=0$ (i.e., the point $x=0$, $y=2$). The parabola is shown in Figure 6-4 along with the parabola whose equation is $y=3x^2$.

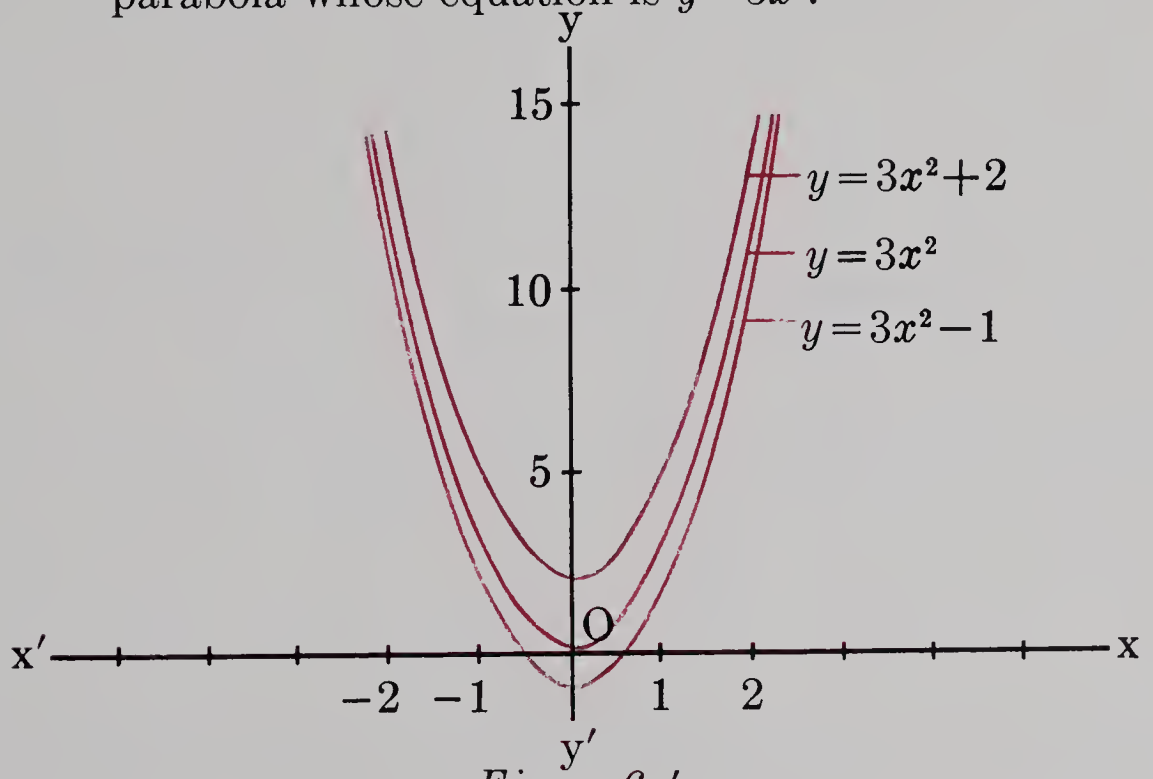


Figure 6-4

(ii) The equation $y=3x^2-1$ may be written

$$y+1=3x^2.$$

This is in the form

$$Y=3X^2$$

where $Y \equiv y+1$ and $X \equiv x$. As in (i), this is the equation of a parabola that is concave upward, symmetric about the line $X=0$ or $x=0$, that is, the y-axis, and that has its vertex at the point $X=0$, $Y=0$ (i.e., the point $x=0$, $y=-1$). The parabola is shown in Figure 6-4.

Example 2

Sketch the curves whose equations are

(i) $y = -3(x-2)^2$ (ii) $y = -3(x+2)^2$.

Solution: (i) The equation $y = -3(x-2)^2$ is in the form

$$Y = -3X^2$$

where $Y \equiv y$ and $X \equiv x-2$. But this is the equation of a parabola that is concave downward, symmetric about the line $X = 0$ (i.e., the line $x-2=0$), and that has its vertex at the point $X = 0$, $Y = 0$ (i.e., the point $x = 2$, $y = 0$). The parabola is shown in Figure 6-5, along with the parabola whose equation is $y = -3x^2$.

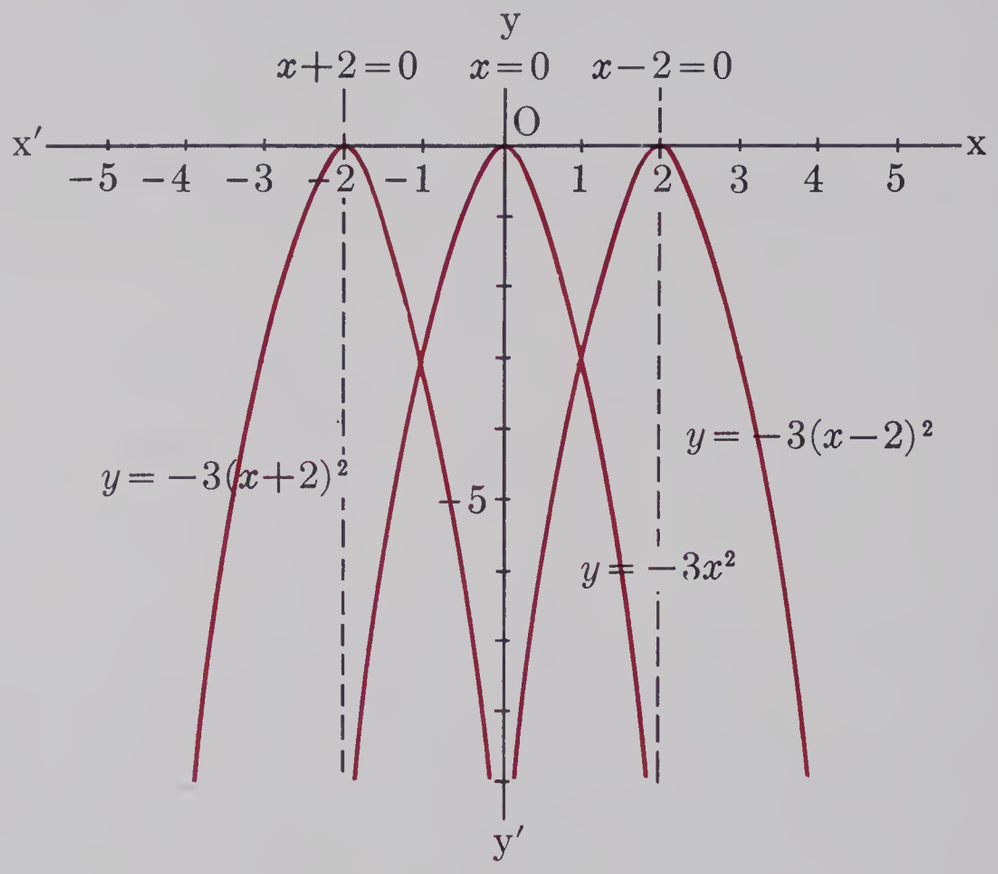


Figure 6-5

(ii) The equation $y = -3(x+2)^2$ is in the form

$$Y = -3X^2$$

where $Y \equiv y$ and $X \equiv x+2$. As above, this is the equation of a parabola that is concave downward, symmetric about the line $X = 0$ (i.e., the line $x+2=0$), and that has its vertex at the point $X = 0$, $Y = 0$ (i.e., the point $x = -2$, $y = 0$). The parabola is shown in Figure 6-5.

Summary

The equation $y=ax^2+d$ or $y-d=ax^2$, by comparison with the equation $Y=aX^2$ (where $Y\equiv y-d$ and $X\equiv x$), is the equation of a parabola whose vertex is the point $(0, d)$ and whose axis of symmetry is the line $x=0$ (the y-axis); the parabola is concave upward if $a>0$ and concave downward if $a<0$.

The equation $y=a(x-m)^2$, by comparison with the equation $Y=aX^2$ (where $Y\equiv y$ and $X\equiv x-m$), is the equation of a parabola whose vertex is the point $(m, 0)$ and whose axis of symmetry is the line $x=m$; the parabola is concave upward if $a>0$ and concave downward if $a<0$.

EXERCISE 6—2

(A) Sketch the sets of parabolas (each set on a single diagram) whose equations are given below, showing clearly the vertex and axis of symmetry of each parabola. Draw the first parabola of each set carefully (use a table of values if necessary), and use it to sketch the others.

1. $y=x^2$, $y=x^2+5$, $y=x^2-3$, $y=x^2+\frac{3}{2}$
2. $y=-2x^2$, $y=-2x^2+3$, $y=-2x^2-\frac{3}{2}$, $y=-2x^2-5$
3. $y=4x^2$, $y=4x^2+7$, $y=4x^2+\frac{1}{4}$, $y=4x^2-5$
4. $y=-\frac{1}{2}x^2$, $y=-\frac{1}{2}x^2+4$, $y=-\frac{1}{2}x^2-2$, $y=-\frac{1}{2}x^2-\frac{3}{2}$

(B) Sketch the parabolas whose equations are given below, marking clearly the vertex and axis of symmetry of each (do not use a table of values):

5. $y=2x^2-4$
6. $y=-3x^2+5$
7. $y=\frac{1}{2}x^2+4$
8. $y=-\frac{3}{4}x^2-\frac{1}{4}$
9. $y=3x^2+\frac{1}{2}$
10. $y=-x^2+\frac{3}{2}$

Apply the instructions of (A) above to the following sets of parabolas:

11. $y=x^2$, $y=(x+2)^2$, $y=(x-4)^2$, $y=(x-\frac{3}{2})^2$
12. $y=-3x^2$, $y=-3(x+1)^2$, $y=-3(x-3)^2$, $y=-3(x+\frac{4}{3})^2$
13. $y=6x^2$, $y=6(x+3)^2$, $y=6(x-3)^2$, $y=6(x+\frac{5}{2})^2$
14. $y=-\frac{1}{2}x^2$, $y=-\frac{1}{2}(x+4)^2$, $y=-\frac{1}{2}(x-5)^2$, $y=-\frac{1}{2}(x-\frac{5}{3})^2$

Apply the instructions of (B) above to the following parabolas:

15. $y=(x-5)^2$
16. $y=-(x+3)^2$
17. $y=\frac{1}{2}(x-5)^2$
18. $y=-\frac{3}{2}(x+2)^2$
19. $y=4(x+\frac{1}{2})^2$
20. $y=-8(x-\frac{3}{2})^2$

6 4 The Family of Parabolas with Equations $y=a(x-m)^2+d$

Example

Sketch the curve whose equation is

$$y=-2(x-3)^2+5.$$

Solution: The equation $y=-2(x-3)^2+5$ may be written

$$y-5=-2(x-3)^2.$$

This is in the form

$$Y=-2X^2$$

where $Y\equiv y-5$ and $X\equiv x-3$. But this is the equation of a parabola that is concave downward, symmetric about the line $X=0$ (i.e., the line $x-3=0$), and that has its vertex at the point $X=0$, $Y=0$ (i.e., the point $x=3$, $y=5$). The parabola is shown in Figure 6-6 along with the parabola whose equation is $y=-2x^2$.

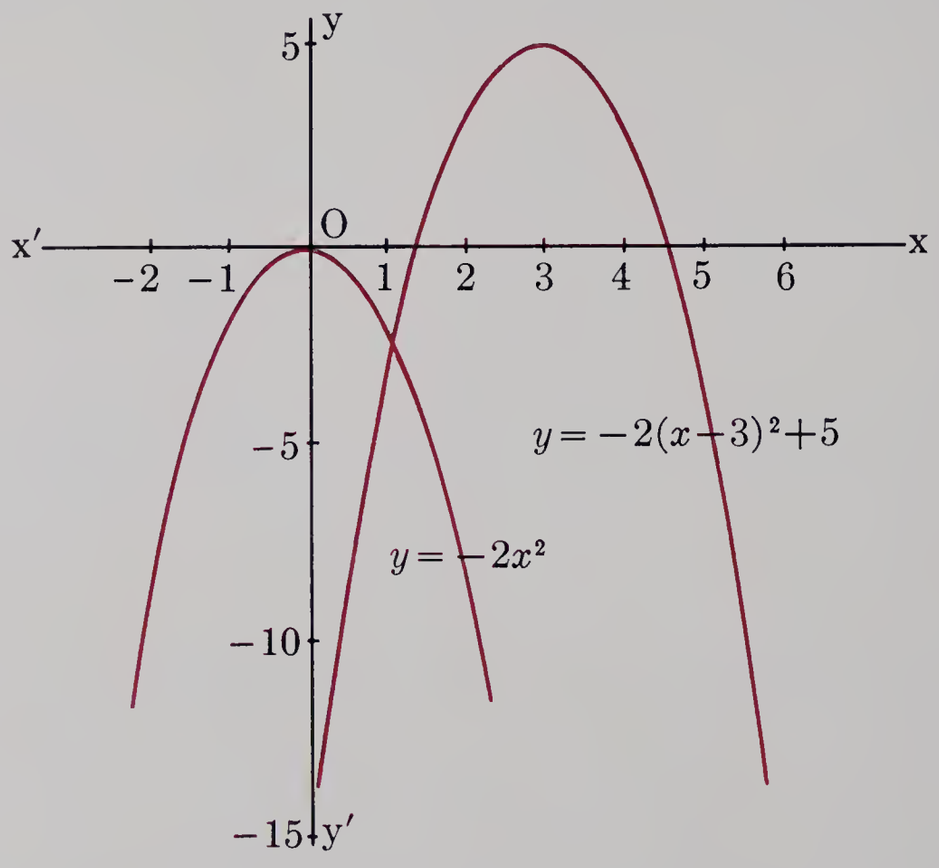


Figure 6-6

Summary

The equation $y=a(x-m)^2+d$ or $y-d=a(x-m)^2$, by comparison with the equation $Y\equiv aX^2$ (where $Y\equiv y-d$ and $X\equiv x-m$), is the equation of a parabola whose vertex is the point (m, d) and whose axis of symmetry is the line $x=m$; the parabola is concave upward if $a>0$ and concave downward if $a<0$.

EXERCISE 6—3

Sketch the sets of parabolas (each set on a single diagram) whose equations are given below, showing clearly the vertex and axis of each parabola. Draw the first parabola of each set carefully (use a table of values if necessary), and use it to sketch the other.

1.	$y=x^2$,	$y=(x-2)^2+3$,	$y=(x-2)^2-4$,	$y=(x-2)^2+\frac{5}{2}$,
2.	$y=x^2$	$y=(x+3)^2-2$,	$y=(x+3)^2+4$,	$y=(x+3)^2-\frac{7}{4}$
3.	$y=-3x^2$,	$y=-3(x-1)^2+1$,	$y=-3(x-1)^2-4$,	$y=-3(x-1)^2+\frac{7}{2}$,
4.	$y-3x^2$,	$y=-3(x+\frac{3}{2})^2+4$,	$y=-3(x+\frac{3}{2})^2-2$,	$y=-3(x+\frac{3}{2})^2-\frac{5}{2}$
5.	$y=6x^2$,	$y=6(x+2)^2+2$,	$y=6(x+2)^2-2$,	$y=6(x+2)^2-\frac{3}{2}$,
6.	$y=6x^2$,	$y=6(x-\frac{3}{2})^2+\frac{1}{2}$,	$y=6(x-\frac{3}{2})^2+6$,	$y=6(x-\frac{3}{2})^2-\frac{3}{2}$
7.	$y=-\frac{1}{2}x^2$,	$y=-\frac{1}{2}(x-2)^2+3$,	$y=-\frac{1}{2}(x-2)^2-1$,	$y=-\frac{1}{2}(x-2)^2-\frac{3}{2}$,
8.	$y=-\frac{1}{2}x^2$,	$y=-\frac{1}{2}(x+\frac{5}{2})^2+\frac{1}{2}$,	$y=-\frac{1}{2}(x+\frac{5}{2})^2-\frac{3}{2}$,	$y=-\frac{1}{2}(x+\frac{5}{2})^2+6$
9.	$y=\frac{4}{3}x^2$,	$y=\frac{4}{3}(x+3)^2-2$,	$y=\frac{4}{3}(x-2)^2+3$,	$y=\frac{4}{3}(x+2)^2-3$,
10.	$y=\frac{4}{3}x^2$,	$y=\frac{4}{3}(x-3)^2+2$,	$y=\frac{4}{3}(x-\pi)^2-\frac{1}{2}$,	$y=\frac{4}{3}(x-\pi)^2-\pi$

State the parabolas, with equations of the form $y=ax^2$, which you would use as an aid in sketching the following parabolas, and then sketch the pairs of parabolas.

11. $y=(x+2)^2-5$	12. $y=-4(x-3)^2$	13. $2(x+3)^2-2+5y=0$
14. $2y=(x-\frac{1}{2})^2+3$	15. $3y+(x+2)^2-2=0$	16. $6y-4(x+k)^2+3=0$

6 5 $y=ax^2+bx+c$ Completing the Square

We notice that
$$x^2+4x+5=x^2+4x+4+1$$
$$=(x+2)^2+1,$$
so that the equation
$$y=x^2+4x+5$$
may be written in the form
$$y-1=(x+2)^2,$$
that is, $Y=X^2$, with $Y\equiv y-1$ and $X\equiv x+2$. This is the equation of a parabola that is concave upward and that has vertex $(-2, 1)$ and axis of symmetry the line $x=-2$. It is then a simple matter to sketch the graph of the equation $y=x^2+4x+5$.

It happens to be true that ax^2+bx+c for $a\neq 0$ can always be written in the form $a(x-m)^2+d$ (the process is called completing the square); therefore the graph of the function defined by $y=ax^2+bx+c$ is a parabola and can be sketched immediately.

THEOREM: $ax^2+bx+c=a(x-m)^2+d$

Proof:
$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \text{ since } a\neq 0. \qquad \text{(i)}$$

Now let us suppose that x^2 and $\frac{b}{a}x$ are the first two terms in the expansion of a perfect square; that is, for some h,

$$(x+h)^2=x^2+2hx+h^2=x^2+\frac{b}{a}x+\square.$$

Then $2h=\frac{b}{a}$ or $h=\frac{b}{2a}$; that is, h is one half the coefficient of x in the form $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, and so

$$(x+h)^2=x^2+2hx+h^2=x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2.$$

Thus we may complete the square by adding and subtracting the term $\left(\frac{b}{2a}\right)^2$ in the right-hand side of equation (i).

We have
$$\begin{aligned} ax^2+bx+c &= a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right] \\ &= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] \\ &= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \\ &= a(x-m)^2+d \end{aligned}$$

where $m=\frac{-b}{2a}$ and $d=\frac{4ac-b^2}{4a}$.

The student should note carefully that we have described the procedure for completing the square; we have not just produced a formula for memorization.

Steps in completing the square:

(1) Write $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$.

(2) Add and subtract the square of half the *new* coefficient of x:
$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left[x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right].$$

(3) Express the result in the required form $a(x-m)^2+d$:
$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

Example 1

Express the following in the form $a(x-m)^2+d$ by completing the square.

(i) x^2+4x+5 (ii) $3x^2-2x+7$

Solution: (i) $x^2+4x+5=x^2+4x+(2)^2-(2)^2+5$
$=(x+2)^2+1.$

(ii) $3x^2-12x+7=3(x^2-4x+\frac{7}{3})$
$=3[x^2-4x+(-2)^2-(-2)^2+\frac{7}{3}]$
$=3[(x-2)^2-\frac{5}{3}]$
$=3(x-2)^2-5.$

Example 2

Sketch the graph of the parabola whose equation is $y=-2x^2+5x-4$.

Solution: $-2x^2+5x-4=-2(x^2-\frac{5}{2}x+2)$
$=-2[x^2-\frac{5}{2}x+(-\frac{5}{4})^2-\frac{25}{16}+2]$
$=-2[(x-\frac{5}{4})^2+\frac{7}{16}]$
$=-2(x-\frac{5}{4})^2-\frac{7}{8}$

Then $y=-2x^2+5x-4$ can be written in the form
$y=-2(x-\frac{5}{4})^2-\frac{7}{8}$, or $y+\frac{7}{8}=-2(x-\frac{5}{4})^2$;

this is the equation of a parabola that is concave downward, with vertex $(\frac{5}{4},-\frac{7}{8})$ and with axis of symmetry $x=\frac{5}{4}$ (Figure 6-7). Note that the range of the function is $\{y|y\leq-\frac{7}{8}\}$.

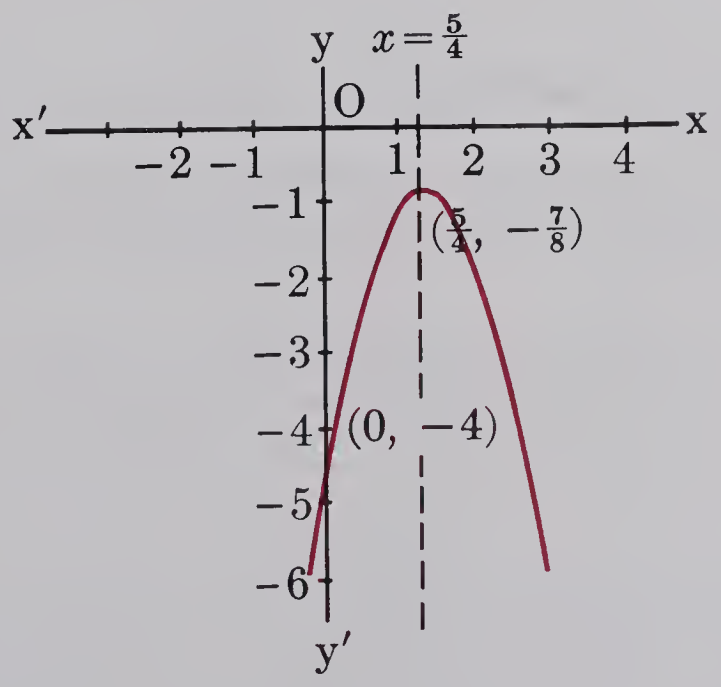


Figure 6-7

The fact that ax^2+bx+c can be written in the form $a(x-m)^2+d$ shows that the quadratic function defined by $y=ax^2+bx+c$ over the domain of real numbers has range $\{y|y\geq d\}$ if $a>0$ (since then (m, d) is a minimum point on the parabola) and range $\{y|y\leq d\}$ if $a<0$ (since then (m, d) is a maximum point on the parabola).

In sketching the parabola whose equation is $y=a(x-m)^2+d$, it is always useful to plot the point $(0, am^2+d)$ where the parabola intersects the y-axis (e.g., the point $(0, -4)$ in Figure 6-7). This helps to tie down the location of the parabola.

EXERCISE 6—4

Complete the square to write the following expressions ax^2+bx+c in the form $a(x-m)^2+d$. Use the result to sketch the graph of the function defined by $y=ax^2+bx+c$, stating the range of the function in each case. Mark clearly the point where the parabola intersects the y-axis.

1. x^2+3x	2. x^2-5x	3. x^2+3x-4
4. x^2-5x+7	5. $-x^2+4x-2$	6. $-x^2-3x+5$
7. $x^2-\frac{1}{2}x+3$	8. $-x^2-\frac{1}{3}x+4$	9. $-x^2-6x-7$
10. $2x^2+6x+3$	11. $5x^2+10x+3$	12. $4x^2+17x+2$
13. $6x^2-5x+2$	14. $-3x^2-6x+5$	15. $-2x^2+3x-2$
16. $4x^2-\frac{3}{2}x-\frac{1}{2}$	17. $-6x^2-12x-5$	18. $-\frac{1}{2}x^2-\frac{3}{2}x+2$

6.6 The Regions Defined by Quadratic Inequalities

We have already seen that a line divides the plane into three regions: the set of points on the line itself and the sets of points in the two open half planes.

We have been considering parabolas that are either concave upward or concave downward. Such parabolas also divide the plane into three regions: the set of points on the parabola and the sets of points above and below the parabola.

The regions of a plane determined by the line whose equation is $y=mx+b$ correspond to the expressions

(1) $y<mx+b$ (2) $y=mx+b$ (3) $y>mx+b$.

Which open half plane corresponds to which inequality is determined by examining the values of y and $mx+b$ for the coördinates of any one point not on the line.

Similarly, the parabola whose equation is $y=ax^2+bx+c$ divides the plane into three sets of points

(1) $\{(x, y)|y>ax^2+bx+c\}$ (the points above the parabola)

(2) $\{(x, y)|y=ax^2+bx+c\}$ (the points on the parabola)

(3) $\{(x, y)|y<ax^2+bx+c\}$ (the points below the parabola)

Again, the inequality associated with a particular region can be determined by examining the coördinates of a single point not on the parabola.

Example

Sketch the regions determined by the inequalities

(i) $y>3x^2-6x+2$, (ii) $y\geq -2x^2+6x-3$.

Solution: (i) $y=3x^2-6x+2$
$=3(x-1)^2-1$

Now $y+1=3(x-1)^2$ is the equation of a parabola that is concave upward, that has vertex $(1, -1)$, axis of symmetry $x=1$ (Figure 6-8), and point of intersection $(0, 2)$ with the y-axis.

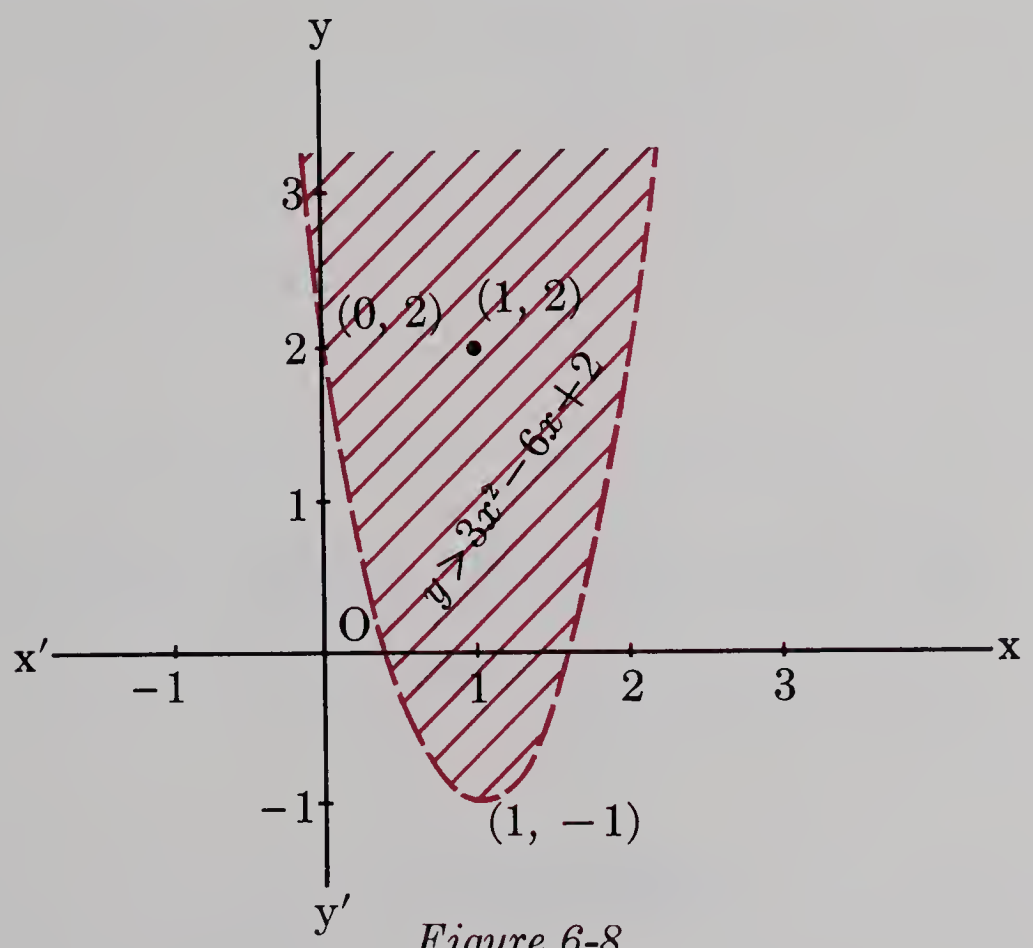


Figure 6-8

Now let us choose a point "inside" the parabola, e.g., (1, 2). Do the coördinates of this point satisfy $y>3x^2-6x+2$ or $y<3x^2-6x+2$? Obviously $3-6+2=-1$ and $2>-1$; therefore the region containing this point corresponds to the inequality $y>3x^2-6x+2$. The curve is drawn in Figure 6-8 with a broken line to indicate that its points are not part of the region.

(ii) $y = -2x^2+6x-3$
$= -2(x-\frac{3}{2})^2+\frac{3}{2}$.

Then $y-\frac{3}{2} = -2(x-\frac{3}{2})^2$ is the equation of a parabola that is concave downward, that has vertex $(\frac{3}{2}, \frac{3}{2})$, axis of symmetry $x=\frac{3}{2}$ (Figure 6-9), and point of intersection $(0, -3)$ with the y-axis.

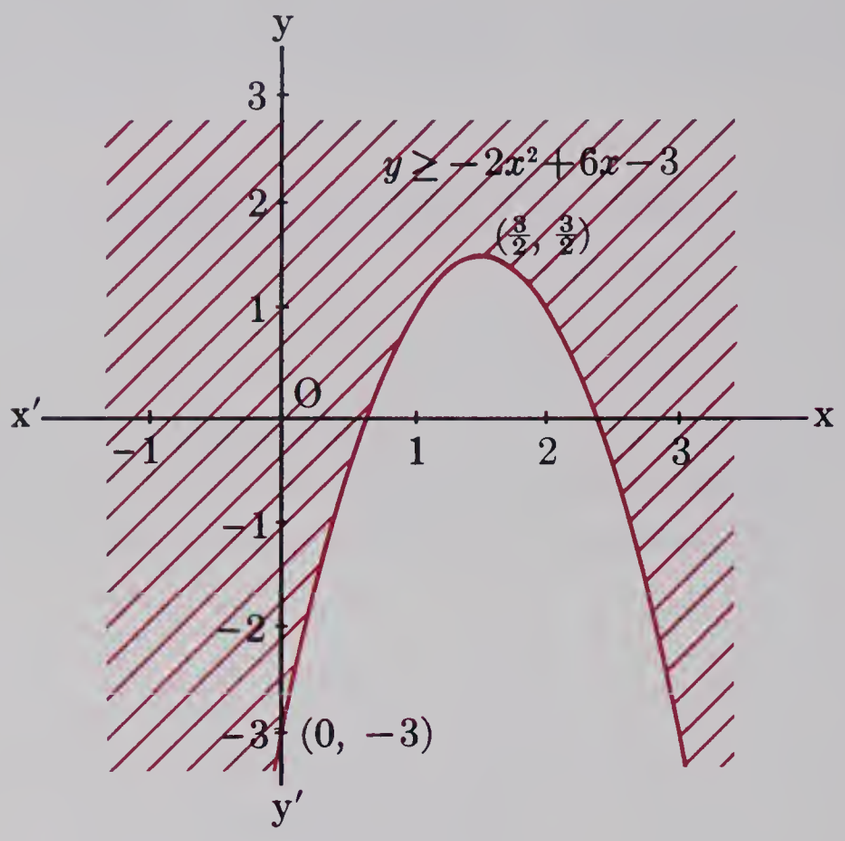


Figure 6-9

We observe that the coördinates of the origin satisfy the inequality $0 > -2(0)+6(0)-3$; therefore the points "outside" or above the parabola, together with the points on the curve, correspond to the inequality $y \geq -2x^2+6x-3$. The curve is drawn using a solid line to indicate that its points *are* points of the region.

EXERCISE 6—5

Sketch the regions of the plane determined by the following inequalities and describe them in words:

1. $y < x^2$
2. $y \geq -x^2-2$
3. $y > 6x^2-3x$
4. $y \leq x^2-2x+5$
5. $y > 3x^2-6x+1$
6. $y \leq -2x^2+4x-3$
7. $y > -\frac{1}{2}x^2-3x-2$
8. $y+2 \geq x^2-\frac{1}{2}x$
9. $y-6 > 3x^2+2x$

Sketch, if possible, the regions determined by the following pairs of inequalities:

10. $y>x^2$, $y<-x^2+1$
11. $y<-2x^2$, $y>3x^2-2$
12. $y>x^2-2x-4$, $y<-3x^2+x+3$
13. $y<x^2$, $y>-x^2$
14. $y<3x^2-6x+4$, $y>-2x^2+x-3$
15. $y\leq x^2$, $y\geq -x^2$
16. $y>x^2$, $y<-x^2$
17. $y<3x^2-6x+4$, $y<-2x^2+x-3$

6 7 Maximum and Minimum Values of the Quadratic Function

When $f(x)=ax^2+bx+c$ is written in the form $f(x)=a(x-m)^2+d$, it is easy to see that the function has a minimum value d at $x=m$ if $a>0$ (for then the graph of the function is a parabola concave upward with minimum point at the vertex (m, d)) or a maximum value d when $x=m$ if $a<0$ (for then the graph is a parabola concave downward with maximum point at the vertex (m, d)). Algebraically,

$$a(x-m)^2+d\geq d \text{ if } a>0 \text{ and } a(x-m)^2+d\leq d \text{ if } a<0.$$

It is easy to remember that the quadratic function whose value is $f(x)=ax^2+bx+c$ has a maximum or minimum value for $x=m=-\frac{b}{2a}$; the maximum or minimum value can be found by evaluating $f(x)=ax^2+bx+c$ for $x=-\frac{b}{2a}$.

Example 1

Find the maximum or minimum value of $f(x)=3x^2-6x+1$.

Solution:

$$\begin{aligned} f(x) &= 3(x^2-2x+\tfrac{1}{3}) \\ &= 3(x^2-2x+1-1+\tfrac{1}{3}) \\ &= 3(x-1)^2-2 \\ &\geq -2 \text{ since } 3(x-1)^2\geq 0. \end{aligned}$$

The function has a minimum value of -2 when $x=1$ and no maximum value.

Alternatively, the function has a maximum or minimum value when $x=-\frac{b}{2a}$. Here, $a=3$ (hence the function has a minimum) and $b=-6$, so that a minimum value occurs when $x=\frac{-(-6)}{6}=1$. The minimum value is $f(1)=3(1)^2-6(1)+1=-2$.

Example 2

A farmer wishes to enclose as large a rectangular lot as possible with 400 ft. of fencing. What must be the dimensions of the lot?

Solution: The perimeter of the lot is 400 ft., and its area is to be a maximum.
Let the width be x feet;
then the length is $200-x$ feet, $0<x<200$.
Let y (square feet) be the area;
then $y=x(200-x)=-x^2+200x$.
This quadratic function, $f(x)=-x^2+200x$, has a maximum value (since $a=-1$) when

$$x=\frac{-b}{2a}=\frac{-200}{2(-1)}=100.$$

The lot will have a maximum area when the width is 100 ft. and the length is $200-100$ or 100 ft.
For maximum area, the lot must be square.

EXERCISE 6—6

Find the maximum or minimum values (specify which) for the functions that have the following values $f(x)$:

1. x^2+8x+3
2. $-x^2+6x-5$
3. $x^2-3x-10$
4. $2x^2+6x-5$
5. $3x^2-9x+2$
6. $-6x^2+3x-2$
7. $-\frac{1}{2}x^2+2x-3$
8. $-6x^2-3x+2$
9. $-x^2-x+6$
10. $3x^2-4x$
11. $6x-5x^2$
12. $4x^2+x-6$
13. $3-2x^2+4x$
14. $-16x+1+8x^2$
15. $4-5x-10x^2$
16. Find the dimensions of the rectangular lot having maximum area that can be enclosed by a fence of length (a) 300 ft (b) 600 feet.
17. The sum of two real numbers is 20. Find their value if their product is a maximum.
18. Find two nonzero real numbers whose difference is 4 and whose product is a minimum.
19. The height, y, in feet, at the end of t seconds, of a ball thrown upward with an initial velocity of 96 ft. per sec., is given by the formula

$$y=96t-16t^2.$$

Find (a) when it reaches its highest point,
(b) the maximum height it reaches,
(c) when the ball reaches the ground.

20. A farmer wishes to enclose a rectangular field that is already bounded on one side by a river (this side needs no fence). What are the dimensions of the field of maximum area that can be enclosed with 300 ft. of fence?

21. A rain gutter, open at the top and rectangular in shape, is to be made from roofing tin that is 12 inches wide. How should the tin be bent to produce a gutter with greatest carrying capacity?

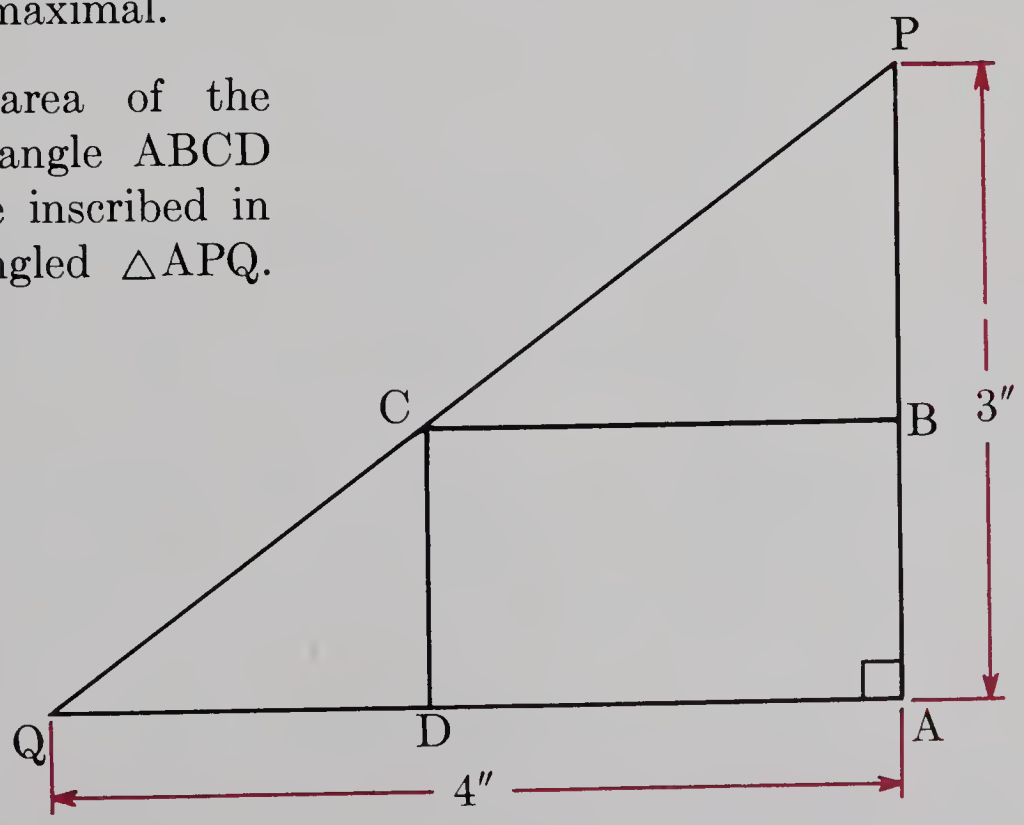
22. A bullet is shot directly upward with a velocity of 1600 ft. per sec. Its distance, y, above ground at the end of t seconds is given by the formula $y=1600t-16t^2$. Find how high the bullet goes and when it reaches its maximum height.

23. In $\triangle ABC$, the sum of the lengths of the side AB and the altitude from C to AB is 10 inches. Find these lengths so that the area of $\triangle ABC$ is maximal.

24. Every week that a farmer waits, his potato crop increases 100 bushels, but the price drops 10¢ a bushel. If he digs his potatoes today, he would have 600 bushels worth \$1 per bushel. Show that he should dig and sell his potatoes at the end of two weeks.

25. A circular sector of radius x inches has a perimeter of 12 inches. The area of the sector is $6x-x^2$ square inches. Find the radius, x, for which the area is maximal.

26. Find the area of the largest rectangle ABCD that can be inscribed in the right-angled $\triangle APQ$.



27. The height, y, in feet, at the end of t seconds, of an object thrown upward with an initial velocity of u ft. per sec., is $y=ut-16t^2$. How high does the object rise and at what time does it reach this maximum height?

6 8 Curves Whose Equations Are $(y-k)^2=a(x-h)$; $y=k\pm\sqrt{ax-b}$ (Supplementary)

Example 1

Sketch the curve whose equation is $(y+1)^2=4(x-3)$.

Solution: We may write $(y+1)^2=4(x-3)$ in the form $X^2=4Y$ where $X\equiv y+1$ and $Y\equiv x-3$. But this is the equation of a parabola with vertex $X=0$, $Y=0$ (that is, $y=-1$ and $x=3$), axis of symmetry the line whose equation is $X=0$ ($y=-1$), and opening along the positive direction of this axis of symmetry.

We conclude that $(y+1)^2=4(x-3)$ is the equation of a parabola with vertex (3, 1), axis of symmetry the line whose equation is $y=-1$, and opening in the positive direction along this line (Figure 6-10). The point $(\frac{13}{4}, 0)$ where the curve crosses the x-axis, found by substituting 0 for y in the equation and solving for x, is useful in sketching the curve.

Notice that, since $(y+1)^2$ is always nonnegative, x must be no less than 3, so that the curve is entirely to the right of the line whose equation is $x=3$ (except for the point $(3, -1)$).

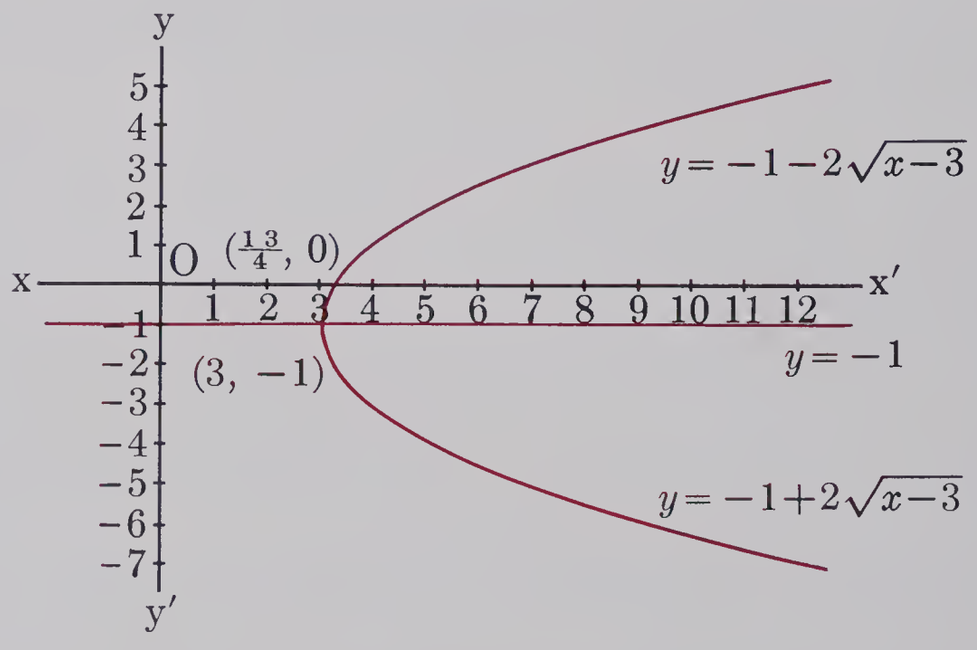


Figure 6-10

Just as in Example 1, we may show that $(y-k)^2=a(x-h)$ is the equation of a parabola with vertex (h, k), axis of symmetry the line whose equation is $y=k$, and the opening in the positive direction of this line if $a>0$ and in the negative direction if $a<0$.

Example 2

Sketch the curve whose equation is $(y-2)^2=-3(x+4)$.

Solution: This is a parabola with vertex $(-4, 2)$, axis of symmetry the line whose equation is $y=2$, and opening in the negative direction along this line (Figure 6-11).

Notice that, since $(y-2)^2$ is always nonnegative, $x+4\leq 0$, that is, $x\leq -4$. The curve crosses the x-axis at the point $(-\frac{16}{3}, 0)$.

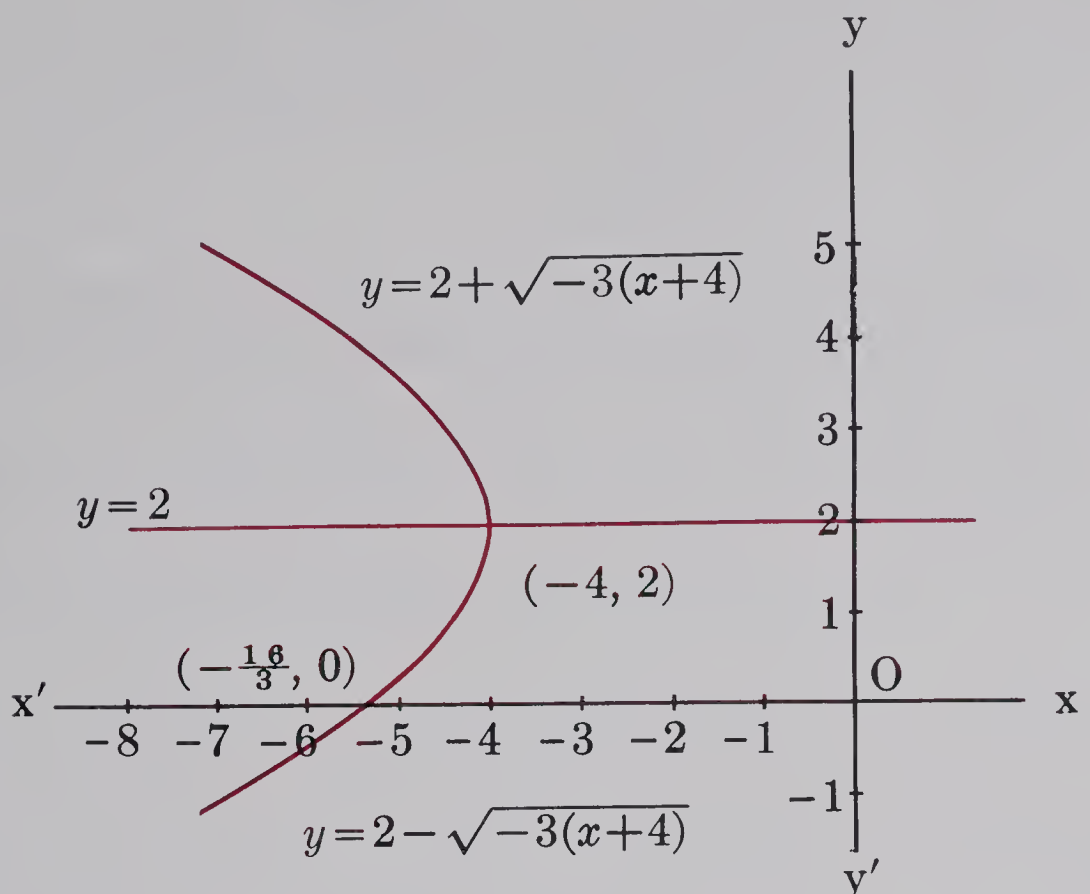


Figure 6-11

Of course the relations in Examples 1 and 2 *do not* define functions since, for $x\geq 3$ in Example 1 and for $x\leq -4$ in Example 2, there are two values of y corresponding to each value of x. However, suppose that we consider the upper and lower branches of the parabola in Figure 6-10 separately. For points on the *upper* branch, y has the value -1 *plus* a positive quantity; for points on the *lower* branch, y has the value -1 *minus* a positive quantity. If

$$(y+1)^2=4(x-3),$$

then $y+1=2\sqrt{x-3}$ or $y+1=-2\sqrt{x-3}$,

that is $y=-1+2\sqrt{x-3}$ or $y=-1-2\sqrt{x-3}$.

Thus $y=-1+2\sqrt{x-3}$ is the equation of the upper branch,

and $y=-1-2\sqrt{x-3}$ is the equation of the lower branch.

These two relations *do* define functions in the domain $x\geq 3$.

Similarly, in Example 2,

$$(y-2)^2=-3(x+4),$$

and therefore $y=2+\sqrt{-3(x+4)}$ or $y=2-\sqrt{-3(x+4)}$,

and these are the equations of the upper and lower branches respectively of the parabola. Both of these relations define functions ih the domain $x \leq -4$.

Example 3

Sketch the curve whose equation is $y=-1-\sqrt{2x+3}$.

Solution: This is the *lower* branch of the parabola whose equation is

$$(y+1)^2=2x+3$$

or $$(y+1)^2=2(x+\tfrac{3}{2}).$$

But this is the parabola with vertex $(-\frac{3}{2}, -1)$, axis of symmetry the line $y=-1$, and opening along the positive direction of this line. The parabola crosses the x-axis at the point $(-1, 0)$ (Figure 6-12). Notice that the curve is to the right of the line $x=-\frac{3}{2}$.

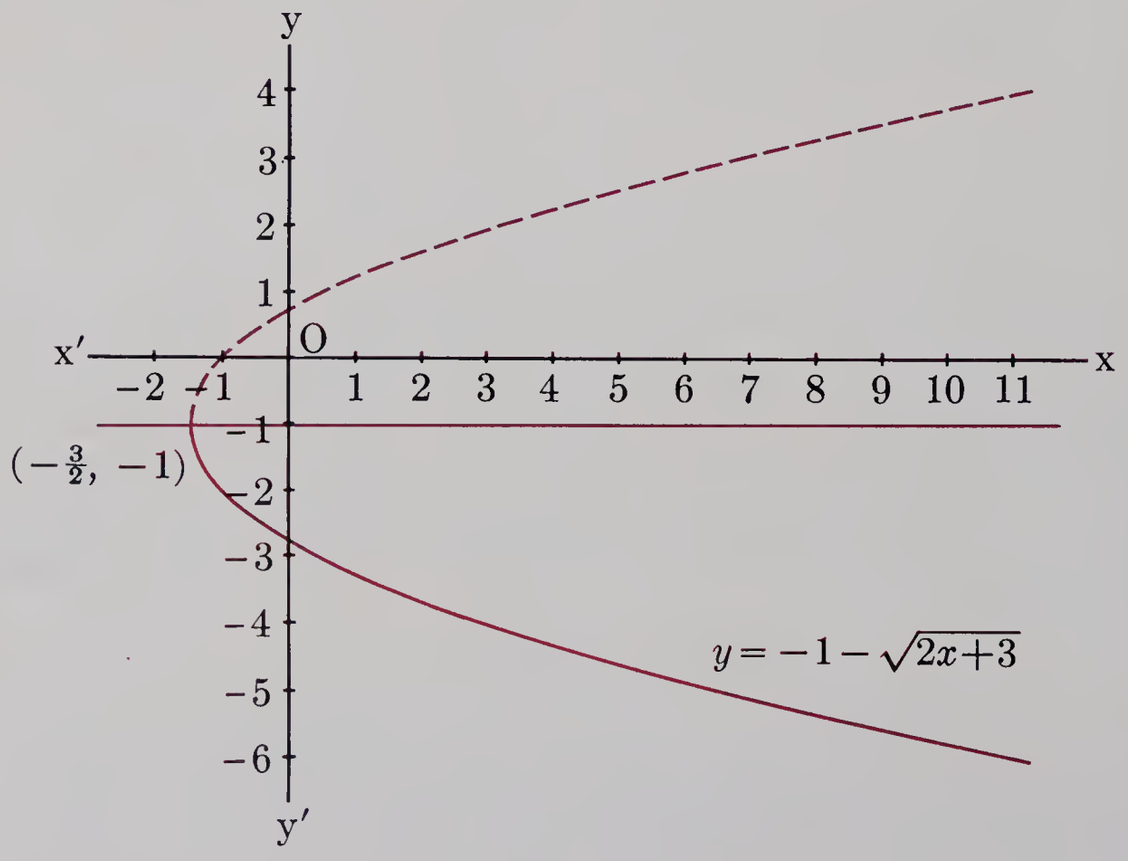


Figure 6-12

EXERCISE 6—7

Sketch the parabolas whose equations are given in (1)-(12), marking clearly the vertex and axis of symmetry in each case and stating any restrictions on x:

1. $y^2=9x$
2. $y^2=-16x$
3. $y^2=3(x+2)$
4. $y^2=2x+5$
5. $y^2=-3x+2$
6. $y^2=-2x+4$
7. $(y-2)^2=4x$
8. $(y+1)^2=-16x$
9. $(y+\frac{1}{2})^2=2(x+\frac{1}{2})$
10. $(y-4)^2=4x-3$
11. $(y+2)^2=-4+8x$
12. $(y-\frac{3}{2})^2=-3x+5$

Sketch the curves whose equations are given in (13)-(24), stating the restrictions on x in each case and the range of values for y. State the equation of the parabola of which each curve is a branch.

13. $y=\sqrt{x-2}$
14. $y=-\sqrt{x+3}$
15. $y=\sqrt{2-x}$
16. $y=-\sqrt{3-x}$
17. $y=\sqrt{2x+3}$
18. $y=\sqrt{3-2x}$
19. $y=2+\sqrt{3+x}$
20. $y=-1-\sqrt{x-2}$
21. $y=-3+\sqrt{3-x}$
22. $y=3-\sqrt{2x-1}$
23. $y-\sqrt{2-3x}=6$
24. $y+\sqrt{4x-1}-5=0$
25. Sketch the curves whose equations are $y=\sqrt{x-3}$ and $y=\sqrt{4-x}$ on the same graph. Estimate their point of intersection. This provides an approximation to the solution to the equation $\sqrt{x-3}=\sqrt{4-x}$.
26. Repeat (25) for the following pairs of curves. State the corresponding equation and an approximation to its solution.
    (a) $y=3+\sqrt{x+2}$, $y=1+\sqrt{2-x}$
    (b) $y=3-\sqrt{x+2}$, $y=1-\sqrt{2-x}$
    (c) $y=-2+\sqrt{x-1}$, $y=3-\sqrt{x-4}$

6 9 Chapter Summary

The graph of the function defined by

$$y=ax^2+bx+c,\ x, a, b, c\in \mathrm{Re},\ a\neq 0,$$

is a parabola; $y=ax^2+bx+c$ is the equation of the parabola, and the graph is the set of points $\{(x, y)|y=ax^2+bx+c; x, a, b, c\in \mathrm{Re}, a\neq 0\}$.

The parabola whose equation is $y=ax^2$

(i) is concave upward if $a>0$, concave downward if $a<0$;

(ii) has vertex at (0, 0) (this is a minimum point if $a>0$, a maximum point if $a<0$);

(iii) has axis of symmetry the y-axis; that is, the line $x=0$.

The parabola whose equation is	is of the form $Y=aX^2$ where	has vertex at	axis of symmetry	and is concave
$y=ax^2+d$ or $y-d=ax^2$	$Y\equiv y-d$ $X\equiv x$	$(0, d)$	$x=0$	upward if $a>0$ or downward if $a<0$
$y=a(x-m)^2$	$Y\equiv y$ $X\equiv x-m$	$(m, 0)$	$x=m$	
$y=a(x-m)^2+d$ or $y-d=a(x-m)^2$	$Y\equiv y-d$ $X\equiv x-m$	(m, d)	$x=m$	

In general, the point where the parabola intersects the y-axis is useful in sketching the parabola; if the equation of the parabola is $y=ax^2+bx+c$, then this point is $(0, c)$.

We may readily find the graph of the function which is defined by $y=ax^2+bx+c$ by completing the square and obtaining

$$y=f(x)=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

The domain of the quadratic function defined by $y=a(x-m)^2+d$ is the entire set Re; its range is $\{y|y\geq d\}$ if $a>0$ and $\{y|y\leq d\}$ if $a<0$.

The quadratic function defined by $y=f(x)=ax^2+bx+c$ has a minimum value for $x=-\frac{b}{2a}$ if $a>0$ and a maximum value for $x=-\frac{b}{2a}$ if $a<0$. The maximum or minimum value is $f\left(-\frac{b}{2a}\right)$.

REVIEW EXERCISE 6—8

1. Sketch the parabola whose equation is $y=\frac{1}{5}x^2$, naming the vertex and axis of symmetry (use a table of values if necessary).

2. Use the result in (1) to sketch the parabolas with the following equations, again naming the vertex and axis of symmetry of each and the point of intersection of each with the y-axis:

(a) $y=\frac{1}{5}x^2+2$ (b) $y=-\frac{1}{5}x^2$ (c) $y=\frac{1}{5}(x+3)^2$

(d) $y=\frac{1}{5}(x-1)^2+3$ (e) $y-1=-\frac{1}{5}(x+2)^2$ (f) $y+\frac{1}{2}=\frac{1}{5}(x-2)^2$

3. Sketch the parabola whose equation is $y=-2x^2$, and use this result to sketch the following parabolas. Name the vertex and axis of symmetry of each, state whether the parabola is concave upward or downward, and show the point of intersection of each parabola with the y-axis:

(a) $y=-2x^2-2$ (b) $y=-2(x+3)^2$

(c) $y=2x^2$ (d) $y=-2(x+2)^2-1$

(e) $y+3=-2(x-2)^2$ (f) $y-3=-2(x+1)^2+2$

In problems (4) to (12) complete the square to write the following quadratic expressions in the form $a(x-m)^2+d$:

4. x^2+7x 5. $x^2-\frac{1}{2}x$ 6. x^2+2x-4

7. x^2-5x-2 8. $-x^2-2x+3$ 9. $-2x^2+x+1$

10. $3x^2-\frac{1}{2}x+2$ 11. $3x^2-\frac{1}{3}x+\frac{2}{5}$ 12. $-\frac{1}{2}x^2-2x+\frac{1}{3}$

Use the results of problems (4) to (12) to sketch the graphs of the function defined by the following. In each case state the range of the function and whether the functions have a maximum or minimum value.

13. $y=x^2+7x$ 14. $y=x^2-\frac{1}{2}x$ 15. $y=x^2+2x-4$

16. $y=x^2-5x-2$ 17. $y=-x^2-2x+3$ 18. $y=-2x^2+x+1$

19. $y=3x^2-\frac{1}{2}x+2$ 20. $y=3x^2-\frac{1}{3}x+\frac{2}{5}$ 21. $y=-\frac{1}{2}x^2-2x+\frac{1}{3}$

Find the maximum or minimum values (specify which) of the functions for which $f(x)$ has the following values.

22. x^2+3x+5 23. $-x^2-2x+4$ 24. $2x^2-5x+6$

25. $-3x^2-x-6$ 26. $\frac{1}{2}-2x+3x^2$ 27. $4x^2-5$

28. Find two integers whose sum is 98 and whose product is a maximum.

29. Find the number which exceeds its square by the greatest possible amount.

30. A circular sector of radius x inches has perimeter 20 inches and area $10x-x^2$ square inches. What radius produces maximum area?

Supplementary

State the equations of the parabolas of which the curves whose equations are given in (31)-(33) are branches.

31. $y=\sqrt{3-x}$ 32. $y=-2+\sqrt{x-2}$ 33. $y+\sqrt{1-2x}=4$

34. Sketch the curves whose equations are given in (31)-(33), stating any restrictions on x and the range of values for y.

35. Find geometrically an approximation to the solution of the equation

$$4-\sqrt{x-3}=-2+\sqrt{3x-1}.$$

Chapter 7

The Quadratic Equation

7 1 Introduction

To draw the graph of a linear equation, we need locate only two points on the line. If the line is not parallel to either axis, the two points usually determined are the points of intersection of the line and the coördinate axes. The coördinates of these points are found by replacing x and y, in turn, by zero in the linear equation and solving for the remaining variable.

In general, the points where a curve crosses the axes are very important in locating the position of the curve. In the last chapter we have used the point where the parabola crosses the y-axis for this purpose. This is the point whose coördinates satisfy simultaneously the equations

$$y = ax^2 + bx + c \qquad \text{(equation of the parabola)}$$

and

$$x = 0 \qquad \text{(equation of the y-axis),}$$

that is, the point $(0, c)$.

We now wish to consider the points of intersection of the parabola and the x-axis. Knowing these points will give us extra information with which to sketch the parabola. They will be the points whose coördinates satisfy simultaneously the equations

$$y = ax^2 + bx + c \qquad \text{(equation of the parabola)}$$

and

$$y = 0 \qquad \text{(equation of the x-axis),}$$

that is, the point $(x, 0)$, where x is a real number satisfying the equation

$$ax^2 + bx + c = 0.$$

This equation is called the general quadratic equation.

The real values of x for which $ax^2+bx+c=0$ are called the real roots of the quadratic equation $ax^2+bx+c=0$. These values of x constitute the solution set $\{x \in \mathrm{Re} | ax^2+bx+c=0\}$.

In the following sections we will consider the problem of solving the general quadratic equation. This will provide the method for finding the points of intersection of a parabola and the x-axis.

If we have the graph of the parabola, we may *estimate* these points. A study of Figure 7-1 will indicate that some parabolas may have two distinct points of intersection with the x-axis, some only one, and some none. This will give us a hint as to what to expect in studying the solution of the general quadratic equation; sometimes there will be two distinct real roots, sometimes only one, and sometimes none.

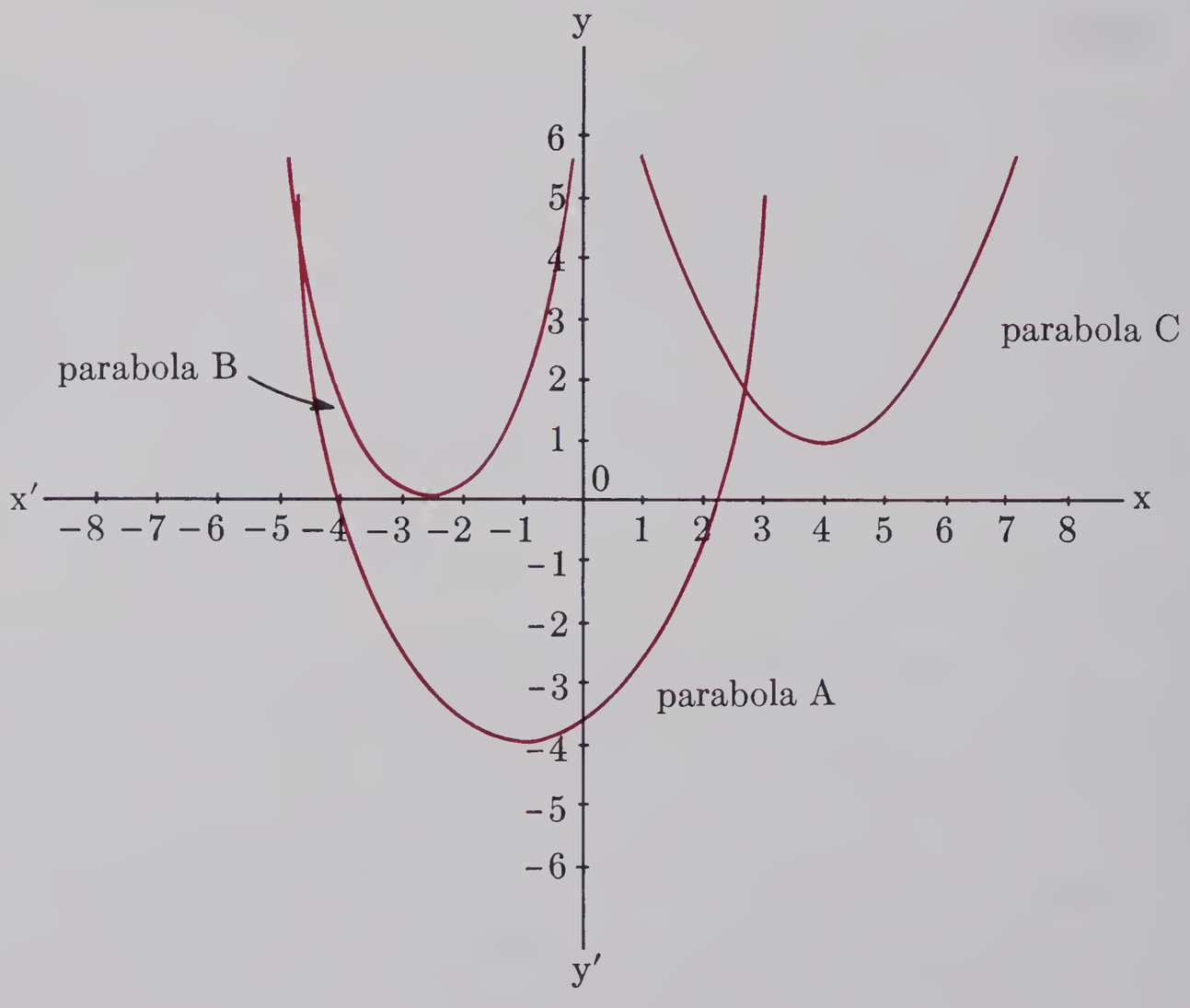


Figure 7-1

Parabola A in Figure 7-1 has two distinct points of intersection with the x-axis. Assuming that the graph of the parabola is a fairly accurate one, we could say that these points are approximately (2.25, 0) and (−4.25, 0). If the graph is only a sketch, we would say that the points are approximately (2, 0) and (−4, 0). The corresponding quadratic equation has two distinct real roots; its "approximate" solution set is {2.25, −4.25} or {2, −4} depending on the accuracy possible.

Parabola B seems to have a single point of contact with the x-axis near the point (−2.5, 0), although it could have two distinct points of intersection near this point. The corresponding quadratic equation has two real roots which are either identical or nearly equal.

Finally, parabola C obviously has no points of intersection with the x-axis. This will always be the case if the parabola is concave upwards and its vertex is above the x-axis or if the parabola is concave downwards and its vertex is below the x-axis. The corresponding quadratic equation has no real roots.

EXERCISE 7–1

Find the points of intersection with the y-axis of the parabolas whose equations follow. Sketch the parabolas and estimate their points of intersection with the x-axis, pointing out whether there seem to be two distinct points, only one, or none. State the associated quadratic equation in each case and the approximate values of its real roots.

1. $y=6x^2$
2. $y=-4x^2$
3. $y=3(x+2)^2$
4. $y=-4(x-1)^2$
5. $y=6x^2+3$
6. $y=-4x^2+8$
7. $y=2x^2-4$
8. $y=-x^2-9$
9. $y=x^2+4x-3$
10. $y=x^2+4x+9$
11. $y=x^2-6x+2$
12. $y=x^2-6x-12$
13. $y=-x^2+4x-1$
14. $y=2x^2+8x-5$
15. $y=-3x^2+6x+4$
16. $3y=4-2x+6x^2$
17. Estimate the points of intersection with the lines $y=3$ and $y=-3$ of the parabolas with equations (1)-(16). State the associated quadratic equations in each case and the approximate values of their real roots.

Express the following as quadratic equations in the form $ax^2+bx+c=0$, stating the values of a, b, and c:

18. $2x^2=6x-7$ 19. $x=6x^2-7$ 20. $3x^2-5=x^2+2$

21. $x^2=9$ 22. $3x-4=5x^2$ 23. $0=6-4x-x^2$

24. $(x-2)(x+3)=5$ 25. $(x+7)(x-1)=6x+2$

26. $(3x-1)(x+2)=x-5$ 27. $(2x+1)(3x-4)=(2-x)^2$

28. $(x-1)^2=(2x-1)^2$ 29. $(x-2)(x+1)=(2x-1)^2$

30. $x+\frac{1}{x}=5$ 31. $3x+\frac{1}{2x}=6$ 32. $x-\frac{1}{4x}-5=0$

33. $\frac{1}{x-1}+\frac{1}{x+1}=5$ 34. $\frac{1}{x-3}-\frac{1}{x+3}-4=0$ 35. $\frac{1}{x+1}+\frac{1}{x+2}=6$

36. $\frac{1}{x^2-4}+\frac{1}{x+2}=\frac{1}{2-x}$ 37. $x-\frac{1}{x}=\frac{5}{2x}$ 38. $\frac{x-1}{x}-\frac{x}{x-1}=2$

39. What restrictions must be placed on x in the equations of (30)-(38)?

7 2 Solving the Quadratic Equation by Factoring ax^2+bx+c

We make use of the principle that the product of two real numbers can equal zero only if at least one of the numbers is zero.

If $rs=0$, then $r=0$ or $s=0$ or $r=s=0$.

The roots of the equation $(x-2)(x+3)=0$ are obviously $x=2$ and $x=-3$, since $(x-2)(x+3)$ has value 0 if either $x-2=0$ or $x+3=0$. Note that $(x-2)(x+3)\neq 0$ for any other value of x, since neither factor will equal zero if x is different from 2 and -3.

Thus if ax^2+bx+c can be factored into $(\alpha x+\beta)(\gamma x+\delta)$, the roots of the quadratic equation $ax^2+bx+c=0$ are obviously $\frac{-\beta}{\alpha}$ and $\frac{-\delta}{\gamma}$ and no other numbers. In this section we will review the factoring of ax^2+bx+c into the form $(\alpha x+\beta)(\gamma x+\delta)$, *where this is possible*.

The question "Is it always possible to factor ax^2+bx+c into $(\alpha x+\beta)(\gamma+\delta)$ where α, β, γ, and δ are real numbers?" may be answered "no" by recourse to the geometry. If it were possible, then every quadratic equation $ax^2+bx+c=0$ would have two real roots and so every parabola $y=ax^2+bx+c$ would intersect the x-axis. But this is obviously not true as we have seen (Figure 7-1).

Suppose that

$$ax^2+bx+c=(\alpha x+\beta)(\gamma x+\delta)$$
$$=\alpha\cdot\gamma x^2+(\beta\cdot\gamma+\alpha\cdot\delta)x+\beta\cdot\delta.$$

Then

(i) the coefficient of x is $\beta\gamma+\alpha\delta$, the sum of two numbers whose product is $(\beta\gamma)(\alpha\delta)$;

(ii) the product of the coefficient of x^2 and the constant term is $(\alpha\gamma)(\beta\delta)$ which is equal to $(\beta\gamma)(\alpha\delta)$.

Recall also that $\alpha\gamma x^2+(\alpha\delta+\beta\gamma)x+\beta\delta$ can be factored by grouping.

$$\alpha\gamma x^2+(\alpha\delta+\beta\gamma)x+\beta\delta=(\alpha\gamma x^2+\alpha\delta x)+(\beta\delta x+\beta\delta)$$
$$=\alpha x(\gamma x+\delta)+\beta(\gamma x+\delta)$$
$$=(\alpha x+\beta)(\gamma x+\delta).$$

These observations provide the following

Rule for Factoring ax^2+bx+c:
To factor ax^2+bx+c, find the two factors p and q of ac such that $p+q=b$. Then write $ax^2+bx+c=(ax^2+px)+(qx+c)$ and factor each bracketed expression on the right side.

Example 1

Solve the quadratic equation $12x^2-11x-15=0$.

Solution: To factor $12x^2-11x-15$, we note that $ac=12(-15)=-180$. We must find two factors p and q of -180 whose sum is $b=-11$; by inspection these are -20 and 9. Then

$$12x^2-11x-15=(12x^2-20x)+(9x-15)$$
$$=4x(3x-5)+3(3x-5)$$
$$=(4x+3)(3x-5).$$

The roots of $12x^2-11x-15=0$ are then the roots of $4x+3=0$ and $3x-5=0$, namely $-\frac{3}{4}$ and $\frac{5}{3}$.

These values may be checked in the usual way:

$$12(-\tfrac{3}{4})^2-11(-\tfrac{3}{4})-15=12(\tfrac{9}{16})+\tfrac{33}{4}-15=\tfrac{27+33-60}{4}=0;$$
$$12(\tfrac{5}{3})^2-11(\tfrac{5}{3})-15=12(\tfrac{25}{9})-\tfrac{55}{3}-15=\tfrac{100-55-45}{3}=0.$$

Example 2

Find the points of intersection of the parabola whose equation is $y=12x^2+17x+2$ and the line whose equation is $y=7$.

Solution: The abscissae of the points of intersection are the real roots of the equation

$$12x^2+17x+2=7$$

or

$$12x^2+17x-5=0.$$

To find the roots we proceed as follows:

Factor $12x^2+17x-5$.

$pq=-60$ and $p+q=17$

By inspection, $p=20$ and $q=-3$.

$$\begin{aligned}12x^2+17x-5&=12x^2+20x-3x-5\\&=4x(3x+5)-(3x+5)\\&=(4x-1)(3x+5).\end{aligned}$$

So the equation $12x^2+17x-5$ is equivalent to $(4x-1)(3x+5)=0$ and the solution set for the equation is $\{\frac{1}{4}, -\frac{5}{3}\}$.
The required points of intersection are $(\frac{1}{4}, 7)$ and $(-\frac{5}{3}, 7)$.
These solutions should be checked.

Example 3

Solve the real quadratic equation $x^2+x+1=0$.

Solution: To factor x^2+x+1, we would have $pq=1$ and $p+q=1$.

If $pq=1$, then p and q are either both positive or both negative.

They cannot be both negative, for then their sum could not equal 1 (why?).

Suppose p and q are both positive. If $p=q=1$, then $p+q\neq 1$. If $p\neq q$, then either p or q must be greater than 1 (why?) and so their sum cannot equal 1.

Thus it is impossible to find real numbers p and q whose product and sum is 1, and so it is impossible to factor x^2+x+1 and to find real roots of the equation $x^2+x+1=0$.

The real solution set of this equation is the empty set.

Example 4

Solve the quadratic equation $x^2+x-1=0$.

Solution: To factor x^2+x-1, we have $pq=-1$ and $p+q=1$.

It is not an easy matter to find, by inspection, two numbers p and q whose sum is 1 and whose product is -1. Thus, this method of solving quadratic equations by factoring leaves something to be desired. This equation does have real solutions and they will be found in Section 7-4.

EXERCISE 7—2

Find the real roots of the following equations by factoring:

1. $x^2-64=0$
2. $x^2=16x$
3. $x^2-3x-4=0$
4. $x^2-5x=0$
5. $y^2+2y=15$
6. $12+a=a^2$
7. $x^2-2x=15$
8. $y^2+6y=91$
9. $x^2-8x+16=0$
10. Prove that if the absolute term c is zero, the quadratic equation $ax^2+bx+c=0$ can always be solved by factoring. Are the roots necessarily rational numbers?

Find the real roots of the following equations:

11. $x^2-x-6=0$
12. $6x^2+17x+5=0$
13. $2y^2+13y+15=0$
14. $3a^2+13a-10=0$
15. $6x^2-25x-9=0$
16. $12y^2+11y-15=0$
17. $6x^2-7x+1=0$
18. $2x^2+7x+3=0$
19. $3x^2+x-2=0$
20. $(x-3)(x-2)=6(x-3)$
21. $5(y-1)=(y-1)(y+1)$
22. $2a(4a-5)=3$
23. $\frac{5}{6}x-\frac{1}{3}=\frac{1}{2}x^2$
24. $2x-\frac{10}{3}=\frac{1}{6}x^2$

Find the points of intersection of the following parabolas with the given lines:

25. $y=8x^2+10x$, $y=3$
26. $y=2x^2-5x+2$, $y=-1$
27. $y=8x^2+14x-10$, $y=5$
28. $y=6x^2+19x+10$, $y=-5$
29. $y=3x^2-5x+6$, $y=8$
30. $y=-x^2+2x-9$, $y=4$

Solve for x:

31. $(x-a)^2-(x-a)=12$
32. $(x-2m)^2+4(x-2m)-5=0$
33. $2x^2-7\sqrt{2}x+12=0$
34. $2x^2-4\sqrt{6}x-15=0$

7 3 Special Quadratic Equations

The variable in a quadratic equation is not always a single letter. The examples will show how such equations are treated.

Example 1

Solve the equation $2\sin^2 x - 1 = -\sin x$ for x in radians if $0 < x < \pi$.

Solution: Recalling that $\sin^2 x$ is an abbreviation for $(\sin x)^2$, we see that the given equation is a quadratic equation in the variable $\sin x$:

$$2(\sin x)^2 + (\sin x) - 1 = 0.$$

This equation may be solved by factoring:

$$(2\sin x - 1)(\sin x + 1) = 0.$$

$\therefore 2\sin x - 1 = 0$ *or* $\sin x = -1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

The solution set is $\{\frac{\pi}{6}, \frac{5\pi}{6}\}$.

This equation has no real solution because there is no real value of x in the domain $0 < x < \pi$ for which $\sin x = -1$.

Sometimes it is convenient to replace a variable, such as $\sin x$, by a single letter. For example, we could replace $\sin x$ by z and the equation

$$2(\sin x)^2 + \sin x - 1 = 0 \text{ becomes } 2z^2 + z - 1 = 0.$$

This quadratic equation may be solved to give $z = \frac{1}{2}$ or -1; that is, $\sin x = \frac{1}{2}$ or -1 as before.

Example 2

Solve the equation $2^{2x} - 4 = 3(2^x)$.

Solution: Recalling that $2^{2x} = (2^x)^2$, we see that this equation may be written as a quadratic in the variable 2^x:

$$(2^x)^2 - 3(2^x) - 4 = 0$$

$$(2^x - 4)(2^x + 1) = 0$$

$\therefore 2^x - 4 = 0$ *or* $2^x + 1 = 0$.

$2^x = 4$

$x = 2$

Since 2^x is positive for all real values of x, there is no real x for which $2^x = -1$.

This quadratic equation has a single real solution, 2.

Again if we begin by replacing 2^x by z, the equation

$$(2^x)^2 - 3(2^x) - 4 = 0 \text{ becomes } z^2 - 3z - 4 = 0.$$

This quadratic equation may be solved to give $z = 4$ or -1; that is, $2^x = 4$ or -1 as before.

EXERCISE 7—3

Find θ in radians if $0 \leq \theta \leq \frac{\pi}{2}$:

1. $2 \sin \theta \cos^2 \theta = 0$
2. $2 \cos^2 \theta = \cos \theta$
3. $\cot^2 \theta + \cot \theta = 0$
4. $2 \sin^2 \theta - \sin \theta - 1 = 0$
5. $2 \cos^2 \theta - 5 \cos \theta = 3$
6. $\cos^2 \theta \sin^2 \theta = \frac{1}{2}$
7. $2 \cos^2 \theta + 1 = \sin \theta$
8. $\cos \theta + 2 = 2 \sin^2 \theta$
9. $\cos^2 \theta + 3 \cos \theta = 6$
10. $\sin \theta \cos \theta + \cos \theta = 1 + \sin \theta$
11. List the solutions to the above problems if $\frac{\pi}{2} < \theta \leq \pi$.

Find the real roots of the following equations:

12. $2^{2x} - 8(2^x) + 16 = 0$
13. $2^{2x} - 17(2^x) + 16 = 0$
14. $4(2^{2x}) - 17(2^x) + 4 = 0$
15. $2^{2x+2} - 17(2^x) + 4 = 0$
16. $2^{2x} - 7(2^x) - 8 = 0$
17. $2^{2x} + 9(2^x) + 8 = 0$
18. $32(2^{2x}) - 12(2^x) + 1 = 0$
19. $2^{2x} + 7(2^x) - 8 = 0$
20. $3^{2x} - 4(3^x) + 3 = 0$
21. $27(3^{2x}) + 26(3^x) - 1 = 0$
22. $5^{2x} - 24(5^x) - 25 = 0$
23. $7^{2x} + 5(7^x) + 6 = 0$
24. Solve for θ if $0 < \theta < \pi$: $\dfrac{1}{\sin^2 \theta} - \dfrac{3}{\sin \theta} + 2 = 0$.

7 4 Solving a Quadratic Equation · Completing the Square

The method of factoring ax^2+bx+c in order to solve the quadratic equation $ax^2+bx+c=0$ is, in general, useful only when the factors are $(\alpha x+\beta)(\gamma x+\delta)$, where α, β, γ, and δ are small integers. Otherwise the factors are normally quite awkward to determine by inspection. Finally as we have pointed out, there are situations in which ax^2+bx+c does not have real linear factors.

A better method for solving the quadratic equation $ax^2+bx+c=0$ comes from using the method of completing the square discussed in Section 6-5.

We first note that, when $a \neq 0$,

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right],$$

and the right side can equal zero if and only if the factor $x^2+\frac{b}{a}x+\frac{c}{a}$ equals zero. This means that $ax^2+bx+c=0$ if and only if $x^2+\frac{b}{a}x+\frac{c}{a}=0$ and so the equations $ax^2+bx+c=0$ and $x^2+\frac{b}{a}x+\frac{c}{a}=0$ are equivalent (they have the same solution set).

Example 1

Solve the quadratic equation $x^2-6x-27=0$.

Solution:

$$x^2-6x-27=0$$
$$x^2-6x=27$$

The left member can be made into a perfect square by adding $(-\frac{6}{2})^2$ or 9 to both members.

$$x^2-6x+9=27+9$$
$$(x-3)^2=36$$

$$x-3=+6 \quad \textit{or} \quad x-3=-6$$
$$x=9 \quad \textit{or} \quad x=-3$$

The solution set is $\{9, -3\}$. These answers are easily checked.

Example 2

Solve $x^2+x-1=0$.

Solution: This is the equation of Example 4 of Section 7-2. You will recall that we were unable to find its roots by the method of factoring.

$$x^2+x-1=0$$
$$x^2+x=1$$

The left member becomes a perfect square with the addition of $(\frac{1}{2})^2$ or $\frac{1}{4}$.

$$x^2+x+\tfrac{1}{4}=1+\tfrac{1}{4}$$
$$(x+\tfrac{1}{2})^2=\tfrac{5}{4}$$

$$x+\tfrac{1}{2}=\sqrt{\tfrac{5}{4}} \quad \text{or} \quad x+\tfrac{1}{2}=-\sqrt{\tfrac{5}{4}}$$
$$x=-\tfrac{1}{2}+\tfrac{\sqrt{5}}{2} \qquad x=-\tfrac{1}{2}-\tfrac{\sqrt{5}}{2}$$
$$=\tfrac{-1+\sqrt{5}}{2} \qquad =\tfrac{-1-\sqrt{5}}{2}$$

Check:

$$(\tfrac{-1+\sqrt{5}}{2})^2+(\tfrac{-1+\sqrt{5}}{2})-1=\tfrac{1-2\sqrt{5}+5}{4}+\tfrac{-1+\sqrt{5}}{2}-1$$
$$=\tfrac{3-\sqrt{5}-1+\sqrt{5}-2}{2}$$
$$=0$$

$$(\tfrac{-1-\sqrt{5}}{2})^2+(\tfrac{-1-\sqrt{5}}{2})-1=\tfrac{1+2\sqrt{5}+5}{4}+\tfrac{-1-\sqrt{5}}{2}-1$$
$$=\tfrac{3+\sqrt{5}-1-\sqrt{5}-2}{2}$$
$$=0$$

The solution set is $\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$.

Example 3

Solve $2x^2-7x+4=0$.

Solution: Replace $2x^2-7x+4=0$ by the equivalent equation

$$x^2-\tfrac{7}{2}x+2=0.$$
$$x^2-\tfrac{7}{2}x=-2$$

Add $(-\frac{7}{4})^2$ or $\frac{49}{16}$ to both members to make the left side a perfect square.

$$x^2-\tfrac{7}{2}x+\tfrac{49}{16}=-2+\tfrac{49}{16}$$
$$(x-\tfrac{7}{4})^2=\tfrac{17}{16}$$
$$x-\tfrac{7}{4}=\tfrac{\sqrt{17}}{4} \quad \textit{or} \quad x-\tfrac{7}{4}=-\tfrac{\sqrt{17}}{4}$$
$$x=\tfrac{7+\sqrt{17}}{4} \quad \textit{or} \quad x=\tfrac{7-\sqrt{17}}{4}$$

The solution set is $\{\frac{7+\sqrt{17}}{4}, \frac{7-\sqrt{17}}{4}\}$.

The checking is left as an exercise for the student.

Example 4

Solve $x^2+x+1=0$.

Solution: This is the equation of Example 3 of Section 7-2. There it was shown that this equation has no real roots.

$$x^2+x+1=0$$
$$x^2+x=-1$$
$$x^2+x+\tfrac{1}{4}=-1+\tfrac{1}{4}$$
$$(x+\tfrac{1}{2})^2=-\tfrac{3}{4}$$
$$x+\tfrac{1}{2}=\sqrt{-\tfrac{3}{4}} \quad \textit{or} \quad x+\tfrac{1}{2}=-\sqrt{-\tfrac{3}{4}}$$

But $\sqrt{-\frac{3}{4}}$ is not a real number; there is no real number whose square is a negative number.

Thus if the equation $x^2+x+1=0$ is to have roots, they must involve some new type of number outside the real number system.

The desire that every real equation should have roots leads to the introduction of a new number system, the set of complex numbers. This set is an extension or enlargement of the set of real numbers but it will not be studied yet. (See Sections 7-8 and 9-3 to 9-6.)

At the present time, our conclusion is simply that the quadratic equation $x^2+x+1=0$ has no real roots.

Example 5

Solve $x^2+10x+25=0$.

Solution: Following our work in the preceding examples, we would write $x^2+10x=-25$, and complete the square by adding $(5)^2$ to both sides:

$$x^2+10x+25=-25+25.$$
$$(x+5)^2=0$$
$$x+5=0 \quad \textit{or} \quad x+5=0$$
$$x=-5 \quad \textit{or} \quad x=-5$$

The solution set is $\{-5, -5\}$; the equation has two identical roots.

Of course if we notice that $x^2+10x+25$ is itself a perfect square, then we may write the original equation as $(x+5)^2=0$ immediately.

EXERCISE 7—4

Find the real solution sets of the following quadratic equations by the method of this section. Specify when the equation has no real roots.

1. $x^2+6x+5=0$
2. $x^2-8x-20=0$
3. $x^2-15=2x$
4. $x^2+4x+1=0$
5. $x^2+6x-4=0$
6. $x^2-6x=9$
7. $x^2-10x+3=0$
8. $x^2+8x=7$
9. $x^2-5x+2=0$
10. $12=4x+x^2$
11. $x^2-7x+3=0$
12. $x^2-5x=10$
13. $x^2=12x+8$
14. $x^2=11x+7$
15. $x^2-6x+1=0$
16. $3x^2-5x+2=0$
17. $12=4x+5x^2$
18. $4x^2+7x=2$
19. $4x^2=5x+6$
20. $3x+4=2x^2$
21. $6x^2+12=17x$
22. $12=5x+2x^2$
23. $3x^2=3x-1$
24. $6x^2=11x+10$
25. $(x-2)(x+3)=2(x+11)$
26. $x^2-2ax=3a^2$
27. $x^2-bx=b^2$
28. $x^2+2ax-b=0$
29. $x^2-12ax+18a^2=0$
30. $ax^2+bx+c=0$

7 5 The Quadratic Formula

We will now solve the general quadratic equation $ax^2+bx+c=0$ by the method of completing the square. The result gives us the important quadratic formula. The solution proceeds as follows:

$$ax^2+bx+c=0 \qquad (a\neq 0)$$

first step
$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

second step
$$x^2+\frac{b}{a}x = -\frac{c}{a}$$

third step
$$\begin{cases} x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a} \\ \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \end{cases}$$

fourth step
$$x+\frac{b}{2a}=+\frac{\sqrt{b^2-4ac}}{2a}$$

or
$$x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$$

fifth step
$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

or
$$x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The two roots are usually combined in a single expression known as the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This formula may be memorized and used to write down at sight the solution of a quadratic equation, bearing in mind that a is the coefficient of x^2, b is the coefficient of x, and c is the constant or absolute term.

The equation will have two real roots if the number b^2-4ac under the root sign is nonnegative.

The equation will have two roots if b^2-4ac is negative; they will not be real numbers, but numbers from a set that we have not yet studied.

If $b^2-4ac=0$, the equation will have two identical real roots. The significance of b^2-4ac is discussed more fully in the next section.

Example 1

Use the quadratic formula to solve the equation $3x^2-x-12=0$.

Solution: In this equation $a=3$, $b=-1$, $c=-12$.

Replace a, b, and c by these values in the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(3)(-12)}}{2(3)}$$

$$=\frac{1\pm\sqrt{145}}{6}.$$

Check: $3\left(\frac{1+\sqrt{145}}{6}\right)^2-\left(\frac{1+\sqrt{145}}{6}\right)-12$

$$=\frac{3}{36}\,(146+2\sqrt{145})-\frac{1+\sqrt{145}}{6}-12$$

$$=\frac{1}{12}[146+2\sqrt{145}-2-2\sqrt{145}-144]=0$$

The second root may be checked similarly.

The solution set is $\left\{\frac{1+\sqrt{145}}{6},\ \frac{1-\sqrt{145}}{6}\right\}$.

The student should note that $\frac{1+\sqrt{145}}{5}$ and $\frac{1-\sqrt{145}}{6}$ are the *exact* roots of this equation. Decimal approximations to any required degree of accuracy may be found. For example, $\sqrt{145}\simeq 12.04159$, so that the roots are approximately 2.17360 and -1.84027. Decimal approximations would be used in solving practical problems, for example, in finding a definite measurement when it is required.

A rough check of the correctness of these roots can be obtained by replacing x by 2.2 and -1.8 in turn in the left member of the equation; the result should be a number close to zero.

$$3(2.2)^2-(2.2)-12=14.52-2.2-12=0.32$$

$$3(-1.8)^2-(-1.8)-12=9.72+1.8-12=-0.48$$

Example 2

Solve the equation $3x^2-2x+6=0$.

Solution: In this equation $a=3$, $b=-2$, and $c=6$.
Replace a, b, and c by these values in the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(6)}}{2(3)}$$

$$=\frac{2\pm\sqrt{-68}}{6}$$

The equation has no real roots.

Example 3

Find θ to the nearest degree if $0\leq\theta\leq90°$ and $\cot^2\theta-2\cot\theta-2=0$.

Solution: In this quadratic equation, $(\cot\theta)^2-2(\cot\theta)-2=0$,
the variable is $\cot\theta$, and $a=1$, $b=-2$, and $c=-2$.
Applying the quadratic formula yields

$$\cot\theta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{2\pm\sqrt{12}}{2}$$

$$=1\pm\sqrt{3}.$$

$\cot\theta=1+\sqrt{3}$

$\tan\theta=\dfrac{1}{1+\sqrt{3}}\cdot\dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$=\dfrac{1-\sqrt{3}}{-2}$

$\simeq\dfrac{-0.732}{-2}$

$=0.366$

$\therefore\ \theta\simeq20°$ from the tables.

or $\cot\theta=1-\sqrt{3}$

$\simeq1-1.732$

$=-0.732$

There is no angle θ in the first quadrant for which $\cot\theta$ is negative.

There is only one root θ satisfying the given restriction $0\leq\theta\leq90°$. This root, to the nearest degree, is 20°.

The student should note that the quadratic formula provides two roots for each quadratic equation (although the roots need not be real).

EXERCISE 7—5

Use the quadratic formula to find the real roots of the following equations. Specify when the roots are not real. If the roots are not rational, leave your answers in radical form unless otherwise instructed. Remember that the answer in radical form is exact, but that a decimal approximation is not.

1. $x^2-7x+6=0$
2. $x^2+12x-28=0$
3. $2x^2-x=21$
4. $3a^2-19a-14=0$
5. $10y^2-13y-3=0$
6. $3x^2-27=0$
7. $6a^2+17a+5=0$
8. $5t^2+3t=2$
9. $2x^2-15x-21=0$
10. $4b^2-9b=55$
11. $3x^2+6x-2=0$
12. $7x^2-2x=1$

Find the real roots of the following equations to the nearest tenth:

13. $x^2-2x-1=0$
14. $x^2+3x=8$
15. $a^2=6a+4$
16. $y^2-2y=2$
17. $h^2-8h+3=0$
18. $2x^2-3x+1=0$
19. $x^2-\frac{2}{3}x-1=0$
20. $x^2+0.2x-0.3=0$
21. $x^2+4x=-2$

Find the real roots of the following equations by any method:

22. $(3x-4)(x-2)=(2x-1)^2$
23. $(y+2)^2+(y-3)^2-15=0$
24. $x+\frac{1}{x}=\frac{13}{6}$
25. $\frac{2}{y-1}-\frac{1}{y+1}=1$
26. $\frac{4}{x-2}+\frac{1}{x+2}=1$
27. $\frac{4}{x}+\frac{x}{4}=\frac{5}{2}$
28. $\frac{14}{h+2}-\frac{2}{h-4}=1$
29. $\frac{3}{x+2}=\frac{1}{4}-\frac{1}{x-4}$
30. $\frac{x^2+4}{4x}=\frac{4}{x}+\frac{1}{4}$
31. $\frac{x^2-6}{3}+\frac{7}{2}=\frac{x+10}{12}$
32. $\frac{5}{h}=\frac{3}{h+1}+\frac{8}{7}$
33. $\frac{3x}{2}-\frac{x+5}{x-1}=x$
34. $\frac{3}{x+1}+\frac{4x}{x^2-1}=3-\frac{2}{1-x}$
35. $\frac{50}{x}-\frac{40}{x+10}=1$
36. $(2x-1)(5x-2)-(x-2)^2=10$
37. $(x+6)(x-1)+x^2=10x+9$
38. $x^2+0.06x=0.8$
39. $0.1x-0.02x^2+0.32=0$

Find the members of the set in each of the following:

40. $\{\sin x \mid 6\sin^2 x - 2\sin x - 3 = 0\}$

41. $\{\cos x \mid 6\cos^2 x + 15\cos x + 5 = 0\}$

42. $\{\sin x \mid 3\sin^2 x - 4\sin x - 5 = 0\}$

43. $\{\tan x \mid \tan^2 x + \tan x = 30\}$

44. $\{\cos x \mid (2\cos x + 1)^2 - 17\cos x - 15 = 0\}$

45. $\{\csc x \mid 3\csc^2 x + \frac{1}{3}\csc x = 1\}$

46. $\{\cot x \mid 3\cot^2 x - 2\cot x + 5 = 0\}$

Solve the following equations for θ to the nearest degree under the restriction $0 \leq \theta \leq 90°$:

47. $\sin\theta - 2 = \csc\theta$

48. $2\tan^2\theta = 7 - \sec\theta$

49. $2\sin^2\theta - 5\sin\theta + 1 = 0$

50. $3\cot^2\theta - 2 = 3\cot\theta$

51. Check that $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ are roots of the equation $ax^2 + bx + c = 0$.

7 6 General Quadratic Equations

We have now studied three methods for solving a quadratic equation (i) factoring, (ii) completing the square, (iii) formula. Which is the easiest to use?

The solution of the quadratic equation by the method of completing the square is seldom used. Its main use lies in the development of the quadratic formula although the technique of completing the square is used in other mathematical topics. Given a quadratic equation to solve, the student would normally use either factoring or formula. If the factors of $ax^2 + bx + c$ are apparent or readily obtainable, the factoring method should be used. However, remember that the formula *always* produces an answer.

The first step in applying either method is to write the quadratic in the form $ax^2 + bx + c = 0$.

Example 1

Solve for x the quadratic equation $pqx^2-(p^2+q^2)x+pq=0$.

Solution: In this equation $a=pq$, $b=-(p^2+q^2)$, and $c=pq$.

$$\text{Then } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-(p^2+q^2))\pm\sqrt{(-(p^2+q^2))^2-4(pq)(pq)}}{2(pq)}$$

$$=\frac{p^2+q^2\pm\sqrt{p^4+2p^2q^2+q^4-4p^2q^2}}{2pq}$$

$$=\frac{p^2+q^2\pm\sqrt{p^4-2p^2q^2+q^4}}{2pq}$$

$$=\frac{p^2+q^2\pm\sqrt{(p^2-q^2)^2}}{2pq}$$

$$x=\frac{p^2+q^2+(p^2-q^2)}{2pq} \qquad \textit{or} \qquad x=\frac{p^2+q^2-(p^2-q^2)}{2pq}$$

$$=\frac{2p^2}{2pq}=\frac{p}{q} \qquad\qquad =\frac{2q^2}{2pq}=\frac{q}{p}$$

These roots could have been found more easily by factoring and it would be preferable to solve the equation in this way.

$$pqx^2-(p^2+q^2)x+pq=0$$

$$(px-q)(qx-p)=0$$

$$px-q=0 \qquad \textit{or} \qquad qx-p=0$$

$$x=\frac{q}{p} \qquad\qquad x=\frac{p}{q}$$

Example 2

Find the real roots of the equation $2(x^2+1)^2+3(x^2+1)-6=0$.

Solution: Let x^2+1 be replaced by z; our equation becomes $2z^2+3z-6=0$.

Using the formula with $a=2$, $b=3$, and $c=-6$, we obtain

$$z=\frac{-3\pm\sqrt{57}}{4}$$

$$\text{Thus } x^2+1=\frac{-3+\sqrt{57}}{4} \qquad \textit{or} \qquad x^2+1=\frac{-3-\sqrt{57}}{4}$$

$$x^2=\frac{-7+\sqrt{57}}{4} \qquad\qquad x^2=\frac{-7-\sqrt{57}}{4}$$

$$\simeq 0.137 \qquad\qquad \text{There are no real values of } x$$

$$x\simeq\pm 0.37 \quad \text{(tables)} \qquad \text{satisfying this equation.}$$

Approximations to the real roots are ± 0.37.

EXERCISE 7–6

Find the real roots of the following equations by any method (state which equations have no real roots):

1. $x^2-5=0$
2. $x^2+64=0$
3. $a^2-2a-3=0$
4. $h^2+4h-1=0$
5. $x^2+2x-3=0$
6. $3x^2-7x+5=0$
7. $20x^2+x=15$
8. $(x-3)(x+3)=6x$
9. $\frac{1}{2}(x-1)(x+2)=x-3$
10. $x^2-974x=975$
11. $(2x-3)^2+17x=15$
12. $(h-\frac{1}{2})(h+3)=39$
13. $x^2-2=2x\sqrt{3}$
14. $y^2+2\sqrt{2}y-6=0$
15. $\dfrac{2x+5}{2x+1}=\dfrac{x}{x+8}$
16. $\dfrac{4}{3}+\dfrac{x+1}{2x+1}=\dfrac{2x-1}{x+2}$
17. $8x(x+3)-3x(x+2)=3(x+1)(x+4)$
18. $\dfrac{x-1}{x-2}+2=\dfrac{2x+5}{3}$
19. $(a+3)^2-(a-3)^2+(a-3)(a+3)=23$
20. $2x^2+(x-4)^2=5x(x-4)+26$
21. $3x^2+5x-2hx=2+hx^2$
22. $x^2+2kx+k^2-h^2=0$
23. $rsx^2+rs=x(r^2+s^2)$
24. $(a^2-b^2)x^2+1=2ax$
25. $2(x-2)^2+7(x-2)-15=0$
26. $2(x^2-2)^2+7(x^2-2)-15=0$
27. $3(x^2-8)^2+6(x^2-8)-24=0$
28. $x^2+5x-\dfrac{6}{x^2+5x}=5$
29. $\dfrac{x^2+2}{x^2-2}+\dfrac{3x^2-6}{x^2+2}=-4$
30. $\dfrac{6}{x^2}-\dfrac{3}{x}+2=0$

7.7 The Character of the Roots of a Quadratic Equation· The Discriminant

We have seen in our examples that the general real quadratic equation $ax^2+bx+c=0$ has two roots

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

which are either real and distinct, real and identical, or nonreal. We have also seen that the roots of $ax^2+bx+c=0$ give the abscissae of the points of intersection of the parabola $y=ax^2+bx+c$ with the x-axis. Corresponding to the three algebraic possibilities for the roots of the quadratic equation $ax^2+bx+c=0$, we have the three geometric possibilities for the corresponding parabola whose equation is $y=ax^2+bx+c$; the parabola either intersects the x-axis in two distinct points, touches the x-axis (intersects in two identical points), or does not meet the x-axis.

It is sometimes important to know the character of the roots of a quadratic equation without actually finding them. For example, we might want to know whether the roots are real or nonreal, or whether they are equal or unequal.

Consider the number b^2-4ac in the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac=0$, the roots are $\frac{-b+0}{2a}$ and $\frac{-b-0}{2a}$; that is, the roots are equal.

If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is not a real number and the quadratic equation has no real roots.

If $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ is a real number and the equation $ax^2+bx+c=0$ has two real and unequal roots.

The number b^2-4ac which determines the character of the roots, that is, whether the roots are real or nonreal and equal or unequal is called the discriminant D of the quadratic equation:

$$\mathrm{D}=b^2-4ac.$$

If a, b, and c are rational, then b^2-4ac is rational, and the roots of the equation $ax^2+bx+c=0$ will be rational if b^2-4ac is the square of a rational number and irrational if it is not.

Summary

If the discriminant $D=b^2-4ac$ is	and $a, b, c,$ are	then the roots of $ax^2+bx+c=0$ are
<0	real	nonreal numbers
$=0$	real	equal real numbers
>0	real	unequal real numbers
<0	rational	nonreal numbers
$=0$	rational	equal rational numbers
>0 and the square of a rational number	rational	unequal rational numbers
>0 and not the square of a rational number	rational	unequal irrational numbers

This table need not be memorized. The student should understand the formation of the table and the role of the discriminant and be able to state the character of the roots of a given quadratic equation.

Example 1

State the character of the roots of the following equations:

(i) $3x^2+5x-1=0$ (ii) $16x^2-8x+1=0$ (iii) $4x^2-5x+3=0$

(iv) $4x^2+8x-5=0$ (v) $\sqrt{2}x^2+\sqrt{5}x-\frac{1}{\sqrt{2}}=0.$

Solution: (i) $D=(5)^2-4(3)(-1)=37$

$\therefore$ the roots are unequal and irrational.

(ii) $D=(-8)^2-4(16)(1)=0$

$\therefore$ the roots are equal and rational.

(iii) $D=(-5)^2-4(4)(3)=-23$

$\therefore$ the roots are nonreal.

(iv) $D=(8)^2-4(4)(-5)=144$

$\therefore$ the roots are unequal and rational.

(v) $D=(\sqrt{5})^2-4(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right)=9$

$\therefore$ the roots are unequal and irrational (D is a perfect square, but $a=\sqrt{2}$ and $b=\sqrt{5}$).

Example 2

State whether the parabolas with the following equations intersect the x-axis, touch the x-axis, or do not intersect the x-axis:

(i) $y=3x^2+5x-1$ (ii) $y=16x^2-8x+1$ (iii) $y=4x^2-5x+3$

(iv) $y=4x^2+8x-5$ (v) $y=\sqrt{2}x^2+\sqrt{5}x-\frac{1}{\sqrt{2}}$.

Solutions: Using the results of the previous example we conclude that:

the parabola in (i) meets the x-axis in two distinct points;

the parabola in (ii) touches the x-axis (two identical points of intersection);

the parabola in (iii) does not meet the x-axis;

the parabolas in (iv) and (v) each meet the x-axis in two distinct points.

Example 3

Find k so that the roots of the equation $x^2-3x+k=0$ are (i) equal (ii) nonreal (iii) real and unequal.

Solution: In this equation $a=1$, $b=-3$, and $c=k$, so that $\mathrm{D}=9-4k$.

(i) The roots will be equal if

$$9-4k=0;$$

that is, $k=\frac{9}{4}$.

(ii) The roots will be nonreal if

$$9-4k<0$$
$$4k-9>0$$
$$4k>9$$
$$k>\tfrac{9}{4}.$$

(iii) The roots will be real and unequal if

$$9-4k>0$$
$$4k-9<0$$
$$4k<9$$
$$k<\tfrac{9}{4}.$$

EXERCISE 7–7

1. What can be said about the roots of the equation $ax^2+bx+c=0$ if a, b, and c are rational and b^2-4ac equals
   (a) zero (b) $\frac{9}{4}$ (c) $-\frac{9}{4}$ (d) $\frac{7}{4}$ (e) 1.44 (f) -1.44?

Without solving, describe the roots of the following equations:

2. $3x^2-7x-5=0$
3. $3x^2-7x+1=0$
4. $2x-x^2=5$
5. $9x^2+25=30x$
6. $5x^2-7=0$
7. $6x^2-7x=-5$
8. $h^2=7h-11$
9. $3a^2+7a-3=0$
10. $x^2=\frac{1}{2}x+\frac{3}{5}$
11. $x^2-4x+k=0$, $k\leq 4$
12. $x^2+4x+k=0$, $k\leq 4$

For what values of k will the following equations have equal roots?

13. $9x^2+kx+1=0$
14. $3x^2+2kx-2k=0$
15. $(k-1)x^2+1=kx$
16. $kx^2+4x=k-5$
17. $kx^2-24x+16=0$
18. $x^2-(k+3)x+3k=0$
19. $(k+1)y^2=1-k-ky$
20. $km^2+5km+25=0$
21. $x^2-kx=1+k$
22. $(2k-1)x^2+(1-3k)x+(2-k)=0$
23. $(3k+1)x^2+2(k+1)x+k=0$
24. $k^2x^2+kx+1=0$

Find k so that the roots of the following equations are
   (a) nonreal (b) real and unequal:

25. $4kx^2-3x+1=0$
26. $3x^2-2kx+3=0$
27. $6x^2-x-3k=0$
28. $(k+1)x^2-6x+4=0$
29. $x^2+(k+2)x+5=0$
30. $3x^2-x+k-4=0$
31. Show that the equation $x^2-x+a=0$ has real roots for all negative values of a. Does it have real roots for any other values of a?
32. Show that the equation $x^2+ax-1=0$ has real roots for all real values of a.
33. For what values of m will the equation $x^2+nx+m=0$ have real roots?
34. Does the parabola $y=25x^2-1340x+17{,}956$ cross the x-axis?

7 8 Nonreal Roots of a Quadratic Equation

We have seen that if $D<0$, where D is the discriminant of a quadratic equation, the equation has no real roots. We have also indicated that the equation will still have two roots but that these roots will be numbers from a set different from the set of real numbers. There is an introduction to the study of these numbers, called complex numbers, in the supplementary Sections 9-3 to 9-6. Students who study mathematics or any of its applications (science, engineering, etc.) at university will find that complex numbers are very important. They are used particularly in the study of such subjects as electricity, elasticity, hydrodynamics and nuclear physics.

In this section we will simply mention that a complex number is one that can be expressed in the form $a+ib$ where a and b are real numbers and i is a symbol with the property that $i^2=-1$, that is, $i=\sqrt{-1}$. This section will give the student practice in writing the nonreal roots of a quadratic equation in this form, but we will not study the arithmetic of these numbers (how to add and multiply them, etc.) at this time.

The main thing to notice is that, if $a>0$, $\sqrt{-a}$ can be written

$$\sqrt{-a}=\sqrt{(-1)a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$$

This is verified by noticing that $(i\sqrt{a})^2=i^2a=-a$, since $i^2=-1$, and so $i\sqrt{a}$ is a square root of $-a$. Of course $-i\sqrt{a}$ is also a square root.

Example 1

(i) $\sqrt{-9}=\sqrt{9(-1)}=\sqrt{9}\sqrt{-1}=3i$

(ii) $\sqrt{-5}=\sqrt{5(-1)}=\sqrt{5}\sqrt{-1}=\sqrt{5}i$

(iii) $\sqrt{-72}=\sqrt{72}i=6\sqrt{2}i$

(iv) $\sqrt{-\frac{9}{4}}=\frac{3}{2}i$

Example 2

Express the roots of the quadratic equation

$$2x^2-x+4=0$$

in the form $a+ib$.

Solution: The quadratic formula gives the roots:

$$x=\frac{1\pm\sqrt{-31}}{2}$$

$$=\frac{1\pm i\sqrt{31}}{2}$$

$\therefore$ the solution set is $\left\{\frac{1}{2}+i\frac{\sqrt{31}}{2}, \frac{1}{2}-i\frac{\sqrt{31}}{2}\right\}$.

EXERCISE 7—8

Find the roots of the following equations; if they are nonreal, express them in the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$.

1. $x^2+x+1=0$
2. $x^2-x+1=0$
3. $x^2+4=0$
4. $y^2+2y+4=0$
5. $2x^2-6x+5=0$
6. $2h^2-h+\frac{1}{4}=0$
7. $-3x^2+x-1=0$
8. $4x^2-\frac{1}{2}x+\frac{3}{2}=0$
9. $4x^2-\frac{1}{2}x-\frac{3}{2}=0$
10. $10y^2+y+2=0$
11. $6m^2-2m+3=0$
12. $-4t^2+3t-2=0$

7 9 Equations Involving Radicals

The solution of equations involving second-order radicals sometimes gives rise to quadratic equations. The following examples will emphasize the care that must be taken to check the results obtained, since algebraic manipulations sometimes produce equations whose roots are not roots of the original equation. For example, the only root of the equation $x=2$ is 2, but squaring both members of the equation produces the equation $x^2=4$ which has two roots 2 and -2. Of course, -2 is not a root of the original equation. Be careful; check all possible solutions.

Example 1

Solve the equation $\sqrt{x^2-9}+4=0$.

Solution: If $\sqrt{x^2-9}+4=0$,
then $\sqrt{x^2-9}=-4$.
$x^2-9=16$ (squaring)
$x^2=25$

The roots of this equation are 5 and -5.

But we must check to see if 5 and -5 are solutions of $\sqrt{x^2-9}+4=0$.

(i) $\sqrt{(5)^2-9}+4=\sqrt{16}+4=4+4\neq 0$

(ii) $\sqrt{(-5)^2-9}+4=\sqrt{16}+4=4+4\neq 0$

Hence 5 and -5 are not roots of the original equation. But there can be no others; these are the only values of x which can possibly be roots.

The equation $\sqrt{x^2-9}+4=0$ has no roots. In the example the result is immediately obvious from the equation. Why?

Example 2

Solve $\sqrt{x}+\sqrt{3x+4}=2.$

Solution: If $\sqrt{x}+\sqrt{3x+4}=2,$

then $\sqrt{3x+4}=2-\sqrt{x}.$ (isolating one radical)

$3x+4=4-4\sqrt{x}+x$ (squaring both members)

$x=-2\sqrt{x}$ (isolating one radical)

$x^2=4x$

The roots of the equation $x^2-4x=0$ are 0 and 4.

Are these roots of the original equation? (They are the only *possible* roots.)

Check: (i) $\sqrt{0}+\sqrt{3(0)+4}=2$ (ii) $\sqrt{4}+\sqrt{3(4)+4}=2+4$

The only root of the original equation is $x=0$; $x=4$ is not a root.

Example 3

Solve $\sqrt{x+1}+\sqrt{x+4}=\sqrt{x+9}.$

Solution: $\sqrt{x+1}+\sqrt{x+4}=\sqrt{x+9}$ (one radical is already isolated)

$x+1+2\sqrt{x+1}\sqrt{x+4}+x+4=x+9$ (squaring)

$2\sqrt{x+1}\sqrt{x+4}=-x+4$ (isolating the remaining radical)

$4(x+1)(x+4)=x^2-8x+16$ (squaring)

$4(x^2+5x+4)=x^2-8x+16$

$3x^2+28x=0$

$3x(x+\frac{28}{3})=0$

$x=0$ or $x=-\frac{28}{3}$

Check: (i) L.S. $=\sqrt{0+1}+\sqrt{0+4}=1+2=3$

R.S. $=\sqrt{0+9}=3$

$\therefore x=0$ is a root.

(ii) L.S. $=\sqrt{-\frac{28}{3}+1}+\sqrt{-\frac{28}{3}+4}=\sqrt{-\frac{25}{3}}+\sqrt{-\frac{16}{3}}=\frac{5}{\sqrt{3}}i+\frac{4}{\sqrt{3}}i$

R.S. $=\sqrt{-\frac{28}{3}+9}=\sqrt{-\frac{1}{3}}=\frac{1}{\sqrt{3}}i$

Does $\frac{5}{\sqrt{3}}i+\frac{4}{\sqrt{3}}i=\frac{1}{\sqrt{3}}i$? It doesn't appear likely, does it? Remember that we do not as yet know how to add and multiply complex numbers; so we cannot check this possible answer.

(Actually $-\frac{28}{3}$ is *not* a root.)

In solving equations involving second-order radicals:

(i) replace the equation by an equivalent one in which one member is a single radical;

(ii) square both members;

(iii) if any radicals remain, repeat steps (i) and (ii);

(iv) solve the resulting radical-free equation;

(v) check the values found by substitution in the original equation.

EXERCISE 7—9

Solve the following equations. Check all possible real roots. (Use approximations for irrational roots.) Express the possible nonreal roots in the form $a+bi$.

1. $\sqrt{x^2-9}-4=0$
2. $\sqrt{x^2+7}-4=0$
3. $\sqrt{x^2+5}-6=0$
4. $3\sqrt{x-1}+1=2x$
5. $3\sqrt{x-1}+2x=1$
6. $\sqrt{x+5}+\sqrt{3-x}=4$
7. $\sqrt{2x+9}+\sqrt{2x+1}=2\sqrt{x}$
8. $\sqrt{x+8}-\sqrt{5-x}=-1$
9. $\sqrt{x+8}+\sqrt{5-x}=-1$
10. $\sqrt{x+7}-\sqrt{3x+19}=\sqrt{2x}$
11. $\sqrt{x+5}+\sqrt{x-3}=4$
12. $\sqrt{3x}+\sqrt{x-2}=2\sqrt{x}$
13. $\sqrt{x-5}+\sqrt{3x+9}=8$
14. $-\sqrt{2x-4}+\sqrt{4x-7}=1$
15. $\sqrt{2x+1}+\sqrt{x+6}=\sqrt{3x}$
16. $\sqrt{x+9}-\sqrt{2x+16}=-1$
17. $\sqrt{x-3}+\sqrt{x-8}=5\sqrt{x}$
18. $\dfrac{3}{\sqrt{x}}-\sqrt{x}=\sqrt{5+x}$
19. $\sqrt{x+2}-\sqrt{4x+3}=\sqrt{5-3x}$
20. $\sqrt{3x+1}=1-\sqrt{x+3}$
21. $\dfrac{1}{\sqrt{3x+1}}+\sqrt{2x}=\sqrt{3x+1}$
22. $\sqrt{x+2}+\sqrt{2x+3}=\sqrt{2x+5}$
23. $\sqrt{x+3}+\sqrt{x+7}=\sqrt{2x+10}$
24. $\sqrt{3x-\frac{1}{2}}-\sqrt{6x+1}=\sqrt{x+\frac{1}{2}}$
25. $\sqrt{4-4x}-\sqrt{4-x}=\sqrt{5-3x}$
26. $\sqrt{x+5}-1=\sqrt{x+15}$
27. $\sqrt{m+x}+\sqrt{n-x}=\sqrt{2m+2n}$
28. $\sqrt{x^2+4x+15}=3+\sqrt{x^2+4x-12}$

7 10 Problems Giving Rise to Quadratic Equations

When a practical problem gives rise to a quadratic equation, only the real roots are of interest to us. If the equation has no real roots, the problem has no solution.

There may also be situations in which real solutions to the quadratic equation may not be solutions to the problem. For example, length and speed cannot be negative, and a number of people cannot be fractional. These limitations will be exemplified in the following problems. **In a given problem the limitations on the variable used should be clearly stated.**

Example 1

Find a real number such that the product of two less than the number and three more than twice the number is 4.

Solution: Let x be the real number; that is, $x \in \mathrm{Re}$.
Then the only other restriction on x is that

$$(x-2)(2x+3)=4.$$
$$2x^2-x-6=4$$
$$2x^2-x-10=0$$
$$(2x-5)(x+2)=0$$
$$x=\tfrac{5}{2} \quad \textit{or} \quad x=-2$$

Check: (i) $x=\frac{5}{2}$

Two less than the number is $\frac{5}{2}-2=\frac{1}{2}$.
Three more than twice the number is $3+2(\frac{5}{2})=8$.
Their product is $\frac{1}{2}(8)=4$.
$\frac{5}{2}$ is a solution.

(ii) $x=-2$

Two less than the number is $-2-2=-4$.
Three more than twice the number is $3+2(-2)=-1$.
Their product is $(-1)(-4)=4$.
-2 is also a solution.

$\therefore$ the real number is $\frac{5}{2}$ or -2.

Example 2

The sum of the square of a certain negative integer and seven times the integer is eighteen. What is the integer?

Solution: Let the negative integer be x; that is, $x \in \mathrm{I}^-$.

Then $x<0$ and $x^2+7x=18$.
$$x^2+7x-18=0$$
$$(x+9)(x-2)=0$$
$$x=-9 \quad \textit{or} \quad x=2$$

Check: The root 2 may be rejected immediately as a solution to the problem since $2 \notin \mathrm{I}^-$; that is, it is not a negative integer.
We then check $x=-9$ as a possible solution.

The square of -9 is 81.
Seven times -9 is -63.
Their sum is 18.

Then $x=-9$ is the only solution.

Example 3

A triangle in which the base is four inches shorter than the altitude to that base has an area of $\frac{21}{2}$ square inches. Find the length of this base and the corresponding altitude.

Solution: Let the length of the altitude be x inches, $x \in \mathrm{Ra}^+$.
The length of the base is then $(x-4)$ inches.
Obviously $x>4$ and

$$\frac{1}{2}x(x-4)=\frac{21}{2}.$$
$$x^2-4x=21$$
$$x^2-4x-21=0$$
$$(x-7)(x+3)=0$$
$$x=7 \quad \text{or} \quad x=-3$$

Again we may reject the root $x=-3$ as a solution because $-3 \in \mathrm{Ra}^+$; this corresponds to the geometric condition that the altitude cannot have a negative length. The length of the altitude is 7 inches and that of the corresponding base is 3 inches.

Example 4

Is it possible to bend a piece of wire ten inches long to form the legs of a right triangle of area fifteen square inches?

Solution: Suppose the wire be bent at a point x inches from the one end. The legs of the right triangle will be x and $(10-x)$ inches in length respectively.
Then $x \in \mathrm{Re}^+$ and $0<x<10$ and the area of the triangle is 15 square inches so that

$$\frac{1}{2}x(10-x)=15.$$
$$x(10-x)=30$$
$$x^2-10x+30=0$$
$$x=\frac{10\pm\sqrt{-20}}{2}$$
$$=\frac{10\pm 2i\sqrt{5}}{2}$$
$$=5\pm i\sqrt{5}$$

But $5\pm i\sqrt{5} \notin \mathrm{Re}$. There are no real solutions to the quadratic equation we set up to solve our problem. This means that it is not possible to bend the wire in the required way.

EXERCISE 7—10

Use a quadratic equation to solve the following problems. In each case state any limitations on the variable used.

1. Find two integers whose product is 1728 and whose sum is 96.
2. A given integer and nine less than twice the integer have -9 as product. Determine the integer.
3. Find two integers whose product is 144 and whose difference is 10.
4. The sum of two numbers is 24 and the sum of their squares is 416. What are the numbers?
5. Find the dimensions of a rectangle of area 28 square inches if the width is 12 inches less than the length.
6. Find two numbers such that their sum is 25 and the difference of their squares is 75.
7. Each side of a given square is increased by two inches, thereby increasing its area by 44 sq. in. Find the area of the original square.
8. Find two consecutive odd integers whose product is 255.
9. Find two consecutive even integers whose product is 288.
10. Find two consecutive even integers whose product is 1093.
11. The sum of the squares of two consecutive odd integers is 290. Find the integers.
12. The sum of the squares of two consecutive odd integers is 493. Find the integers.
13. The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.
14. The average of a number and its reciprocal is $\frac{17}{8}$. Find the number.
15. A rectangular box is constructed with equal height and width and the length one inch greater than the width. The total surface area of the box is 240 square inches. Find the dimensions.
16. The altitude of a triangle is 2 inches longer than the corresponding base and the area is 40 square inches. Find this altitude and base.
17. Is it possible to have a right-angled triangle whose hypotenuse measures 17 inches and whose legs differ by 7 inches? If so, what are the lengths of the legs?

18. Two teachers, Mr. Joy and Mr. Cheer, were given 71 pupils to share. Each teacher wanted to have a square array of students' desks in his room, and Mr. Cheer, having the larger room, agreed that the side of his square would contain one desk more than that in Mr. Joy's room. Were the teachers able to place the pupils in their rooms in the way they wanted? (To be continued in (19).)

19. Mr. Joy, being rather quick at mathematics, said "Look here, Cheer, I'm willing to give up a square array; I'll have one more row than you, but each of my rows will contain one fewer desk than your rows." How then would the pupils be distributed?

20. The sum of the squares of three consecutive integers is 50. What are the integers?

21. The sum of the squares of a number, twice that number, and one plus twice that number, is 246. What is the number?

22. The sum of the squares of a number, that number plus a, and that number plus $2a$ is $14a^2$. What is the number?

23. Otto Basher and his neighbour Kurt E. S. Driver both drive 20 miles home from work each day on the freeway. The day before their holidays started Otto said to Kurt, "You drive home at your usual rate of speed; I'll average 20 m.p.h. faster than you and I'll get home in 10 minutes less time." What was Kurt's average rate of speed?

24. While visiting Otto in the hospital that evening, Kurt mentioned that he would be driving 150 miles to his cottage the next day. Smiling bravely, Otto pointed out that if he were able to travel he could leave an hour and a quarter after Kurt and, driving on the average 20 m.p.h. faster, he would arrive at the same time. What is Kurt's average speed on the highway?

25. If the average speed of a plane had been 40 m.p.h. less, the plane would have taken one hour longer to fly 1200 mi. What was the speed of the plane?

26. A sidewalk x' wide is built on the inside of a rectangular (30′ by 40′) grass plot. The resulting grassed area is then only half the original. How wide was the sidewalk?

27. The participants in the school concert were photographed in the gymnasium before the performance. The chairs in the gymnasium were set in a square array. Mr. Screech's Glee Club filled all but four of the rows. Mr. Squawk's Orchestra filled six fewer rows. A total of 240 were photographed. How many were in the orchestra?

28. What is the length of the diagonal of a square of area 17 square inches?

29. If 7 be added to a number, the sum is 30 times the reciprocal of the number. What is the number?

30. If a ball is thrown vertically upwards with an initial velocity of 88 ft. per sec., its distance d ft. above ground after t seconds is given by the formula $d=88t-16t^2$. Find when the ball will be 120 feet above the ground. Explain the two solutions.

31. The sum of the reciprocal of an integer and three times the reciprocal of the next integer is $\frac{11}{24}$. What is the integer?

32. Albert dropped out of school and sold his school books for \$24; he lost as much per cent of their cost as the books cost in dollars. What was this cost?

33. A violent storm bent a 50-foot-high T.V. mast standing on a flat roof so that its top touched the roof 15 ft. from the foot of the mast. What was the height of the part left standing?

34. In a certain construction job the relation between height h ft. and thickness w ft. is given by the relation $w^2+0.19\ wh-0.18h^2=0$. Find the width w (to the nearest foot) if $h=4$.

35. A local train takes one hour longer than an express train to travel 210 mi. The express train travels 20 m.p.h. faster than the local. Find, to the nearest tenth of a mile per hour, the speed of each train.

36. The formula $$\frac{Hd}{L}=\frac{4v^2+5v-2}{1200}$$ gives the flow of water in a long horizontal pipe connected with the bottom of a reservoir. H (in feet) is the depth of water in the reservoir, d (in inches) the diameter of the pipe, L (in feet) the length of the pipe, and v (in feet per second) the velocity of the water flowing through the pipe. Find the velocity (to the nearest tenth of a foot per sec.) of water in a pipe 600 ft. long and 4 inches in diameter, if the reservoir contains 36 ft. of water.

37. The students of Brandex High School all agreed to purchase a copy of the yearbook, at a price that would just cover the publication cost of \$3300. It was found that the printer forgot to include the picture of the 50-man Glee Club, and the members of the club, in high dudgeon, refused to buy the book, thereby increasing the cost to the remaining students by 50¢ per book. How many students were there at Brandex?

38. Earlier in the season a number of students chartered a bus for \$80 for an out-of-town football game. The Students' Council, at the last minute, refused to pay for tickets for the eight cheerleaders and they, in high dudgeon, decided to stay home. Each of the other students then had to pay 50 cents extra. How many students went on the trip?

39. Last year the School Board granted the students of Dudgeon High a certain per capita rate for their next year's extracurricular activities and the resulting amount of \$12,000 was used to plan the budget. This year, 50 additional students enrolled and the per capita rate dropped by 40¢ per student. How many students enrolled this year?

40. Find the dimensions of a rectangle of perimeter 62 inches if a diagonal has a length of 25 inches.

41. The sum S_n of the n natural numbers is

$$S_n = 1+2+3+ \ldots +n = \frac{n(n+1)}{2}.$$

Albert has started adding the natural numbers and has reached 2278. How many numbers has he counted?

42. Norbert started adding the natural numbers but became distracted and forgot the sum. Thereupon he added the *next* 67 natural numbers and reached a sum of 3618. Which natural numbers did he add before the distraction occurred?

43. The equation of a certain curve is $2x^2-5xy+y^2+6x+3y-5=0$. Are there any points on the curve with integral coordinates (x, y) such that $-2 \leq y \leq 2$?

7 11 Quadratic Inequalities (Supplementary)

Consider the problem of finding the values of k for which the equation

$$3x^2+2kx-2k=0$$

has real roots. The discriminant

$$\begin{aligned} D &= (2k)^2-4(3)(-2k) \\ &= 4k^2+24k \\ &= 4k(k+6) \end{aligned}$$

must be positive or zero. The product $k(k+6)$ will be positive if both factors are positive or both negative.

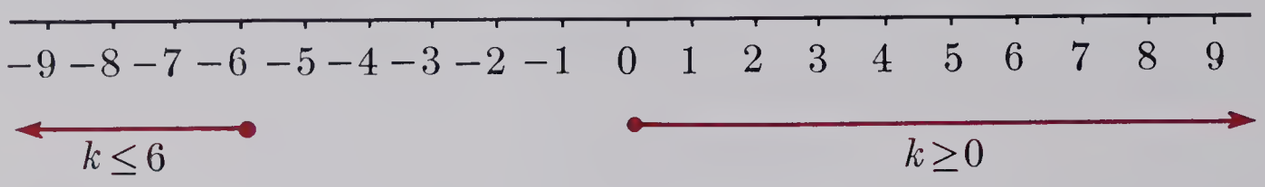
$$\therefore\ k \geq 0 \text{ and } k+6 \geq 0 \quad \text{or} \quad k \leq 0 \text{ and } k+6 \leq 0.$$

The "and" in each case indicates the intersection of two sets and the "or" indicates a union:

$$\begin{aligned} &\{k|k \geq 0\} \cap \{k|k \geq -6\} \cup \{k|k \leq 0\} \cap \{k|k \leq -6\} \\ &= \{k|k \geq 0\} \cup \{k|k \leq -6\} \end{aligned}$$

The set of required values of k is thus $\{k|k \leq -6 \text{ or } k \geq 0\}$.

In this problem we have solved a particular form of a quadratic inequality, the inequality $ax^2+bx+c \geq 0$ in which ax^2+bx+c can be factored. In this case the quadratic inequality was $k^2+6k=k(k+6) \geq 0$. The diagram shows the solution set.



Example

Solve the quadratic inequality $x^2-5x-6<0$.

Solution: The quadratic inequality in factored form is

$$(x-6)\,(x+1)<0.$$

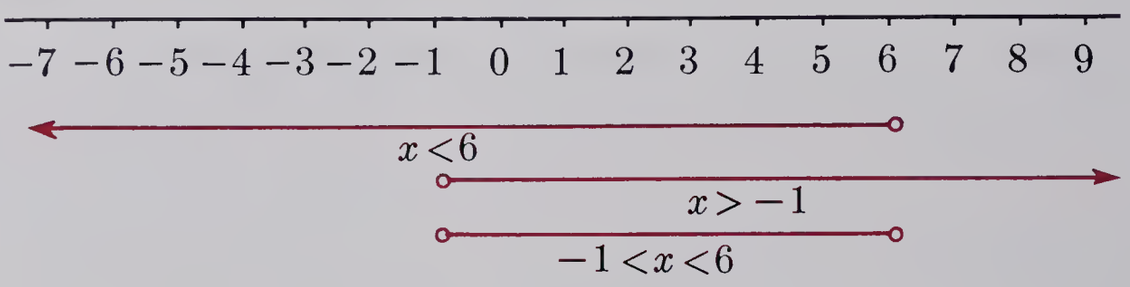
In this case, the two factors must be opposite in sign:

$x-6<0$ and $x+1>0$ *or* $x-6>0$ and $x+1<0$.

The solution set is then

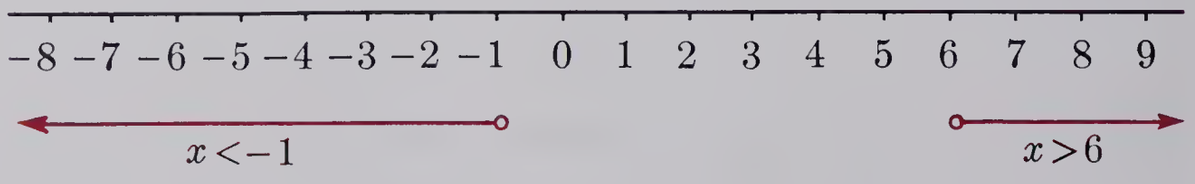
$$\{x|x<6\} \cap \{x|x>-1\} \cup \{x|x>6\} \cap \{x|x<-1\}.$$

Consider first the set $\{x|x<6\} \cap \{x|x>-1\}$. The diagram shows that this set contains all real numbers x lying between (but not including) -1 and 6 and hence is the set $\{x|-1<x<6\}$.



$$\{x|x<6\} \cap \{x|x>-1\} = \{x|-1<x<6\}$$

Now consider the set $\{x|x>6\} \cap \{x|x<-1\}$. The diagram shows that this set is the empty set.



$$\{x|x>6\} \cap \{x|x<-1\} = \phi$$

Thus the solution set of the quadratic inequality $x^2-5x-6<0$ is $\{x|-1<x<6\} \cup \phi$; that is, the solution set is $\{x|-1<x<6\}$.

EXERCISE 7—11

Find the graph of each of the following sets:

1. $\{x|x^2-4>0\}$
2. $\{x|x^2-4<0\}$
3. $\{x|x^2+4>0\}$
4. $\{x|x^2+4<0\}$
5. $\{x|x^2-x-12<0\}$
6. $\{x|x^2-x-12>0\}$
7. $\{x|x^2-7x>18\}$
8. $\{x|x^2-7x<18\}$
9. $\{x|x^2+1\leq 2x\}$

Solve the following real quadratic inequalities:

10. $2x^2-x-3>0$
11. $2x^2-x-3<0$
12. $6x^2+5x-6<0$
13. $30x^2-11x+1<0$
14. $3x^2+8x-3>0$
15. $(x+1)(x-2)<(x+1)^2$
16. $x^2-7x+12\geq 0$
17. $x^2-7x+12\leq 0$
18. $6x^2+7x-20\leq 0$

Find k so that the roots of the following equations are
(a) nonreal (b) real and unequal:

19. $9x^2+kx+1=0$
20. $(k-1)x^2+1=kx$
21. $kx^2+4x+5=k$
22. $x^2-(k+3)x+3k=0$
23. $kx^2+5kx+25=0$
24. $x^2+2(k-2)x-(k-2)=0$
25. $kx^2+(2k-1)x-2=0$
26. $k^2x^2+kx+1=0$

7.12 Chapter Summary

The general quadratic equation $ax^2+bx+c=0$.

Roots of the quadratic equation.

Correspondence between the roots of $ax^2+bx+c=0$ and the points of intersection of the parabola whose equation is $y=ax^2+bx+c$ and the x-axis.

Graphical estimation of the roots of a quadratic equation.

Solution of the quadratic equation by factoring.

Special quadratic equations (the variable is $\sin x$, 2^x, etc.).

Solution of the quadratic equation by completing the square.

The quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The discriminant $D=b^2-4ac$ and its use in determining the character of the roots:

DISCRIMINANT	a, b, c	ROOTS
$D<0$	real	nonreal
$D=0$	real	equal real
$D>0$	real	unequal real
$D<0$	rational	nonreal
$D=0$	rational	equal rational
$D>0$ and the square of a rational number	rational	unequal rational
$D>0$ and not the square of a rational number	rational	unequal irrational

Expression of nonreal roots in the form $a+ib$.

Equations involving radicals; check all results.

Problems giving rise to quadratic equations (disregard roots of the equation which do not provide solutions to the problem).

Quadratic inequalities (supplementary).

REVIEW EXERCISE 7—12

Find the points of intersection with the y-axis of the parabolas with the following equations. Sketch the curves involved and estimate their points of intersection with the x-axis, indicating whether there seem to be two distinct points, only one, or none.

1. $y=4x^2-12x+9$ 2. $y=3x^2-14x-5$ 3. $y=x^2+x+4$

4. Estimate the points of intersection of the above parabolas with the lines $y=3$ and $y=-3$.

Solve by factoring:

5. $6x^2-17x+5=0$ 6. $-2x^2+3x+2$ 7. $8x^2+18x-5=0$

Solve by completing the square:

8. $x^2-5x+16=0$ 9. $3x^2-2x+5=0$ 10. $x^2-6x-10=0$

Solve using any method:

11. $-x^2+3x+4=0$ 12. $x^2-4x-\frac{1}{2}=0$ 13. $6x^2+4x-1=0$

14. Find the points of intersection of the parabola $y=5x^2+4x-3$ with the lines (a) $y=4$ (b) $y=-4$.

15. Solve the equation $2\tan^2 x+3\tan x-1=0$ for x in radians if $0\leq x\leq\frac{\pi}{2}$.

16. Solve for x if

$$4^{2x}-3(4^{x+1})-64=0.$$

17. Develop the quadratic formula.

Solve for x:

18. $3x(x-2)-2x(x+5)=(x+2)(x+3)$

19. $2(x^2-8)^2+(x^2-8)-3=0$

20. $\frac{2}{3}+\frac{x+1}{2x-3}=\frac{3x}{x+1}$

21. $\frac{3}{x+2}-\frac{1}{2x-1}=6$

Without solving, describe the roots of the following equations:

22. $6x^2+x+2=0$

23. $6x^2-10\sqrt{6}x+25=0$

24. $4x^2-7x-10=0$

25. $\sqrt{2}x^2+6x-\sqrt{3}=0$

26. For what values of k will the equation $2x^2-x+k-6=0$ have (a) equal roots (b) real roots (c) nonreal roots?

Find real values of x (if possible) such that:

27. $\sqrt{x-2}+\sqrt{4x+1}=\sqrt{11-x}$

28. $\sqrt{3x-1}=1-\sqrt{x-3}$

29. $\frac{1}{\sqrt{2x+1}}+\sqrt{3x}=\sqrt{2x+1}$

30. $\sqrt{x-3}+\sqrt{x-2}=\sqrt{x-1}$

31. The product of an integer decreased by 5 and the same integer increased by 1 is twice the next integer. What is the integer?

32. Does it follow that $x-1=3$ if $(x-1)(x-2)=3(x-2)$?

33. The sum of the squares of two integers differing by 3 is 425. What are the integers?

34. The sum of the reciprocals of two integers differing by 3 is $\frac{7}{36}$. What are the integers?

35. The athletic banquet at Dudgeon High cost \$450 and the committee decided that the cost would be shared equally by all those attending. Seventy-five of those eligible to attend did not and, as a result, the cost to each of those attending was 50 cents higher than it would otherwise have been. How many were eligible to attend?

36. Kurt drove 20 miles at his usual average speed, but Otto, driving 10 m.p.h. faster, drove 50 miles in a half hour longer. What were their rates of speed?

37. If $s=40t-16t^2$ gives the distance s (in feet) above ground at time t (in seconds) of a ball thrown vertically upwards, when is the ball nine feet above ground? Explain the two answers.

38. Find points with integral coordinates (x, y) on the curve

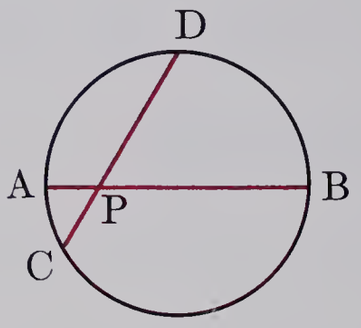
$$3x^2-xy+y^2-4x+5y-2=0$$

with $-2\leq x\leq 2$.

39. We learn in geometry that if two chords AB and CD of a circle intersect at point P, then

$$PA\times PB=PC\times PD.$$

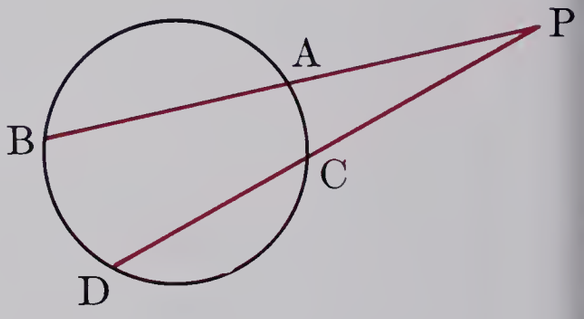
Find PC if PA = 2, PB = 16, and CD = 12 inches.



40. Again from geometry we learn that if two secants PAB and PCD be drawn from an external point P to a circle, then

$$PA\times PB=PC\times PD.$$

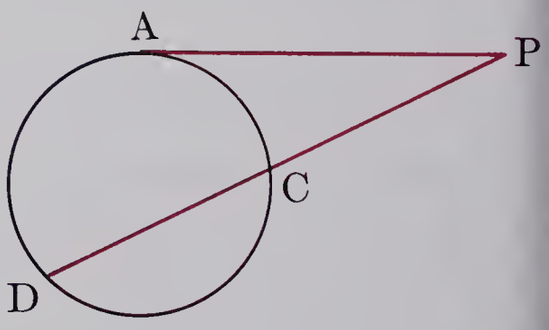
Find PC, given that PA = 5, AB = 3, and CD = 6 inches.



41. If the points of intersection, A and B, of secant PAB coincide, then PA is called a *tangent* line to the circle, and then

$$\overline{PA}^2=PC\times PD.$$

Find PD, given that $PA=\frac{4}{3}$ and CD = 2 inches.



(Supplementary) Solve the following quadratic inequalities:

42. $15x^2-8x+1>0$

43. $2x^2+7x-15\leq 0$

44. For what values of k will the equation

$$(k+\tfrac{5}{4})x^2+3kx+9=0$$

have nonreal roots?

Chapter 8

Polynomial Equations

8 1 Introduction

In this chapter we will study certain results in connection with the real quadratic equation

$$ax^2+bx+c=0;$$

many of these results will also apply to the real cubic equation

$$ax^3+bx^2+cx+d=0,$$

to the real quartic equation

$$ax^4+bx^3+cx^2+dx+e=0,$$

and even to the general real polynomial equation of degree n,

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ \ldots\ +a_{n-1}x+a_n=0.$$

(In all of these equations, the coefficients are real numbers.)

A polynomial in the indeterminate or variable x (that is, x is an element of a certain given set) is an expression of the form

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ \ldots\ +a_{n-1}x+a_n,$$

in which n is a nonnegative integer. A real polynomial is one in which the coefficients are real numbers. (A rational polynomial is one in which the coefficients are rational numbers, etc.) The equations we discuss in this chapter are all real polynomial equations, that is, real polynomials set equal to zero, and the variable x will be from the set of real numbers. For example, $x^2-3x+2=0$ and $\frac{1}{2}x^5-\pi x+\sqrt{2}=0$ are real polynomial equations.

The highest power n of the indeterminate x in a polynomial is called the degree of the polynomial. For example,

$3x+2$ is a polynomial of degree 1 (or a linear polynomial);

x^2-3x+2 is a polynomial of degree 2 (or a quadratic polynomial);

x^3-3x+2 is a polynomial of degree 3 (or cubic polynomial);

x^4-3x+2 is a polynomial of degree 4 (or quartic polynomial); and finally

x^n-3x+2 is a polynomial of degree n (or n-ic polynomial).

A real number such as 2 may be regarded as a polynomial of degree 0.

We also speak of the degree of a polynomial equation. This is the degree of the polynomial which has been set equal to zero to produce the polynomial equation. Thus $3x+2=0$, $x^2-3x+2=0$, $x^3-3x+2=0$, $x^4-3x+2=0$, and $x^n-3x+2=0$ are polynomial equations of degree 1, 2, 3, 4, and n respectively.

It is often convenient to have a symbol to use in referring to a particular polynomial. If a polynomial is known, for example the polynomial x^2-2x-3, we may refer to it as $P(x)$ and say that

$$P(x)=x^2-2x-3.$$

Further we shall agree that $P(a)$ will mean the value of the polynomial, that is, the number obtained, when x is replaced by a.

Then, for this particular polynomial $P(x)$, we have:

$$P(2)=(2)^2-2(2)-3=-3$$

$$P(-1)=(-1)^2-2(-1)-3=0$$

$$P(0)=(0)^2-2(0)-3=-3$$

$$P(a)=(a)^2-2(a)-3=a^2-2a-3$$

$$P(y)=(y)^2-2(y)-3=y^2-2y-3$$

$$P\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)^2-2\left(\frac{1}{x}\right)-3=\frac{1}{x^2}-\frac{2}{x}-3.$$

In practice each middle step above is usually performed mentally. It is obvious that an unlimited number of values of the polynomial can be found in this way by assigning all real values to x in turn.

If there is a value of x for which the polynomial $P(x)$ has the value 0, that value of x is called a zero of the polynomial. In our example, -1 is a zero of the polynomial x^2-2x-3.

Now consider the corresponding polynomial equation

$$x^2-2x-3=0.$$

Here again it is possible to assign any real value of x, but the statement expressed, that x^2-2x-3 equals 0, will usually not be true. For example,

$$(2)^2-2(2)-3\neq 0.$$

The only values of x for which the statement will be true are the zeros of the polynomial. These values of x are called the roots of the polynomial equation; that is, they constitute the solution set of the equation. We have seen that

$$(-1)^2-2(-1)-3 \quad \text{and} \quad (3)^2-2(3)-3$$

both equal 0, so that -1 and 3 are zeros of the polynomial x^2-2x-3 or roots of the polynomial equation $x^2-2x-3=0$.

EXERCISE 8—1

State the degree of the following polynomials:

1. $x^3-3x^2+\frac{1}{2}x-6$
2. $3y^2-4y$
3. $5-6x+\frac{1}{2}x^4$
4. $-6x^2+x^{10}-\sqrt{3}x$
5. y^{100}
6. $100-t^{99}$
7. Which of the polynomials above are integral polynomials? rational polynomials? real polynomials?

State the degree of the following polynomial equations:

8. $3x^4-6x^2+5=0$
9. $6x=x^3-3x^2$
10. $t^4+3t=t^4-1$
11. $4a^{99}-a^{88}=6$
12. $\frac{1}{2}x^3-2x=\frac{1}{4}x^3-5x$
13. $3x-2=\frac{1}{2}x^5$

Find the indicated values of the following polynomials:

14. $P(x)=x^2-6x+7$; $P(0)$, $P(1)$, $P(-1)$, $P(k)$, $P\left(\frac{1}{x}\right)$, $P\left(\frac{1}{x-1}\right)$
15. $Q(y)=4y^2-y+\frac{1}{2}$; $Q(0)$, $Q(1)$, $Q(\sqrt{2})$, $Q(\frac{1}{2})$, $Q(x)$
16. $R(t)=t^2-5t-2$; $R(0)$, $R(-1)$, $R(-\frac{1}{2})$, $R(\sqrt{5})$, $R(1+t)$
17. $S(x)=x^3-3x^2+2x-1$; $S(0)$, $S(-1)$, $S(a)$, $S\left(\frac{1}{x}\right)$, $S(\sqrt{2})$
18. $T(y)=y^4-2y+3$; $T(0)$, $T(1)$, $T(-1)$, $T\left(\frac{1}{y}\right)$

Find the zeros of the following polynomials:

19. $P(x)=x^2-6x+7$
20. $P(y)=3y^2-y-4$
21. $P(x)=x^2-x-7$
22. $P(x)=5x^2-x-1$
23. $P(t)=t^3$
24. $P(y)=y^3-y$
25. Write the corresponding polynomial equations in (19)-(24) and state the roots of these equations.

8.2 Division of P(x) by the Linear Polynomial x—a

In this section we review an algebraic operation already known to the student, the division of one polynomial by another. We are particularly interested in the division of a polynomial by a polynomial of the form $x-a$, that is, by a linear polynomial in which the coefficient of x is 1.

We know that division of nonzero integers is a unique operation, although its result can be expressed in different but equivalent forms. Thus

$$\tfrac{2}{7}\,\tfrac{2}{7}=3+\tfrac{1}{7}=2+\tfrac{8}{7}=4-\tfrac{6}{7}=\text{ etc.}$$
$$\tfrac{1}{2}\,\tfrac{2}{2}=6=5+\tfrac{2}{2}=7-\tfrac{2}{2}=\text{ etc.}$$

However when we refer to the division of one integer by another, we normally mean the result in which the remainder is nonnegative and less than the divisor; that is, we write

$$\tfrac{4}{6}\,\tfrac{9}{6}=8+\tfrac{1}{6},$$
$$\text{rather than } \tfrac{4}{6}\,\tfrac{9}{6}=9-\tfrac{5}{6},$$

although both statements are correct.

A similar situation exists in the division of one polynomial by another. The division operation is unique, although the result can be expressed in different, but equivalent, forms. For example,

$$\frac{x^3+3x^2-3x+1}{x}=x^2+\frac{3x^2-3x+1}{x}$$
$$=x^2+3x+\frac{-3x+1}{x}$$
$$=x^2+3x-3+\frac{1}{x}$$

shows that x^3+3x^2-3x+1 divided by x can have a quotient x^2 and remainder $3x^2-3x+1$, or a quotient x^2+3x and a remainder $-3x+1$, or a quotient x^2+3x-3 and a remainder 1. The last expression is the one normally referred to as the result of dividing x^3+3x^2-3x+1 by x, but the other two are equally valid.

In the operation of dividing one polynomial by another, the expression normally desired is that involving a polynomial quotient and a remainder which is a polynomial of degree less than that of the divisor. For example, the operation of dividing $x^4-5x^4+6x^2-2x+7$ by the polynomial x^2-x+2 is normally continued until a remainder is obtained which is a polynomial of degree less than two (the degree of the divisor).

$$
\begin{array}{r}
x^2-4x \\
x^2-x+2\overline{)x^4-5x^3+6x^2-2x+7} \\
\underline{x^4-x^3\ +2x^2} \\
-4x^3+4x^2-2x \\
\underline{-4x^3+4x^2-8x} \\
6x+7
\end{array}
$$

$\therefore$ the quotient is x^2-4x and the remainder is $6x+7$.

$$\therefore \frac{x^4-5x^3+6x^2-2x+7}{x^2-x+2} = x^2-4x+\frac{6x+7}{x^2-x+2}.$$

If we are dividing by a linear polynomial $x-a$, the division is carried out until a polynomial remainder is obtained with degree less than that of the divisor $x-a$. But the only such polynomial is a constant. Thus the division procedure is continued until a constant is obtained as the remainder.

Example

Divide $2x^3-5x^2+4x-7$ by $x+2$.

Solution:

$$
\begin{array}{r}
2x^2-9x+22 \\
x+2\overline{)2x^3-5x^2+4x-7} \\
\underline{2x^3+4x^2} \\
-9x^2+4x \\
\underline{-9x^2-18x} \\
22x-7 \\
\underline{22x+44} \\
-51
\end{array}
$$

$\therefore$ the quotient is $2x^2-9x+22$ and the remainder is -51.

$$\therefore \frac{2x^3-5x^2+4x-7}{x+2} = 2x^2-9x+22+\frac{-51}{x+2} \text{ for } x \neq -2.$$

This last expression may also be written

$$2x^3-5x^2+4x-7 = (x+2)\,(2x^2-9x+22)-51$$

and the division can be checked by "multiplying out" in the right member. This expression gives the dividend as the remainder plus the product of the quotient and the divisor. This is a very important expression.

In general, if a polynomial $P(x)$ be divided by $x-a$ to obtain a quotient $Q(x)$ and remainder R, we may write

$$\frac{P(x)}{x-a} = Q(x)+\frac{R}{x-a}$$

or

$$P(x) = (x-a)Q(x)+R.$$

The expression $(x-a)\ Q(x)+R$ is just another way of writing the polynomial $P(x)$.

EXERCISE 8—2

Divide the following polynomials $P(x)$ by the indicated $x-a$ to obtain a quotient $Q(x)$ and a constant remainder R. In each case express the polynomial $P(x)$ in the form $(x-a)\,Q(x)+R$.

1. $P(x)=x^2-6x+7$;	(i) $x-1$	(ii) $x+3$	(iii) $x+\frac{1}{2}$
2. $P(x)=x^2-13x+36$;	(i) $x+3$	(ii) $x-4$	(iii) $x-\frac{1}{3}$
3. $P(x)=4x^2-6x+3$;	(i) $x-2$	(ii) $x+5$	(iii) $x+\frac{1}{4}$
4. $P(x)=x^3-6x^2+3x-2$;	(i) $x+1$	(ii) $x-3$	(iii) $x-\frac{3}{2}$
5. $P(x)=3x^3-5x^2+x+1$;	(i) $x+2$	(ii) $x-1$	(iii) $x+\frac{1}{3}$
6. $P(x)=4x^3-6x+2$;	(i) $x-1$	(ii) $x+1$	(iii) $x-\frac{1}{2}$
7. $P(x)=6x^4-2x^2+3$;	(i) $x+2$	(ii) $x-2$	(iii) $x+\frac{1}{3}$
8. $P(x)=2x^5-3$;	(i) $x-1$	(ii) $x+2$	(iii) $x+\frac{1}{2}$

8 3 Zeros and Factors of Polynomials

Consider the following values of the polynomial $P(x)=x^2-3x-4$:

$P(-3)=(-3)^2-3(-3)-4=14$ $\quad$ $P(2)=(2)^2-3(2)-4=-6$

$P(-2)=(-2)^2-3(-2)-4=6$ $\quad$ $P(3)=(3)^2-3(3)-4=-4$

$P(-1)=(-1)^2-3(-1)-4=0$ $\quad$ $P(4)=(4)^2-3(4)-4=0$

$P(0)=(0)^2-3(0)-4=-4$ $\quad$ $P(5)=(5)^2-3(5)-4=6$

$P(1)=(1)^2-3(1)-4=-6$ $\quad$ $P(6)=(6)^2-3(6)-4=14$

Notice that -1 and 4 are zeros of the polynomial $P(x)$.

Now examine the factors of $P(x)$.

$$P(x)=x^2-3x-4=(x+1)\,(x-4).$$

Obviously examination of this factored form would tell us immediately that $P(x)$ has value 0 for the values -1 and 4 of x. This is so because one of the factors of $P(x)$ would then have the value 0.

The following values of the polynomial $P(x)=x^3-2x^2+x-2$,

$P(-2)=-20$ $\quad$ $P(1)=-2$

$P(-1)=-6$ $\quad$ $P(2)=0$

$P(0)=-2$ $\quad$ $P(3)=10$,

show that 2 is a zero of the polynomial $P(x)$. Now certainly if $x-2$ were a factor of $P(x)$, 2 would be a zero of $P(x)$ since $P(2)$ would then have a factor $2-2$ or 0. The question that arises is "Is $x-2$ a factor of x^3-2x^2+x-2?" We check by division.

$$\begin{array}{r} x^2+0x+1 \\ x-2\overline{)x^3-2x^2+x-2} \\ \underline{x^3-2x^2} \\ 0+x-2 \\ \underline{0+x-2} \end{array}$$

Thus

$$x^3-2x^2+x-2=(x-2)\,(x^2+1)$$

and so $x-2$ is a factor of x^3-2x^2+x-2.

Example

Find any integral zeros a of the polynomial

$$\mathrm{P}(x)=x^4+2x^3-5x^2-4x+6,$$

in the domain $-4\leq x\leq 4$. For every zero a, check by division to see if $x-a$ is a factor of $\mathrm{P}(x)$.

Solution:

$\mathrm{P}(-4)=70$ $\quad$ $\mathrm{P}(0)=6$
$\mathrm{P}(-3)=0$ $\quad$ $\mathrm{P}(1)=0$
$\mathrm{P}(-2)=-6$ $\quad$ $\mathrm{P}(2)=10$
$\mathrm{P}(-1)=4$ $\quad$ $\mathrm{P}(3)=84$
$\mathrm{P}(4)=294$

The zeros of $\mathrm{P}(x)$ in the required domain are -3 and 1.

By division we find that

$$x^4+2x^3-5x^2-4x+6=(x+3)\,(x^3-x^2-2x+2)$$

and $x^4+2x^3-5x^2-4x+6=(x-1)\,(x^3+3x^2-2x-6)$,

so that $x-a$ is a factor of $\mathrm{P}(x)$ for the zeros a of $\mathrm{P}(x)$ that we have found.

Check that $x-1$ is a factor of x^3-x^2-2x+2 and that $x+3$ is a factor of x^3+3x^2-2x-6.

EXERCISE 8—3

Find any integral zeros a of the polynomial $\mathrm{P}(x)$ in the domain indicated and check by division to see if $x-a$ is a divisor of $\mathrm{P}(x)$. If it is, express $\mathrm{P}(x)$ in the form

$$\mathrm{P}(x)=(x-a)\,\mathrm{Q}(x).$$

In (1)-(6) use the domain $-5\leq x\leq 5$.

1. $\mathrm{P}(x)=x^2+x-6$
2. $\mathrm{P}(x)=x^2-5x+6$
3. $\mathrm{P}(x)=x^2-4x$
4. $\mathrm{P}(x)=x^2+4x+3$
5. $\mathrm{P}(x)=x^3-2x^2+x-2$
6. $\mathrm{P}(x)=x^3+x^2-2x-2$

In (7)-(12) use the domain $-2 \leq x \leq 2$.

7. $P(x) = 2x^3 - 2x^2 + 3x - 3$
8. $P(x) = 3x^3 + 6x^2 - 4x - 8$
9. $P(x) = x^4 + 2x^3 + x + 2$
10. $P(x) = x^4 + 5x^2 + 4$
11. $P(x) = x^4 - 5x^2 + 4$
12. $P(x) = x^4 - 2x^3 + x^2 - x - 2$

8.4 The Factor Theorem and its Converse

The work of the preceding section should convince you of the validity of the following important theorem: (A proof may be obtained from the work of the supplementary Section 9-8.)

The Factor Theorem

If a is a zero of the polynomial $P(x)$, that is, if $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

Example 1

Are (a) $x - 2$ and (b) $x + 4$ factors of $P(x) = x^3 - 4x^2 + 6x - 4$?

Solution: (a) By the Factor Theorem, $x - 2$ is a factor of $P(x)$ if $P(2) = 0$.

$P(2) = (2)^3 - 4(2)^2 + 6(2) - 4 = 0$

$\therefore x - 2$ is a factor of $P(x)$.

Indeed, $x^3 - 4x^2 + 6x - 4 = (x - 2)(x^2 - 2x + 2)$.

(b) $x + 4$ is a factor of $P(x)$ if $P(-4) = 0$.

$P(-4) = (-4)^3 - 4(-4)^2 + 6(-4) - 4$

$= -64 - 64 - 24 - 4$

$\neq 0$

$\therefore x + 4$ is not a factor of $P(x)$.

Now it is certainly true that is $x - a$ is a factor of $P(x)$, that is,

if $P(x) = (x - a)Q(x)$ for some polynomial $Q(x)$,

then a is a zero of $P(x)$,

since then $P(a) = 0 \cdot Q(a) = 0$.

This result is the converse of the Factor Theorem.

Converse of Factor Theorem

a is a zero of $P(x)$, that is, $P(a) = 0$, if $x - a$ is a factor of $P(x)$.

The two theorems of this section may be combined into a single one.

THEOREM: $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Example 2

For what value of k will $x-\frac{1}{2}$ be a factor of

$$P(x)=2x^3-4x^2+3x+k?$$

Solution: By the Factor Theorem, $x-\frac{1}{2}$ is a factor of $P(x)$ if $P(\frac{1}{2})=0$, that is, if

$$P(\tfrac{1}{2})=\tfrac{1}{4}-1+\tfrac{3}{2}+k=0.$$
$$\therefore k=-\tfrac{3}{4}.$$

$\therefore x-\frac{1}{2}$ is a factor of $2x^3-4x^2+3x-\frac{3}{4}$.

EXERCISE 8—4

Use the factor theorem to test if the given $x-a$ are factors of the given polynomial $P(x)$.

1. $P(x)=3x^2-14x+8$ (a) x (b) $x-1$ (c) $x+1$ (d) $x-2$ (e) $x+2$ (f) $x-4$ (g) $x+4$ (h) $x-100$

2. $P(x)=2x^2-9x-18$ (a) x (b) $x-1$ (c) $x+1$ (d) $x-\frac{1}{2}$ (e) $x+\frac{1}{2}$ (f) $x-3$ (g) $x-6$ (h) $x-18$

3. $P(x)=x^3-3x^2-10x+24$ (a) x (b) $x-1$ (c) $x+1$ (d) $x-2$ (e) $x+2$ (f) $x-3$ (g) $x+3$ (h) $x-4$

4. $P(x)=2x^3-7x^2+7x-2$ (a) x (b) $x-1$ (c) $x+1$ (d) $x-2$

5. $P(y)=y^3-6y^2+11y-6$ (a) y (b) $y-1$ (c) $y+1$ (d) $y-2$ (e) $y+2$ (f) $y-3$ (g) $y+3$ (h) $y+100$

In (6)-(16), find the values of k for which the given $x-a$ is a factor of the given polynomial.

6. $P(x)=x^2+5x+k$, $x+2$
7. $P(x)=-x^2+7x-2k$, $x-\frac{1}{2}$
8. $P(x)=x^2-k$, $x-\sqrt{2}$
9. $P(x)=3x^2-kx+\frac{1}{2}$, $x+\frac{1}{2}$
10. $P(x)=2x^2-\sqrt{2}\,x+k$, $x+\sqrt{2}$
11. $P(x)=x^3-5x^2+4k-3$, $x-2$

12. $P(x) = -2x^3+x^2-5k+2$, $x+3$

13. $P(x) = 2x^3+kx^2-6x-4$, $x+2$

14. $P(x) = 2x^3+kx^2-kx-4$, $x+2$

15. $P(x) = x^5-16x^3+4kx-7$, $x+1$

16. $P(x) = -x^5+\frac{1}{2}x^4-kx^3+2x+3k$, $x+\frac{1}{2}$

17. Use the factor theorem to show that $x-y$ is a factor of x^3-y^3.

18. Use the factor theorem to show that $x+y$ is a factor of x^3+y^3.

8 5 Factoring a Quadratic Polynomial Using the Factor Theorem

We have seen that the factors of a polynomial can be stated once we know the zeros of the polynomial. If a is a zero of $P(x)$, that is, if $P(a)=0$, then $x-a$ is a factor of $P(x)$.

Thus it is an easy matter to find the factors of a quadratic polynomial $P(x)=ax^2+bx+c$. To find these factors we need only know the zeros of $P(x)$, that is, the values of x for which $ax^2+bx+c=0$. But these are simply the roots of the quadratic equation $ax^2+bx+c=0$ which we may obtain from the quadratic formula. If this equation has real roots, then the quadratic polynomial has real linear factors as shown in the following examples.

Example 1

Factor $3x^2-5x-2$.

Solution: The zeros of $3x^2-5x-2$ are the roots of $3x^2-5x-2=0$. These are

$$x=\tfrac{1}{6}(5\pm\sqrt{25+24})=\tfrac{1}{6}(5\pm 7), \text{ that is, } 2 \text{ and } -\tfrac{1}{3}.$$

Then $x-2$ and $x+\frac{1}{3}$ are factors of $3x^2-5x-2$.
Does this mean that $3x^2-5x-2=(x-2)(x+\frac{1}{3})$?
Let us check:

$$(x-2)(x+\tfrac{1}{3})=x^2-\tfrac{5}{3}x-\tfrac{2}{3}=\tfrac{1}{3}(3x^2-5x-2).$$

Then $3x^2-5x-2=3(x-2)(x+\frac{1}{3})$
and a set of factors is $\{3, x-2, x+\frac{1}{3}\}$.
If we are interested in *linear* factors only, the set of such factors could be written $\{3x-6, x+\frac{1}{3}\}$ or $\{x-2, 3x+1\}$.

Example 2

Factor x^2-5x-3.

Solution: The roots of $x^2-5x-3=0$ are

$x=\frac{1}{2}(5\pm\sqrt{25+12})$, that is, $x=\frac{1}{2}(5+\sqrt{37})$ and $x=\frac{1}{2}(5-\sqrt{37})$.
Then $x-\frac{1}{2}(5+\sqrt{37})$ and $x-\frac{1}{2}(5-\sqrt{37})$ are factors of x^2-5x-3.

Let us see if their product is x^2-5x-3 or whether it must be multiplied by some constant as in Example 1.

$$\begin{aligned}(x-\tfrac{1}{2}(5+\sqrt{37}))\,(x-\tfrac{1}{2}(5-\sqrt{37})) &= [(x-\tfrac{5}{2})-\tfrac{1}{2}\sqrt{37}]\,[(x-\tfrac{5}{2})+\tfrac{1}{2}\sqrt{37}] \\ &= (x-\tfrac{5}{2})^2-(\tfrac{1}{2}\sqrt{37})^2 \\ &= x^2-5x+\tfrac{25}{4}-\tfrac{37}{4} \\ &= x^2-5x-3\end{aligned}$$

Notice how this product was first rearranged in the form (A−B)(A+B) to facilitate the multiplication. This is a very useful manoeuvre.

Example 3

Factor $2x^2-4x+7$.

Solution: The roots of $2x^2-4x+7=0$ are

$$\begin{aligned}x=\tfrac{1}{4}(4\pm\sqrt{16-56}) &= \tfrac{1}{4}(4\pm\sqrt{-40}) \\ &= 1\pm\tfrac{1}{2}\sqrt{10}\,i.\end{aligned}$$

The roots of this quadratic are not real numbers and so the polynomial $2x^2-4x+7$ has no *real* linear factors.
The quadratic polynomial $2x^2-4x+7$ is called irreducible.

DEFINITION: A real irreducible quadratic polynomial is one that can not be written as the product of two real linear factors.

For example, x^2+1 and x^2+4 are real irreducible quadratics.

EXERCISE 8—5

Find a set of real linear factors for the following quadratic polynomials where possible. In each possible case, check by multiplying the two linear factors to see if a constant factor is required (as in Example 1 above). State which quadratics are irreducible, that is, which do not have real linear factors.

1. x^2-5x+6	2. x^2+6x-5	3. y^2+6y+7
4. x^2+3x+5	5. t^2-4t-8	6. $3x^2+5x+2$
7. $3y^2+5y+1$	8. $4x^2+2x+7$	9. $12a^2-8a+1$
10. $6x^2-x-4$	11. $12b^2-32b+5$	12. $6y^2+2y-3$
13. $4x^2+2x-7$	14. $3x^2-x+2$	15. $5z^2+3z-2$

8 6 Factors of a Polynomial

Now consider the following products:

(1) $(x-a)(x-b)$

(2) $(x-a)(x-b)(x-c)$

(3) $(x-a)(x-b)(x-c)(x-d)$

Of course we could keep on building products in this way. Can you tell what the first term in the expansion of each product will be without completing the multiplication? The expanded expression in each case will be a polynomial and the first term will be x^2 in (1), x^3 in (2), and x^4 in (3).

Here is a slightly more difficult question. Can you tell what the constant term will be in each polynomial? The answer is ab in (1), $-abc$ in (2), and $abcd$ in (3).

Answer these same two questions for

(4) $(x-a)(x-b)(x-c)(x-d)(x-e)$.

The term of highest degree in a polynomial is called the leading term and its numerical coefficient is called the leading coefficient. In the polynomials above, the leading coefficient is 1. Such polynomials are called monic polynomials; x^5-5x^2+2 is a monic polynomial (its leading coefficient is 1), but $3x^4-2x+3$ is not (its leading coefficient is 3).

A consideration of the above examples should indicate to you the truth of the following theorem.

THEOREM (without proof): If $x-m$ is a factor of the monic polynomial $P(x)$, then m is a divisor of the constant term of $P(x)$.

This theorem is useful in finding factors $x-a$ (a integral) of a monic polynomial $P(x)$. This is demonstrated in the following examples.

Example 1

Find the linear factors of the polynomial

$$P(x)=x^3+x^2-5x-6.$$

Solution: If $x-a$ is a factor of $P(x)$, then by the above theorem a must be a divisor of 6; the only possible integral values of a are ±1, ±2, ±3, and ±6. We use the factor theorem which tells us that $x-a$ is a factor if $P(a)=0$.

$P(1)=1+1-5-6=-9$ $\quad\therefore x-1$ is not a factor.

$P(-1)=-1+1+5-6=-1$ $\therefore x+1$ is not a factor.

$P(2)=8+4-10-6=-4$ $\quad\therefore x-2$ is not a factor.

$P(-2)=-8+4+10-6=0$ $\quad\therefore x+2$ *is* a factor.

Now we do not need to check the remaining possible values for a. A better procedure is to divide $P(x)$ by $x+2$ to get the second factor $Q(x)$, and then find the factors of $Q(x)$.

By division we find that $x^3+x^2-5x-6=(x+2)(x^2-x-3)$ and we find the factors of x^2-x-3 as in Section 8-5.

If $x^2-x-3=0$, then $x=\frac{1}{2}(1\pm\sqrt{1+12})$.

Then the factors of x^2-x-3 are $x-\frac{1}{2}(1+\sqrt{13})$ and $x-\frac{1}{2}(1-\sqrt{13})$.

The student may check by multiplication that

$x^3+x^2-5x-6=(x+2)\,[x-\frac{1}{2}(1+\sqrt{13})]\,[x-\frac{1}{2}(1-\sqrt{13})]$.

Example 2

Factor $P(x)=x^3-2x^2-9$.

Solution: Possible linear factors are $x+1$, $x-1$, $x+3$, $x-3$, $x+9$, $x-9$.

$P(1)=1-2-9=-10$ $\quad\therefore x-1$ is not a factor.

$P(-1)=-1-2-9=-12$ $\quad\therefore x+1$ is not a factor.

$P(3)=27-18-9=0$ $\quad\therefore x-3$ *is* a factor.

By division we find that $x^3-2x^2-9=(x-3)(x^2+x+3)$.

What are the factors of x^2+x+3?

If $x^2+x+3=0$, then $x=\frac{1}{2}(-1\pm\sqrt{1-12})$.

But these values of x are not real numbers and so x^2+x+3 is irreducible; it has no real linear factors $x-a$.

Thus the required factored expression is given by

$$x^3-2x^2-9=(x-3)(x^2+x+3).$$

The student should realize that a polynomial equation can be solved immediately if it can be written in factored form.

If $P(x)=(x-a)(x-b)\ldots=0,$

then $a, b, \ldots$ are the roots of the polynomial equation $P(x)=0$.

Example 3

Solve the equations (a) $x^3+x^2-5x-6=0,$

(b) $x^3-2x^2-9=0, \quad x\in \mathrm{Re}.$

Solution: (a) Using the result of Example 1 we may write

$x^3+x^2-5x-6=(x+2)\,(x-\frac{1}{2}(1+\sqrt{13}))\,(x-\frac{1}{2}(1-\sqrt{13}))=0$

The equation has three real roots, -2, $\frac{1}{2}(1+\sqrt{13})$ and $\frac{1}{2}(1-\sqrt{13})$.

(b) Using the result of Example 2 we may write

$x^3-2x^2-9=(x-3)(x^2+x+3)=0$

where x^2+x+3 is irreducible.

This equation has only one real root, namely 3.

EXERCISE 8—6

Write down all *possible* factors $x-a$ (a integral) of the following polynomials:

1. x^3-6x^2+2x-4 2. x^4-6x^2+10 3. $x^4-3x+12$

Factor the following polynomials; that is, express each one as either a product of real linear factors or of real linear and irreducible quadratic factors:

4. x^3+4x^2-3x-2
5. $y^3-11y^2+31y-21$
6. $x^3-7x^2+7x+15$
7. x^3-4x^2+7x-6
8. $2z^3-24z^2+58z-36$
9. $3x^2-24x+9$
10. t^3-5t+4
11. $2x^3+3x^2-18x+8$
12. a^3+8
13. $27z^3-8$
14. b^4-b^2-12
15. x^4-7x^2+1
16. t^4-t
17. t^5-t
18. $y^4+y^3-5y^2-3y+6$
19. y^4+5y^2+6

Solve the following equations: (All variables belong to Re.)

20. $x^3+x^2-9x-9=0$

21. $y^3+4y^2-3y-18=0$

22. $z^3-3z^2-6z+8=0$

23. $x^3+5x^2-6x-30=0$

24. $x^3-2x^2+3x-6=0$

25. $m^3-3m^2-3m-4=0$

26. $a^3-7a-6=0$

27. $x^3-5x^2+3x+9=0$

8 7 The Sum and Product of the Roots of a Quadratic Equation

The roots m and n of the quadratic equation $ax^2+bx+c=0$ are

$$m=\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ and } n=\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

Then $m+n=\dfrac{-b}{a}=\dfrac{\text{the negative of the coefficient of } x}{\text{the coefficient of } x^2}$.

$$mn=\left(\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

This is of the form (A+B) (A−B).

$$=\left(\frac{-b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$

$$=\frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}$$

$$=\frac{4ac}{4a^2}$$

$$=\frac{c}{a}$$

$$=\frac{\text{constant term}}{\text{coefficient of } x^2}$$

We then have the following theorem.

THEOREM: The sum of the roots of a quadratic equation

$$=\frac{\text{negative of coefficient of } x}{\text{coefficient of } x^2}.$$

The product of the roots of a quadratic equation

$$=\frac{\text{constant term}}{\text{coefficient of } x^2}.$$

Example 1

Find the sum $m+n$ and the product mn of the roots of the equations

(a) $2x^2-3x+7=0$ (b) $6x^2-3x=0$ (c) $4x^2-13=0$.

Solution: (a) $m+n=\frac{-(-3)}{2}=\frac{3}{2}$ and $mn=\frac{7}{2}$

(b) $m+n=\frac{-(-3)}{6}=\frac{1}{2}$ and $mn=0$

(c) $m+n=\frac{0}{4}=0$ and $mn=\frac{-13}{4}$

Example 2

Given the equation $(k-2)x^2+(k-5)x-6=0$, find k so that

(a) one root is 2;

(b) the sum of the roots is 3;

(c) the product of the roots is -4;

(d) the roots are numerically equal but opposite in sign.

Solution: (a) 2 is a root if

$$(k-2)(2)^2+(k-5)(2)-6=0$$
$$6k-24=0$$
$$\therefore k=4.$$

Let the roots be m and n in (b), (c), and (d).

(b)
$$m+n=\frac{-(k-5)}{k-2}=3$$
$$-k+5=3k-6$$
$$11=4k$$
$$\therefore k=\frac{11}{4}.$$

(c)
$$mn=\frac{-6}{k-2}=-4$$
$$-4k+8=-6$$
$$-4k=-14$$
$$\therefore k=\frac{7}{2}.$$

(d) Here $m+n=0$, since the roots are numerically equal but opposite in sign.

$$m+n=\frac{-(k-5)}{k-2}=0$$
$$\therefore k=5.$$

EXERCISE 8—7

Without solving, find the sum and product of the roots of the following equations:

1. $x^2-4x-6=0$
2. $3-x^2+4x=0$
3. $y^2=9$
4. $x^2+2x+2=0$
5. $t^2-t+1=0$
6. $x^2+x-1=0$
7. $9-z-4z^2=0$
8. $ax-ax^2=bc$
9. $(1-2x)^2=12-5x$
10. $(2-3x)^2=x-4$
11. $a^2y^2-by=b(a+by^2)$
12. $pz^2+qz=r$
13. What can be said about a, b, or c if each root of $ax^2+bx+c=0$ is the additive inverse of the other?
14. Find k if one root of $3x^2-5kx+k+2=0$ is 0.
15. If one root of $3x^2-x-c=0$ is -2, what is c?
16. For what value of p does the sum of the roots of $4x^2-2px+5=0$ equal the product of the roots?
17. Find the roots of a quadratic equation if their sum is $\frac{1}{6}$ and their product is $-\frac{1}{3}$.
18. Prove that the roots of $ax^2+bx+c=0$ are the reciprocals of the roots of $cx^2+bx+a=0$.
19. (a) For what value of k will one root of $(1-2k)x^2-kx+5=0$ be 2?
    (b) Find the other root.
20. Given the equation $(2k-1)y^2+(k-3)y-2=0$, find k so that
    (a) one root is -2;
    (b) the sum of the roots is 5;
    (c) the product of the roots is -6;
    (d) the product of the roots is -1;
    (e) the roots are numerically equal but opposite in sign;
    (f) the roots are reciprocals of each other;
    (g) the roots are the negative reciprocals of each other.
21. Are there any real values of k for which the sum of the roots of the equation
$$(k+2)y^2+k^2y-2=0$$
is equal to 5?

8 8 Finding a Quadratic Equation With Given Roots

If the leading coefficient of a quadratic equation is unity, that is, if the equation is of the form $x^2+bx+c=0$, and the roots are m and n,

then $m+n=-b=$ negative of the coefficient of x

and $mn=c=$ constant term.

With this knowledge, it is a simple matter to form the monic quadratic equation that has given roots.

Example 1

Form the monic quadratic equation which has roots -2 and -4.

Solution: $m+n=$ sum of roots $=-2-4=-6$

$\therefore -6$ is the negative of the coefficient of x.

$mn=$ product of roots $=(-2)(-4)=8$

$\therefore$ 8 is the constant term in the monic quadratic equation.

The monic quadratic equation with these roots is

$$x^2+6x+8=0.$$

The student should check that this equation has the given roots.

Example 2

Use a different method to obtain the solution in Example 1.

Solution: If -2 is a root of a quadratic equation $ax^2-bx+c=0$, then $x+2$ is a factor of ax^2+bx+c.

Similarly, if -4 is a root, $x+4$ is a factor of ax^2+bx+c.

The required quadratic equation is then $(x+2)(x+4)=0$ or $x^2+6x+8=0$.

Example 3

Form the quadratic equation whose roots are $-\frac{1}{3}$ and $\frac{2}{5}$.

Solution: Here $m+n=-\frac{1}{3}+\frac{2}{5}=\frac{-5+6}{15}=\frac{1}{15}$

and $mn=(-\frac{1}{3})(\frac{2}{5})=-\frac{2}{15}$.

The *monic* quadratic equation with roots $-\frac{1}{3}$ and $\frac{2}{5}$ is

$$x^2-\tfrac{1}{15}x-\tfrac{2}{15}=0.$$

An equivalent quadratic equation, with integral coefficients, is

$$15x^2-x-2=0.$$

Check by using the factor method.

Example 4

Find the quadratic equation whose roots are

$$\tfrac{1}{2}(4+\sqrt{5}) \text{ and } \tfrac{1}{2}(4-\sqrt{5}).$$

Solution: $m+n=\tfrac{1}{2}(4+\sqrt{5})+\tfrac{1}{2}(4-\sqrt{5})=\tfrac{8}{2}=4$

$$mn=\tfrac{1}{4}(4+\sqrt{5})(4-\sqrt{5})=\tfrac{1}{4}(16-5)=\tfrac{11}{4}$$

The *monic* quadratic equation with the given roots is

$$x^2-4x+\tfrac{11}{4}=0$$

and an equivalent equation with integral coefficients is

$$4x^2-16x+11=0.$$

Check by using the factor method. Which method is preferable here?

EXERCISE 8—8

Use the factor method to obtain quadratic equations with the following pairs of roots:

1. $-4, 10$	2. $7, -7$	3. $\frac{2}{3}, -\frac{5}{3}$	4. $2\sqrt{5}, -\sqrt{5}$
5. $2\pm\sqrt{3}$	6. $\frac{1}{2}(2\pm\sqrt{5})$	7. $\frac{p}{2}, \frac{q}{2}$	8. $p+q, p-q$

Use the sum-and-product method to write quadratic equations having the following pairs of roots:

9. $-5, -2$	10. $6, -6$	11. $-\frac{2}{3}, -3$	12. $-\frac{a}{b}, \frac{b}{a}$
13. $4\pm\sqrt{3}$	14. $\frac{1}{3}(3\pm\sqrt{2})$	15. $\frac{p+q}{2}, \frac{p-q}{2}$	16. $\frac{2}{p}, \frac{p}{4}$
17. $2, q$	18. $3q, -2q$	19. $6-q, q+2$	20. $\frac{1}{q}, \frac{1}{q^2}$

21. Find k if one root of the equation $4x^2-24x+k=0$ is three more than the other root.

22. Find k if one root of the equation $9x^2-(k+1)x+8=0$ is three times the other root.

8 9 Finding a Quadratic Equation Whose Roots Are the Negatives, Squares, or Reciprocals of the Roots of a Given Quadratic (Supplementary)

Example 1

Find the quadratic equation whose roots are the negatives of the roots of the equation $6x^2-3x-10=0$.

Solution: Let the roots of $6x^2-3x-10=0$ be m and n and those of the required equation be M and N.

Then $m+n=\frac{-(-3)}{6}=\frac{1}{2}$

and $mn=\frac{-10}{6}=\frac{-5}{3}$.

$\therefore \quad M+N=-m-n=-(m+n)=-\frac{1}{2}$

and $MN=(-m)(-n)=mn=-\frac{5}{3}$.

The required equation is

$$x^2+\tfrac{1}{2}x-\tfrac{5}{3}=0$$

or $6x^2+3x-10=0$.

Example 2

Find the quadratic equation whose roots are the squares of the roots of the equation $6x^2-3x-10=0$.

Solution: Let the roots of $6x^2-3x-10=0$ be m and n and those of the required equation be M and N.

Then $m+n=\frac{1}{2}$

and $mn=-\frac{5}{3}$.

$$\therefore \quad M+N=m^2+n^2=(m+n)^2-2mn$$
$$=(\tfrac{1}{2})^2-2(-\tfrac{5}{3})$$
$$=\tfrac{1}{4}+\tfrac{10}{3}$$
$$=\tfrac{3+40}{12}=\tfrac{43}{12}$$

$$\text{and} \quad MN=m^2n^2=(mn)^2$$
$$=(-\tfrac{5}{3})^2$$
$$=\tfrac{25}{9}.$$

The required equation is

$$x^2-\tfrac{43}{12}x+\tfrac{25}{9}=0$$

or $36x^2-129x+100=0$.

Note that the crucial step in the solution of Example 2 is writing

$$m^2+n^2=(m+n)^2-2mn.$$

This expresses the sum of the roots of the *required* equation in terms of the (known) sum and product of the roots of the *given* equation. The expression

$$m^2n^2=(mn)^2$$

expresses the product of the roots of the required equation in terms of the (known) product of the roots of the *given* equation.

Example 3

Without using (18) Exercise 8-7, find the quadratic equation whose roots are the reciprocals of the roots of the equation $6x^2-3x-10=0$.

Solution: Let the roots of $6x^2-3x-10=0$ be m and n and those of the required equation be M and N.

Then $m+n=\frac{1}{2}$

and $mn=-\frac{5}{3}$.

$$\therefore \quad M+N=\frac{1}{m}+\frac{1}{n}=\frac{m+n}{mn}=\frac{\frac{1}{2}}{-\frac{5}{3}}=-\frac{3}{10}$$

$$\text{and} \quad MN=\frac{1}{m}\times\frac{1}{n}=\frac{1}{mn}=-\frac{3}{5}.$$

The required equation is

$$x^2+\tfrac{3}{10}x-\tfrac{3}{5}=0$$

or $10x^2+3x-6=0.$

Note that the essential step in this solution is writing

$$\frac{1}{m}+\frac{1}{n}=\frac{m+n}{mn}$$

to give the sum of the roots of the required equation in terms of the (known) sum and product of the roots of the given equation.

EXERCISE 8—9

Form the equations whose roots are (a) the negatives, (b) the squares, and (c) the reciprocals of the roots of the following equations:

1. $x^2-6x+7=0$
2. $x^2-9=0$
3. $y^2-3y+4=0$
4. $2x^2-5x+7=0$
5. $-z^2+3z+2=0$
6. $3x^2-4x-1=0$
7. $t^2-\frac{1}{2}t-\frac{1}{3}=0$
8. $5x^2-\frac{1}{2}x+3=0$
9. $-2x^2+4x=9$
10. $3a^2-7=a$
11. $y^2+y+1=0$
12. $z^2+az+b=0$

8 10 Chapter Summary

Real cubic equation, real quartic equation, real equation of degree n, polynomial equation, degree of polynomial, zero of a polynomial, root of polynomial equation.

Division of a polynomial $P(x)$ by $x-a$. $P(x)=(x-a)Q(x)+R$ where $Q(x)$ is a polynomial of degree one lower than $P(x)$ and R is a constant.

The Factor Theorem and its converse.

Factoring a quadratic polynomial using the Factor Theorem; irreducible quadratics.

Leading term of a polynomial; leading coefficient; monic polynomial.

Factors of a polynomial. If $x-m$ is a real factor of a monic polynomial, then m is a divisor of the constant term of the polynomial.

Solution of a polynomial equation of degree 2 by factoring.

The sum and product of the roots of the quadratic equation $ax^2+bx+c=0$; if they are m and n, then $m+n=\frac{-b}{a}$ and $mn=\frac{c}{a}$.

Formation of a quadratic equation with given roots.

Formation of a quadratic equation whose roots are the negatives, squares, or reciprocals of the roots of a given quadratic. (supplementary)

REVIEW EXERCISE 8—10

1. If $P(x)=x^3-3x+1$, find (a) $P(0)$ (b) $P(1)$ (c) $P(-\frac{1}{2})$
   (d) $P(a)$ (e) $P(-m)$ (f) $P(y)$ (g) $P\left(\frac{1}{x}\right)$

2. Define (a) zero of a polynomial
   (b) root of a polynomial equation.

3. If $P(y)=y^3+3y^2+2y-5$, find $P(x-1)$.

Express the following polynomials $P(x)$ in the form $(x-a)Q(x)+R$ for the given $x-a$:

4. $P(x)=5x^2-4x+2$ (a) $x-1$ (b) $x+3$ (c) $x-\frac{1}{2}$

5. $P(x)=4x^3+x^2+x-6$ (a) $x+1$ (b) $x-2$ (c) $x+3$

6. $P(x)=x^5-x^4+2x^3-6x^2+3x-1$ (a) $x+1$ (b) $x+2$ (c) $x-1$

Find the remainder when the following polynomials $P(x)$ are divided by the indicated $x-a$:

7. $P(x)=6x^2-3x+2$ (a) $x-1$ (b) $x-\frac{1}{2}$

8. $P(x)=2x^3-6x+5$ (a) $x+5$ (b) $x+\frac{1}{2}$

9. $P(x)=3x^4-5x^2+6x$ (a) x (b) $x+3$

10. For what values of m will $2x^2-3x+4$ leave a remainder of 3 on division by $x-m$?

11. Use the Factor Theorem to test whether or not (a) $x+2$ (b) $x-5$ are factors of $x^3-6x^2+10x-5$.

Use the Factor Theorem to decide whether or not each of the following is true:

12. $x+3$ is a factor of $3x^3-x^2-22x+24$.

13. $y+2$ is a factor of $3y^3+10y^2+9y+2$.

14. $x-3$ is a factor of $2x^3-5x^2-x+6$.

15. $z-4$ is a factor of z^3+64.

16. $x+3$ is a factor of x^4-81.

17. $t-4$ is a factor of t^3-64.

18. $x+3$ is a factor of x^4+81.

19. $x-y$ is a factor of x^n-y^n for any positive integer n.

20. $x-y$ is not a factor of x^n+y^n for any positive integer n.

21. $x+y$ is not a factor of x^5-y^5.

22. $x+y$ is not a factor of x^8+y^8.

23. Show that x^2+1 is a factor of $P(x)=x^3-2x^2+x-2$.
Is $P(-1)=0$? Is there a contradiction here? Explain.

24. Show that $x-1$ is a factor of $2398x^{1965}-1921x^{965}-477$.

25. Solve the equation $x^3-4x^2+x+6=0$, given that 3 is a root.

26. Factor $P(x)=8x^4-38x^3+20x^2+19x-12$, given that 4 and $\frac{3}{4}$ are zeros of $P(x)$.

27. Factor $P(x)=x^5-3x^4+2x^3-6x^2-3x+9$, given that x^2+3 is an irreducible quadratic factor.

28. Find the real factors of y^3+5y^2+9y+9.

29. Find the real roots of $z^3+5z^2+8z+16=0$.

30. Write the general real quadratic equation that has 4 as one root.

31. Write the factors of a^2+a-7.

32. Find the sum and products of the roots of the following equations:

(a) $3x^2-7x-7=0$ (b) $5z^2-9=0$

(c) $4x^2+3x=0$ (d) $-y^2+2y-5=0$

(e) $\frac{5}{2}-3t+\sqrt{2}t^2=0$ (f) $-4t-\frac{1}{2}t^2+\sqrt{3}=0$

Supplementary.

Write the quadratic equation whose roots are

(a) the negatives (b) the squares (c) the reciprocals

of the roots of the following equations:

33. $4x^2-5x-5=0$

34. $6y^2-7=0$

35. $5z^2+z=0$

36. $-3x^2-2x+5=0$

Chapter 9

Elementary Theory of Equations and Complex Numbers

9 1 Monic Polynomial Equations

In this section we consider a result which is true for every polynomial equation.

Recall that a monic polynomial is one in which the leading coefficient is 1. A monic polynomial equation is a monic polynomial equated to 0. Thus x^3-6x+1 is a monic polynomial and $x^4-3x^2=0$ is a monic polynomial equation.

Recall also that two equations are equivalent if they have the same solution set, that is, the same set of roots.

THEOREM: A polynomial equation is always equivalent to a monic polynomial equation (and hence may be replaced by such an equation in an algebraic discussion).

Proof: If $a_0 \neq 0$, we may write

$$a_0x^n+a_1x^{n-1}+\ \ldots\ +a_{n-1}x+a_n=a_0\left[x^n+\frac{a_1}{a_0}x^{n-1}+\ \ldots\ +\frac{a_{n-1}}{a_0}x+\frac{a_n}{a_0}\right].$$

Since $a_0 \neq 0$, the left side can equal zero only if the factor

$$x^n+\frac{a_1}{a_0}x^{n-1}+\ \ldots\ +\frac{a_{n-1}}{a_0}x+\frac{a_n}{a_0}$$

on the right equals zero and conversely. This shows that

$$a_0x^n+a_1x^{n-1}+\ \ldots\ +a_{n-1}x+a_n=0$$

and

$$x^n+\frac{a_1}{a_0}x^{n-1}+\ \ldots\ +\frac{a_{n-1}}{a_0}x+\frac{a_n}{a_0}=0$$

are equivalent equations.

For example, $3x^2-5x+7=0$ and $x^2-\frac{5}{3}x+\frac{7}{3}=0$ are equivalent equations, as are $\pi x^3-6=0$ and $x^3-\frac{6}{\pi}=0$.

It is important to note that this theorem is about polynomial equations and not about polynomials. It is *not* true that a polynomial can always be replaced by a monic polynomial, although this is a mistake students commonly make.

The polynomials $P(x)=3x^2-5x+2$ and $Q(x)=x^2-\frac{5}{3}x+\frac{2}{3}$ are quite different. Indeed,

$$P(0)=2,\ Q(0)=\tfrac{2}{3};\quad P(2)=4,\ Q(2)=\tfrac{4}{3};\quad P(-1)=10,\ Q(-1)=\tfrac{10}{3};$$

so that $P(x)$ and $Q(x)$ cannot be equal because they have different values for the values 0, 2, and -1 of x.

Note however that the *zeros* of $P(x)$ and $Q(x)$ are the same:

$$P(1)=0,\ Q(1)=0;\quad P(\tfrac{2}{3})=0,\ Q(\tfrac{2}{3})=0.$$

The reason for this is that the zeros of $P(x)$ and $Q(x)$ are the roots of the corresponding equations which are *equivalent* equations. Geometrically, $y=3x^2-5x+2$ and $y=x^2-\frac{5}{3}x+\frac{2}{3}$ are the equations of distinct parabolas which have the same points of intersection with the x-axis.

EXERCISE 9—1

Replace the following equations by equivalent monic equations:

1. $3x^2-5x-7=0$
2. $4x^3-3x-5=0$
3. $100y^{16}-y^8+y^2=0$
4. $-\frac{1}{2}x^2+3x-2=0$
5. $\sqrt{2}z^2-6z-3=0$
6. $\pi a^2-3a+\frac{22}{7}=0$
7. Show that the polynomials
   $P(y)=4y^3-3y^2+y-3$ and $Q(y)=y^3-\frac{3}{4}y^2+\frac{1}{4}y-\frac{3}{4}$
   are not identical by demonstrating that these polynomials have different values for the same value of y.
8. Are the polynomials $P(x)=3x^2-2x+5$ and $Q(y)=3y^2-2y+5$ identical?
9. Show that the polynomials $P(x)=2x^2-5x-6$ and $Q(x)=x^2-\frac{5}{2}x-3$ are not identical but have the same zeros.
10. Sketch on the same diagram the graphs of the parabolas whose equations are $y=2x^2-5x-6$ and $y=x^2-\frac{5}{2}x-3$.
11. What is the mathematical connection between the curve whose equation is $y=x^3-3x-4$ and the equation $x^3-3x-4=0$? (No diagram is required.)

9 2 Formulas for the Roots of a Polynomial Equation

This section is included to show that not all results true for quadratic equations will apply to polynomial equations of every degree.

We know that the only root of the linear equation $ax+b=0$ is $-\frac{b}{a}$ and the solution of the quadratic equation $ax^2+bx+c=0$ is the set

$$\left\{\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right\}.$$

The solution in both cases is a formula for the roots of the equation in terms of the coefficients of the equation and the so-called rational operations of addition, subtraction, multiplication, division, and the extraction of roots. Such a formula allows the student to write down the roots of the equation with, at most, a little arithmetical calculation. For example, the roots of $6x^2-3x-5=0$ are

$$\frac{-(-3)\pm\sqrt{(-3)^2-4(6)(-5)}}{2(6)}.$$

The question is, does such a formula exist for the cubic equation $ax^3+bx^2+cx+d=0$? If we know the coefficients, as in the cubic $4x^3-x^2-x+6=0$, is it possible to write at once the roots of the equation?

The answer is *yes*; the roots of any cubic can be expressed in terms of the coefficients, using the operations of addition, subtraction, multiplication, division, and extraction of roots. However the formulae are more complicated than for the quadratic equation and more calculations are involved. These formulae were discovered by some Italian mathematicians of the sixteenth century (the actual discoverer seems to be one of Scipione del Ferro, Tartaglia, or Girolamo Cardano). These formulae are seldom used; their *existence* is important but they are of little practical value.

The next question is, does the same algebra hold true for the quartic equation $ax^4+bx^3+cx^2+dx+e=0$? The answer again is *yes*. Formulae do exist giving the roots of any quartic equations in terms of the coefficients and the same operations. Again the *existence* of the formulae is important but they are of little practical value. The formulae for the quartic were discovered by Lodovico Ferrari, a pupil of Cardano.

It seems natural to think that such formulae would exist for the quintic equation $ax^5+bx^4+cx^3+dx^2+ex+f=0$ and for equations of higher degrees. This is what mathematicians once thought and they spent many years trying to discover these formulae.

Finally, it occurred to mathematicians that such formulae might not exist and so the emphasis shifted from trying to find them to trying to prove that such formulae did not exist. In the latter they were successful. One of the men generally credited with this discovery was the famous Norwegian mathematician Niels Henrik Abel (1802-1829).

The search for these formulae and finally for the proof that such formulae could not exist led to the discovery of some very remarkable and important mathematics which some of you will study in university.

The algebraic situation here paralleled the situation in geometry. At almost the same time, mathematicians were trying to *prove* Euclid's Parallel Axiom from his other axioms. Finally it was shown that this was impossible by creating geometries in which the Parallel Axiom does not hold (non-Euclidean geometries). Again the attempts to solve this problem led to the discovery of much new mathematics.

Thus we have shown a result in this section that is true for some polynomial equations but not for all. Equations of the first, second, third, and fourth degree are solvable by means of formulae involving the coefficients of the equation and the operations of addition, subtraction, multiplication, division, and the extraction of roots. This is not true, in general, for equations of higher degree.

EXERCISE 9—2

1. If $P(y)=y^3-3y^2+4y-3$, find $P(x+1)$.
2. If $P(y)=y^3+6y^2-y-2$, find $P(x-2)$.
3. If $P(y)=y^3+12y^2-2y+3$, find $P(x-4)$.
4. Show that the cubic equation $y^3+py^2+qy+r=0$ can be reduced to the form $x^3+mx=n$ by substituting $x-\frac{p}{3}$ for y.
5. Reduce $y^3+3y^2-5y+4=0$ to the form $x^3+mx=n$.
6. Reduce $y^3-12y^2+y-6=0$ to the form $x^3+mx=n$.
7. Reduce $2y^3-8y^2+y-3=0$ to the form $x^3+mx=n$.
8. Show that $(a-b)^3+3ab(a-b)=a^3-b^3$.
9. Show that $a-b$ is a root of the equation $x^3+mx=n$ if $3ab=m$ and $a^3-b^3=n$.
10. Show that if $a=\sqrt[3]{\frac{1}{2}n+\sqrt{(\frac{1}{2}n)^2+(\frac{1}{3}m)^3}}$
and $b=\sqrt[3]{-\frac{1}{2}n+\sqrt{(\frac{1}{2}n)^2+(\frac{1}{3}m)^3}}$,
then $3ab=m$ and $a^3-b^3=n$.
11. Use the results of (9) and (10) to find a root of the equation $x^3+9x=6$.
12. Find a root of the equation $x^3+12x-12=0$.
13. Use (4), (9), and (10) to find a root of the equation $x^3+6x^2+18x+18=0$.

9 3 Complex Numbers

In Section 9-2 it was pointed out that the solution set of the quadratic equation $ax^2+bx+c=0$ is

$$\left\{\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right\},$$

and that formulae also existed giving the roots of the cubic and quartic equations in terms of their coefficients.

We have seen in Section 7-8 that the roots of the quadratic equation are not always real numbers; this is true if $b^2-4ac<0$. A similar statement is true for equations of higher degree.

In this section we begin a brief study of the set of numbers which will include the roots of all polynomial equations, the set C of complex numbers. For our purposes the following definition is sufficient.

DEFINITION: A complex number is one that can be represented by the numeral $a+bi$ where a and b are real numbers and i is a symbol that has the property that $i^2=-1$.

Example 1

$2+3i$, $4-\frac{1}{2}i$, $\sqrt{2}+\pi i$ represent, or are numerals for, complex numbers. $3-\sqrt{2}i$ may be written $3-i\sqrt{2}$ to avoid confusion with the root sign.

Example 2

Find the roots of the equation $x^2+x+4=0$.

Solution: The roots are

$$\tfrac{1}{2}(-1\pm\sqrt{1-16})$$

that is, $\frac{1}{2}(-1+i\sqrt{15})$ and $\frac{1}{2}(-1-i\sqrt{15})$.

We have seen before that a rational number is one that can be represented in the form $\frac{a}{b}$ where a and b are integers and $b\neq0$. Further we have seen that $\frac{1}{2}$ and $\frac{2}{4}$ represent the same rational number; in general we said that $\frac{a}{b}=\frac{c}{d}$ if and only if $ad=bc$.

Can a complex number be represented by more than one expression of the form $a+bi$? The answer is "no" and is obtained from the following definition.

DEFINITION: $a+bi=c+di$ if and only if $a=c$ and $b=d$.

Example 3

$2+3i \neq \pi+\sqrt{2}i$

$x+yi=2-\frac{1}{2}i$ if and only if $x=2$ and $y=-\frac{1}{2}$.

Example 4

For what values of m and n will

$$(m+n)+(2m-n)i=3+9i?$$

Solution: These two expressions will represent the same complex number if and only if

$$m+n=3$$
$$2m-n=9.$$

Solving these two equations yields $m=4$, $n=-1$.

The above definition tells us that the essential part in the representation $a+bi$ of the complex number is the pair (a, b) of real numbers. As a matter of fact, a complex number is sometimes defined to be an ordered pair (a, b) of real numbers. The real number a is called the real part or real component or real coefficient of the complex number $a+bi$,

$$a=\mathrm{Re}(a+bi)$$

and the real number b is called the imaginary part or imaginary component or imaginary coefficient of $a+bi$,

$$b=\mathrm{Im}(a+bi).$$

For example, $\mathrm{Re}(3-6i)=3$ and $\mathrm{Im}(3-6i)=-6$.

The name "imaginary coefficient" stems from the fact that i or $\sqrt{-1}$ was originally thought of as an "imaginary" number. Complex numbers have been in use for a very long time and were actually accepted mathematically at a time when people still viewed negative numbers with much suspicion. They have important applications in electricity, elasticity, hydrodynamics, nucleonics, etc., branches of physical science and engineering in which the real number system is not sufficient to describe the physical processes.

If the complex number $x+yi$ is thought of essentially as the pair (x, y) of real numbers, there is an obvious geometric representation of the complex number by the point with coördinates (x, y). However, when we so represent a complex number, we often do not use the usual names x- and y-axes for the perpendicular lines forming the frame of reference. Rather, we call these lines the real axis and the imaginary axis, and we call the resulting plane the complex plane or the Argand plane (named after Jean Robert Argand who, in 1806, gave this geometrical representation of a complex number and used it to show that every algebraic equation has a root).

Example 5

Represent geometrically the complex numbers $1+i$, $-5+4i$, $-3-2i$, $4-3i$, $6+0i$, $0-2i$.

Solution: These complex numbers are represented respectively by the points (1, 1), (−5, 4), (−3, −2), (4, −3), (6, 0), and (0, −2) in the complex or Argand plane (Figure 9-1).

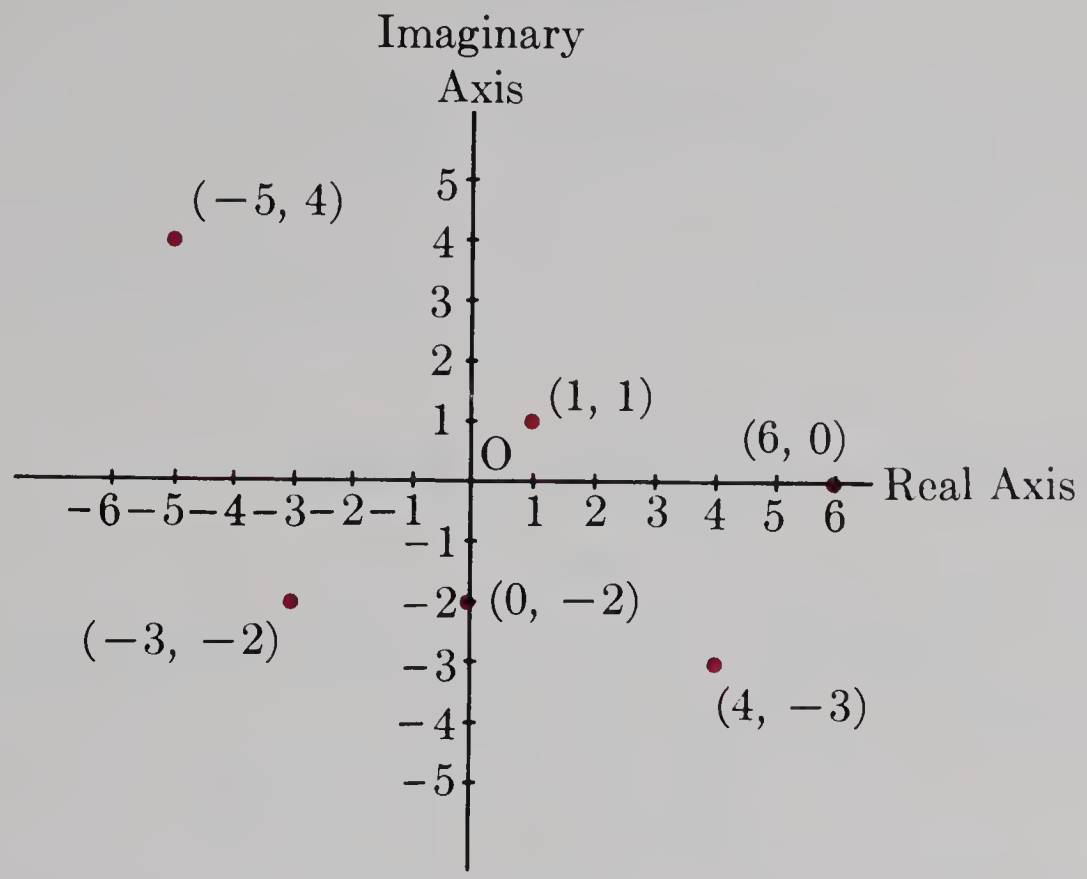


Figure 9-1 Argand plane

Note that complex numbers of the form $a+0i$ are represented by points on the real axis and those of the form $0+bi$ are represented by points on the imaginary axis. Numbers of the form $0+bi$, often written bi, are called pure imaginary numbers ($2i$, $-3i$, πi, etc.).

EXERCISE 9—3

1. Give (i) the real coefficient and (ii) the imaginary coefficient of the following complex numbers:
   (a) $6-\sqrt{2}i$ (b) $\frac{1}{2}+\frac{1}{2}i$ (c) $-\pi+6i$ (d) $2x+3yi$
2. Represent geometrically the following complex numbers:
   (a) $6+4i$ (b) $-3+2i$ (c) $\frac{1}{2}+\frac{3}{2}i$ (d) $-4-2i$
   (e) $0+3i$ (f) $-3+0i$ (g) $\sqrt{2}+\sqrt{2}i$ (h) $\pi+4i$
3. What complex number does the origin of the Argand plane represent?

4. Find the values of x and y so that the complex numbers in the following pairs are equal:

(a) $6x+3yi;\ 4+2i$ (b) $x-3yi;\ -\pi-\sqrt{2}i$

(c) $x-(4-y)i;\ -\frac{1}{2}+\frac{3}{2}i$ (d) $x-(x+y)i;\ 3+0i$

(e) $(x+2y)+3xi;\ -6+0i$ (f) $(x+2y)+3xi;\ 0-6i$

(g) $(x+y)+(2x-3y)i;\ 3-2i$ (h) $(4x-y)+(\frac{1}{2}x+y)i;\ -3-2i$

Find the complex roots of the following equations:

5. $6x^2-x+2=0$ 6. $3x^2-\sqrt{2}x+1=0$

7. $\sqrt{2}x^2-5x+6\sqrt{2}=0$ 8. $\sqrt{3}x^2-x+\frac{4}{\sqrt{3}}=0$

9.4 Addition of Complex Numbers

DEFINITION: $(a+bi)+(c+di)=(a+c)+(b+d)i$.

This definition tells us that we may add complex numbers exactly as we add algebraic expressions; for example,

$$(2+3x)+(4+7x)=6+10x,$$
$$(2+3i)+(4+7i)=6+10i.$$

The definition shows us immediately that the set of complex numbers is closed under addition

According to our definition,

$$(a+bi)+(0+0i)=a+bi$$

so that the complex number $0+0i$ is the identity or zero for addition of complex numbers If there is no confusion, we will write simply 0 for this complex number

Now note that

$$(7-3i)+(-7+3i)=0+0i,$$

so that $(-7+3i)$ is the negative of the complex number $7-3i$:

$$-7+3i=-(7-3i).$$

In general, $(a+bi)+(-a-bi)=(a-a)+(b-b)i=0$

and so $-(a+bi)=-a-bi$ is the additive inverse of $a+bi$.

We may now define subtraction of complex numbers:

DEFINITION: $(a+bi)-(c+di)=(a+bi)+(-c-di)$.

Thus to subtract a complex number, we *add* its additive inverse.

Example

(a) Express $-(-3+2i)$ as a complex number in the form $a+bi$.

(b) Express $(5+4i)-(3+2i)$ as a complex number in the form $a+bi$.

Solution: (a) $-(-3+2i)=3-2i$

(b) $$\begin{aligned}(5+4i)-(-3+2i)&=(5+4i)+[-(-3+2i)]\\&=(5+4i)+(3-2i)\\&=8+2i\end{aligned}$$

The fact that addition of complex numbers is carried out in terms of real numbers (the real and imaginary components of the complex numbers) shows us that addition of complex numbers is commutative and associative.

Addition of complex numbers is commutative; for example,

$$\begin{aligned}(3+2i)+(-7+4i)&=-4+6i\\&=(-7+4i)+(3+2i)\end{aligned}$$

In general
$$\begin{aligned}(a+bi)+(c+di)&=(a+c)+(b+d)i\\&=(c+a)+(d+b)i\\&=(c+di)+(a+bi)\end{aligned}$$

Addition of complex numbers is associative; for example,

$$\begin{aligned}(4+i)+[(-3+2i)+(6-4i)]&=(4+i)+(3-2i)\\&=7-i\\[(4+i)+(-3+2i)]+(6-4i)&=(1+3i)+(6-4i)\\&=7-i\end{aligned}$$

EXERCISE 9—4

Add the following pairs of complex numbers:

1. $4+3i,\ 7+2i$
2. $6-3i,\ -2+4i$
3. $-\frac{1}{2}+3i,\ -\frac{3}{2}-2i$
4. $6-4i,\ 6+4i$
5. $-\frac{1}{2}-\frac{3}{2}i,\ \frac{5}{2}+i$
6. $-4+\pi i,\ 4+\pi i$
7. $\sqrt{2}+i,\ 3+2i$
8. $6\sqrt{2}+i,\ -3+\sqrt{2}i$
9. $\pi+2i,\ \sqrt{3}+\sqrt{2}i$

State the additive inverses of the following in the form $a+bi$:

10. $5-2i$
11. $-3\sqrt{2}+\frac{1}{2}i$
12. $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

In the following, subtract the second complex number from the first:

13. $6+i,\ -4+i$
14. $3-2i,\ -2-5i$
15. $-6-2i,\ -5-4i$
16. $4-\frac{1}{2}i,\ \frac{3}{2}+\frac{1}{2}i$
17. $\frac{1}{2}+3i,\ -2-\frac{1}{2}i$
18. $6+4i,\ -6+4i$
19. $\sqrt{2}+3i,\ 5\sqrt{2}-i$
20. $3\sqrt{2}+4i,\ \pi+\sqrt{2}i$
21. $\frac{1}{\sqrt{3}}+i,\ 2-\sqrt{2}i$

22. Prove that the set of complex numbers is closed under subtraction.

23. Prove that addition of complex numbers is associative.

Solve the following equations for z:

24. $(2+3i)+z=-4+i$

25. $(-1+2i)+z=5-\frac{1}{2}i$

26. $(\sqrt{2}+i)-z=4+\sqrt{2}i$

27. $6-i=z+(5-\sqrt{2})i$

Verify in the following equations $ax^2+bx+c=0$, that the sum of the roots is $-\frac{b}{a}$: (Find the complex roots.)

28. $x^2+5x+8=0$

29. $2x^2-4x+3=0$

30. $6x^2-5x+2=0$

31. $3x^2+x+2=0$

32. Show that the sum of the roots of the real equation $ax^2+bx+c=0$ is always $-\frac{b}{a}$.

33. Represent the complex numbers $z_1=5+i$, $z_2=2+3i$, and their sum z_1+z_2 geometrically.

34. Prove that the origin and the points representing z_1, z_2, and z_1+z_2 in (33) are the vertices of a parallelogram.

35. Prove that the result in (34) is true for every pair of complex numbers z_1 and z_2.

9 5 Multiplication of Complex Numbers

DEFINITION: $(a+bi)\times(c+di)=(ac-bd)+(ad+bc)i$.

This rather complicated looking definition tells us that we may multiply complex numbers in exactly the way that we multiply algebraic expressions, bearing in mind that i^2 may be replaced by -1. For example,

$$\begin{aligned}(2+3x)\times(4+x) &= 8+2x+12x+3x^2\\ &= 8+14x+3x^2\\ (2+3i)\times(4+i) &= 8+2i+12i+3i^2\\ &= 8+14i+3(-1)\\ &= 5+14i\\ (a+bi)\times(c+di) &= ac+adi+bci+bdi^2\\ &= ac+(ad+bc)i+bd(-1)\\ &= (ac-bd)+(ad+bc)i.\end{aligned}$$

The definition shows us immediately that the set of complex numbers is closed under multiplication.

The following examples indicate that multiplication of complex numbers is commutative and associative.

Example 1

Evaluate (a) $(2-3i)\times(5+i)$

(b) $(5+i)\times(2-3i)$.

Solution: (a) $(2-3i)\times(5+i)=10-15i+2i-3i^2$

$=13-13i$

(b) $(5+i)\times(2-3i)=10-15i+2i-3i^2$

$=13-13i$

$\therefore (2-3i)\times(5+i)=(5+i)\times(2-3i)$.

Example 2

Evaluate (a) $(1+2i)\times[(6-i)\times(2+3i)]$

(b) $[(1+2i)\times(6-i)]\times(2+3i)$.

Solution: (a) $(1+2i)\times[(6-i)\times(2+3i)]$

$=(1+2i)\times[12-2i+18i-3i^2]$

$=(1+2i)\times(15+16i)$

$=15+16i+30i+32i^2$

$=-17+46i$

(b) $[(1+2i)\times(6-i)]\times(2+3i)$

$=[6-i+12i-2i^2]\times(2+3i)$

$=(8+11i)\times(2+3i)$

$=16+24i+22i+33i^2$

$=-17+46i$

$\therefore (1+2i)\times[(6-i)\times(2+3i)]=[(1+2i)\times(6-i)]\times(2+3i)$.

Note that

$$(a+bi)\times(1+0i)=a+bi,$$

so that $1+0i$ is the multiplicative identity or unit for the multiplication of complex numbers. When there can be no confusion, we will write 1 instead of $1+0i$.

Now consider the following product

$$(6+2i)\times(6-2i)=40+0i,$$

and in general,

$$(a+bi)\times(a-bi)=(a^2+b^2)+0i.$$

The complex numbers $a+bi$ and $a-bi$ are called conjugate complex numbers.

DEFINITION: Conjugate complex numbers are complex numbers differing only in the sign of their imaginary coefficients.

For example, $6-3i$ and $6+3i$, $-4+0i$ and $-4-0i$, $0-6i$ and $0+6i$ are pairs of conjugate complex numbers.

If you examine closely the behaviour of complex numbers of the form $a+0i$ under addition and multiplication, you will see that they behave in exactly the same way as the real numbers a.

$$(6+0i)+(2+0i)=\ 8+0i \qquad 6+2=\ 8$$
$$(6+0i)\times(2+0i)=12+0i \qquad 6\times 2=12$$

Because of this, mathematicians agree to identify the complex number $a+0i$ with the real number a and even to write $a+0i=a$ (as long as there is no possibility of confusion). With this understanding, we may say that the set of complex numbers contains the set of real numbers and also that the product of a complex number and its conjugate is a real number.

EXERCISE 9—5

Multiply the following pairs of complex numbers:

1. $4+3i,\ 7+2i$
2. $6-3i,\ -2+4i$
3. $-\frac{1}{2}+3i,\ -\frac{3}{2}-2i$
4. $6-4i,\ 6+4i$
5. $-\frac{1}{2}-\frac{3}{2}i,\ \frac{5}{2}+i$
6. $-4+\pi i,\ 4+\pi i$
7. $\sqrt{2}+i,\ 3+2i$
8. $6\sqrt{2}+i,\ -3+\sqrt{2}i$
9. $4\sqrt{2}-3i,\ \sqrt{2}+2i$

Use the following pairs of complex numbers to verify by example that multiplication of complex numbers is commutative:

10. $6+i,\ -4+i$
11. $3-2i,\ -2-5i$
12. $\frac{1}{2}+3i,\ -2-\frac{1}{2}i$
13. Prove that multiplication of complex numbers is commutative.

Use the following triples of complex numbers to verify by example that multiplication of complex numbers is associative:

14. $3-i,\ 5+i,\ 2-3i$
15. $6+2i,\ 5-\sqrt{2}i,\ 5+\sqrt{2}i$
16. (a) Verify that $(1+2i)[(3-2i)+(-5-4i)]$
$=(1+2i)(3-2i)+(1+2i)(-5-4i)$.
(b) What law is exemplified in (a)?

Multiply each of the following complex numbers by its conjugate:

17. $6-i$
18. $-2-4i$
19. $\frac{1}{2}+\frac{1}{2}i$
20. $0+4i$
21. $\frac{1}{2}+0i$
22. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Verify in the following equations $ax^2+bx+c=0$, that the product of the roots is $\frac{c}{a}$: (Find the complex roots.)

23. $x^2+5x+8=0$
24. $2x^2-4x+3=0$
25. $6x^2-5x+2=0$

26. Is $[x-(1+i)][x-(2-i)]=0$ a real quadratic equation?

27. Show that if the roots of the real quadratic equation $ax^2+bx+c=0$ are complex numbers, they are conjugate complex numbers.

28. Show that the product of the roots of the real quadratic equation $ax^2+bx+c=0$ is always $\frac{c}{a}$.

Form quadratic equations with the following pairs of roots:

29. $2\pm i$ 30. $-3\pm 2i$ 31. $0\pm i$

32. $-\sqrt{2}\pm 3i$ 33. $6\pm\sqrt{3}i$ 34. $\frac{1}{2}\pm\frac{1}{2}\sqrt{2}x$

35. Let P represent the complex number $6-i$ and Q the complex number $(0+i)\times(6-i)$. Show that POQ represents an isosceles triangle right-angled at the origin O.

36. Prove that the result in (35) is true if P represents the complex number $a+bi$ and Q the complex number $(0+i)\times(a+bi)$.

9 6 Division of Complex Numbers

In this section we consider the problem of dividing complex numbers.

Example 1

Express $\frac{2-3i}{4+i}$ in the form $a+bi$.

Solution: We use the fact that the product of a complex number and its conjugate is essentially a real number. We multiply the given expression by $\frac{4-i}{4-i}$, since $4-i$ is the conjugate of the denominator.

Then $$\frac{2-3i}{4+i}=\frac{2-3i}{4+i}\times\frac{4-i}{4-i}=\frac{5-14i}{17}=\frac{5}{17}-\frac{14}{17}i,$$

as required. Note that our operation here is similar to "rationalizing the denominator" (although, of course, we are actually "*real*izing" the denominator).

The student may check that $\frac{2-3i}{4+i}=\frac{5}{17}-\frac{14}{17}i$,

by showing that $2-3i=(4+i)\left(\frac{5}{17}-\frac{14}{17}i\right)$.

Division of a complex number by a nonzero complex number is always possible as in the example. The set of complex numbers is closed under division by nonzero complex numbers.

If $(a+bi)(c+di)=1+0i=1$, then $c+di$ is the multiplicative inverse of $a+bi$; by the definition of division as the inverse operation to multiplication, $c+di=\dfrac{1}{a+bi}$.

Example 2

Find the multiplicative inverse of $4-3i$.

Solution: We are asked to find $\dfrac{1}{4-3i}$.

$$\frac{1}{4-3i}=\frac{1}{4-3i}\times\frac{4+3i}{4+3i}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3}{25}i$$

Check: $(4-3i)\left(\dfrac{4}{25}+\dfrac{3}{25}i\right)=\dfrac{16}{25}-\dfrac{12}{25}i+\dfrac{12}{25}i-\dfrac{9}{25}i^2=1+0i=1$

EXERCISE 9—6

Divide the first complex number in each of the following pairs by the second:

1. $4+3i,\ 7+2i$
2. $6-3i,\ -2+4i$
3. $3+2i,\ -4-2i$
4. $-\frac{1}{2}+3i,\ -\frac{3}{2}+2i$
5. $6-4i,\ 6+4i$
6. $-\frac{1}{2}-\frac{3}{2}i,\ \frac{5}{2}+i$
7. $-4+\pi i,\ 4+\pi i$
8. $\sqrt{2}+i,\ 3+2i$
9. $6\sqrt{2}+i,\ -3+\sqrt{2}i$

Evaluate the following quotients in the form $a+bi$; multiply the resulting numbers to check (the product should be 1).

10. $\dfrac{6-3i}{1+2i},\ \dfrac{1+2i}{6-3i}$
11. $\dfrac{2+i}{4-3i},\ \dfrac{4-3i}{2+i}$
12. $\dfrac{1+\sqrt{2}i}{4+\sqrt{3}i},\ \dfrac{4+\sqrt{3}i}{1+\sqrt{2}i}$

Find the multiplicative inverses of the following complex numbers. Check each result.

13. $4+i$
14. $3-2i$
15. $6+3i$
16. $-6+2i$
17. $\frac{1}{2}-\frac{1}{2}i$
18. $\sqrt{3}+i$
19. $2-\sqrt{2}i$
20. $\sqrt{3}+\sqrt{2}i$
21. Find the complex number which is the multiplicative inverse of the nonzero complex number $a+bi$.

Solve the following equations for z:

22. $(1+i)z=3-2i$
23. $(-2+3i)z=2-5i$
24. $(6-2i)z=-\frac{1}{2}+3i$
25. $(4-3i)z=2$

9 7 Synthetic Division

Division of one polynomial by another can be carried out, if done carefully, in terms of the coefficients of the two polynomials, as in the following example.

Example

Divide $6x^3-x^2+3x-2$ by $x-2$ in the standard way.

Solution:

Full Calculation

$$
\begin{array}{r}
6x^2+11x+25 \\
x-2\overline{)\,6x^3-x^2+3x-2} \\
\underline{6x^3-12x^2} \\
11x^2+3x \\
\underline{11x^2-22x} \\
25x-2 \\
\underline{25x-50} \\
48
\end{array}
$$

quotient $6x^2+11x+25$

remainder 48

Coefficients Only

$$
\begin{array}{r}
6 \quad 11 \quad 25 \\
1-2\overline{)\,6 \;-\; 1 \quad 3 \;-\; 2} \\
\underline{6 \;-12} \\
11 \quad 3 \\
\underline{11 \;-22} \\
25 \;-\; 2 \\
\underline{25 \;-50} \\
48
\end{array}
$$

quotient $6 \quad 11 \quad 25$

remainder 48

$$\therefore \frac{6x^3-x^2+3x-2}{x-2} = 6x^2+11x+25+\frac{48}{x-2}.$$

In using this method of detached coefficients the following points must be noted:

(i) Coefficients must be supplied for missing terms in the polynomials; for example, the coefficients of $6x^3+3x-2$ are 6, 0, 3, -2 and not 6, 3, -2, and those of $6x^3-2$ are 6, 0, 0, -2, and not 6, -2.

(ii) It is necessary to determine by inspection the degree of the first term in the quotient. The remaining terms then have successively lower degrees. For example, in dividing x^7-4x^2+3 by $x+5$, the first term of the quotient is of degree 6.

The main value in using this method is the saving of time and effort in not writing the various powers of the variable.

When the divisor is a monic linear polynomial $x-a$, still further time and effort are saved by the use of synthetic division. This is a refinement of the method of detached coefficients. The genesis of this method is demonstrated by reconsidering the above example. The work with the coefficients only in the above example is repeated in Scheme A on the next page.

$$
\begin{array}{rrrrr}
 & 6 & 11 & 25 & \\
1-2\overline{)} & 6 & -1 & 3 & -2 \\
 & 6 & -12 & & \\
\hline
 & & 11 & 3 & \\
 & & 11 & -22 & \\
\hline
 & & & 25 & -2 \\
 & & & 25 & -50 \\
\hline
 & & & & 48
\end{array}
$$

Scheme A

$$
\begin{array}{rrrrr}
 & 6 & 11 & 25 & \\
1-2\overline{)} & 6 & -1 & 3 & -2 \\
 & 6 & -12 & & \\
\hline
 & & 11 & & \\
 & & 11 & -22 & \\
\hline
 & & & 25 & \\
 & & & 25 & -50 \\
\hline
 & & & & 48
\end{array}
$$

Scheme B

In Scheme A, there are a number of terms "brought down" and so repeated. This is unnecessary, and we may rewrite the calculations as in Scheme B.

Also in both Schemes A and B, the terms of the quotient are selected so that certain coefficients (shown in red) will be equal. It is unnecessary to repeat these coefficients and so we can use Scheme C.

$$
\begin{array}{rrrrr}
 & 6 & 11 & 25 & \\
1-2\overline{)} & 6 & -1 & 3 & -2 \\
 & & -12 & & \\
\hline
 & & 11 & & \\
 & & & -22 & \\
\hline
 & & & 25 & \\
 & & & & -50 \\
\hline
 & & & & 48
\end{array}
$$

Scheme C

$$
\begin{array}{rrrrr}
1-2\overline{)} & 6 & -1 & 3 & -2 \\
 & & -12 & & \\
\hline
 & & 11 & & \\
 & & & -22 & \\
\hline
 & & & 25 & \\
 & & & & -50 \\
\hline
 & & & & 48
\end{array}
$$

Scheme D

We note that in any of these Schemes, since we are dividing by a *monic* linear polynomial $x-a$, the coefficients in the quotient will be the same as the coefficients we have just omitted in Scheme C (originally these same coefficients occurred *three times*). In fact there is no need to repeat them twice and we may omit the repetition above the division bar and arrive at Scheme D.

When we divide by $x-a$, the leading coefficient of the divisor will always equal 1 and so we may agree to omit it. Also there is no point in having the remaining work spread out; we condense it as in Scheme E.

$$
\begin{array}{rrrrr}
-2\overline{)} & 6 & -1 & 3 & -2 \\
\hline
 & -12 & -22 & -50 & \\
\hline
 & 11 & 25 & 48 &
\end{array}
$$

Scheme E

$$
\begin{array}{r|rrrr}
-2 & 6 & -1 & 3 & -2 \\
\hline
 & & -12 & -22 & -50 \\
\hline
 & 6 & 11 & 25 & 48
\end{array}
$$

Scheme F

If we now repeat the leading coefficient 6 of the dividend on the third line, as in Scheme F, we see that the members of the third line are the coefficients of the quotient $6x^2+11x+25$ together with the remainder 48.

When we study the calculations, we notice that the second term on the third line of Scheme F is obtained by multiplying the first term on the third line by -2, placing the product on the second line under the second coefficient of the first line, and subtracting. The third term on the third line is obtained by multiplying the second term of the third line by -2, placing the product on the second line, and subtracting. These details are shown in Scheme G.

-2	6	-1	3	-2	
		$6\times(-2)$	$11\times(-2)$	$25\times(-2)$	*Subtract* line 2
	6	11	25	48	from line 1.

Scheme G

The same result could have been obtained by multiplying the third line coefficients by $+2$ and adding columns as in Scheme H. This gives us our final refinement for synthetic division.

2	6	-1	3	-2	
		6×2	11×2	25×2	*Add* line 2
	6	11	25	48	to line 1.

Scheme H

Rule for Synthetic Division

To divide polynomial $P(x)$ by $x-a$,

(1) write down the coefficients of $P(x)$, supplying zeros for any missing terms;

(2) write the number a (*not* $-a$) to the left of these coefficients;

(3) repeat the first coefficient of $P(x)$ on the third line;

(4) multiply this coefficient by a, place the product on the second line under the second coefficient of $P(x)$, and add to get the second term on the third line;

(5) multiply this term by a, place on the second line below the third coefficient of $P(x)$, and add;

(6) continue this operation as long as possible.

The coefficients on the third line, except the last one, will be the coefficients of the quotient $Q(x)$ (a polynomial of degree one less than that of $P(x)$) written in descending powers of x, and the last coefficient will be the constant remainder R.

Example

Divide $P(x)=3x^4+2x^2+3x-2$ by

(i) $x+2$ (ii) $x-3$.

Solution: (i)

-2	3	0	2	3	-2
		-6	12	-28	50
	3	-6	14	-25	48

If $x-a=x+2$, then $a=-2$.

quotient: $3x^3-6x^2+14x-25$ remainder: 48

$3x^4+2x^2+3x-2=(x+2)(3x^3-6x^2+14x-25)+48.$

Check: R.S. $=3x^4-6x^3+14x^2-25x+6x^3-12x^2+28x-50+48$

$=3x^4+2x^2+3x-2=$ L.S.

Solution: (ii)

3	3	0	2	3	-2
		9	27	87	270
	3	9	29	90	268

If $x-a=x-3$, then $a=3$.

quotient: $3x^3+9x^2+29x+90$ remainder: 268

$3x^4+2x^2+3x-2=(x-3)(3x^3+9x^2+29x+90)+268.$

We check by recombining the terms on the right.

EXERCISE 9—7

Use the method of detached coefficients to carry out the division of $P(x)$ by $x-a$ in the following:

1. $P(x)=6x^2-3x+5$ (i) $x-2$ (ii) $x+3$ (iii) $x-\frac{1}{2}$
2. $P(x)=x^3-5x^2+6x-2$ (i) $x+1$ (ii) $x-3$ (iii) $x+\frac{1}{2}$
3. $P(x)=4x^3-6x+3$ (i) $x-2$ (ii) $x+4$ (iii) $x+\frac{1}{3}$
4. $P(x)=x^4-6x^3-6x^2+3x-2$ (i) $x+3$ (ii) $x-1$ (iii) $x+\frac{1}{2}$
5. $P(x)=x^5-5x^3+6x$ (i) $x-1$ (ii) $x+2$ (iii) $x-\frac{1}{2}$
6. Check the results in the above problem using synthetic division.

Use synthetic division to carry out the division of $P(x)$ by $x-a$ in the following. In each case express $P(x)$ in the form $(x-a)Q+R$ and check your work by recombining this expression.

7. $P(x)=2x^3-6x^2+3x-1$ (i) $x+2$ (ii) $x-1$
8. $P(x)=x^4-6x^2+2$ (i) $x+1$ (ii) $x-2$
9. $P(x)=5x^4-3x+1$ (i) $x+3$ (ii) $x-1$

10. $P(x)=2x^5-4x^3+3x-7$ (i) $x-1$ (ii) $x+2$

11. Find k so that when $2x^3-3x^2+4kx-2$ is divided by $x-a$, the remainder is zero.

12. Find k so that when x^3-kx^2+3x+9 is divided by $x+3$, the remainder is zero.

9 8 The Remainder Theorem

We now prove a very elegant result connected with the division of a polynomial $P(x)$ by a monic linear polynomial $x-a$.

We have seen that, given a polynomial $P(x)$ and a monic linear polynomial $x-a$, we may always write

$$P(x)=(x-a)Q(x)+R,$$

where $Q(x)$ is a polynomial of degree one less than that of $P(x)$ and R is a constant. Now $(x-a)Q(x)+R$ is just another way of writing $P(x)$ so that $P(x)$ and $(x-a)Q(x)+$ R will have the same value for all values of x. This fact allows us to prove the following theorem.

REMAINDER THEOREM: When a polynomial $P(x)$ is divided by a monic linear polynomial $x-a$, the constant remainder is the value of the polynomial $P(x)$ when x has the value a; that is, $R=P(a)$.

Proof: Write $P(x)=(x-a)Q(x)+R$.

The two members of this expression are equal for all values of x. When x is replaced by a in this expression

$$\begin{aligned} P(a) &= (a-a)Q(a)+R \\ &= (0)Q(a)+R; \end{aligned}$$

that is, $P(a)=R$, as required.

Example

Find the remainder when $P(x)=x^4-x^3+6x^2-5x+2$ is divided by

(i) $x-2$ (ii) $x+3$.

Solution: (i) By the theorem, $R=P(2)=2^4-2^3+6(2^2)-5(2)+2=24$.

Let us check by synthetic division (or by long division).

2	1	−1	6	−5	2
		2	2	16	22
	1	1	8	11	24

(ii) By the theorem, $R=P(-3)=81+27+54+15+2=179$.

The check is left to the student.

EXERCISE 9–8

Use the Remainder Theorem to find the remainder when each of the following polynomials is divided by the given $x-a$:

1.	$6x^2-2x+1$	(i) $x-2$	(ii) $x+3$	(iii) $x-\frac{1}{2}$
2.	$2x^3-4x^2+3x+5$	(i) $x+1$	(ii) $x-5$	(iii) $x-\frac{1}{3}$
3.	x^3-x+7	(i) $x+6$	(ii) $x-4$	(iii) $x+\frac{3}{2}$
4.	x^4-x^3+3x+7	(i) $x-4$	(ii) $x+3$	(iii) $x+\frac{1}{2}$
5.	x^5-6x+7	(i) $x+2$	(ii) $x-3$	(iii) $x-\frac{1}{3}$
6.	$x^6-6x^5+x^4-x^3+x^2-3x+1$	(i) $x+1$	(ii) $x-1$	(iii) $x-2$

7. Find the values of k for which $x^3-k^2x^2-kx-6$ leaves a remainder of 15 on division by $x-3$.

8. Find m so that $2x^3-3x^2-mx+20$ leaves a remainder of zero on division by (a) $x-5$ (b) $x+4$.

9. Find k so that $kx^4+k^2x^3+4x^2+4x+6$ leaves a remainder of zero on division by $x-3$.

10. Are there any real values for m for which $mx^3-3m^2x^2+4x-16$ leaves
    (a) a remainder of 4 on division by $x-2$?
    (b) a remainder of -8 on division by $x-2$?

11. For what values of m will $4x^2+2x-3$ leave a remainder of 3 on division by (a) $x-m$ (b) $x-2m$?

12. Use the Remainder Theorem to prove the Factor Theorem and its converse.

9 9 The Number of Roots of a Polynomial Equation

We know that if $ax=b$ and $a\neq 0$, then $\frac{1}{a}(ax)=\frac{1}{a}(b)$, that is, $x=\frac{b}{a}$,

so that the real linear equation $ax=b(a\neq 0)$ has exactly one real root.

Similarly, if $a\neq 0$, and if $ax^2+bx+c=0$, then we know that x must have one of the two values

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a};$$

that is, the quadratic equation has exactly two roots. If $b^2-4ac\geq 0$, the equation has two real roots which are equal if $b^2-4ac=0$; if $b^2-4ac<0$, the equation has two (conjugate) complex roots ((27) Exercise 9-5).

Now what can be said about cubic equations? We have seen that a cubic equation can have three distinct real roots. For example,

$$(x-1)(x+1)(x-2)=x^3-2x^2-x+2=0$$

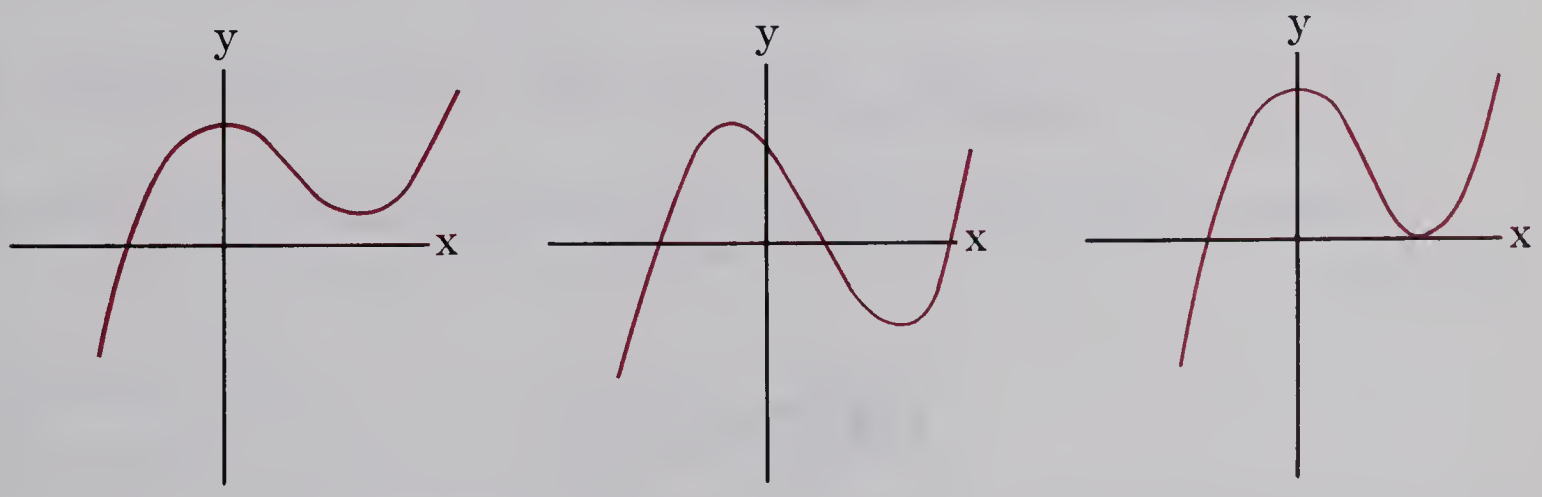
is a cubic equation with three distinct real roots 1, -1, 2.

In fact every cubic equation with real coefficients has at least *one* real root. We can show this is true in an example. Consider the cubic equation $x^3+3x^2+4x+6=0$. Suppose we plot the curve whose equation is $y=x^3+3x^2+4x+6$. If we assign large negative values to x, then y will be negative. For example, if $x=-10$, $y=(-100)^3+3(-100)^2+4(-100)+6$ will certainly be negative (the term $(-100)^3$ ensures that), and if $x=-1000$, $y=(-1{,}000{,}000)^3+3(-1{,}000{,}000)^2+4(-1{,}000{,}000)+6$ will also be negative. On the other hand, if $x=100$, $y=(100)^3+3(100)^2+4(190)+6$ will be positive. This means that part of the curve is below the x-axis and part above and if we assume that the graph is an unbroken curve (and this is always true for the graph whose equation is $y=\mathrm{P}(x)$, where $\mathrm{P}(x)$ is a polynomial, then the curve must cross the x-axis at least once. The value of x at a point where the curve crosses the x-axis will be such that

$$x^3+3x^2+4x+6=0;$$

that is, will be a root of this equation.

As we have stated, this is true for the general cubic equation $ax^3+bx^2+cx+d=0$. If a is positive, the curve $y=ax^3+bx^2+cx+d$ must look like one of the following:



In the first case, the cubic will have one real root; in the second and third, it will have three.

Indeed, having determined one real root of the cubic, say $x=p$, we may write

$$ax^3+bx^2+cx+d=(x-p)\mathrm{Q}(x)$$

where $\mathrm{Q}(x)$ is a quadratic. Then the roots of $\mathrm{Q}(x)=0$ will also be roots of the cubic equation. Since we have already determined that a quadratic

equation has either two real or two complex roots, we see that a cubic equation will have either one real root and two complex roots or three real roots. It cannot have more than three roots.

A quartic equation can certainly have four distinct real roots; for example,

$$(x-1)(x+1)(x-2)(x+2)=x^4-5x^2+4=0$$

is such an equation.

The quartic equation

$$(x-1)(x+1)(x^2+4)=x^4+3x^2-4=0$$

has two real roots 1 and -1 but any other root must satisfy $x^2+4=0$ and this equation has two complex roots.

Finally, the quartic equation

$$(x^2+1)(x^2+4)=x^4+5x^2+4=0$$

has no real roots, since the roots of this equation must satisfy either $x^2+1=0$ or $x^2+4=0$.

A quartic equation can have four, two, or no real roots.

The following statements are made without proof:

(1) A polynomial equation of degree n cannot have more than n distinct real roots.

(2) If n is odd, a polynomial equation of degree n has at least one real root and it must have an odd number of real roots.

(3) If n is even, a polynomial equation of degree n has an even number of real roots (this could be zero).

Of course, if the constant term in a polynomial equation is zero, then 0 is a root.

EXERCISE 9—9

State the possible number of real roots of the following equations. (Do not attempt to solve the equations.)

1. $x^2-6x-7=0$	2. $y^3+3y-4=0$	3. $-6x^3+x^2=2$
4. $3z^4-6z^2+2z=0$	5. $3x^4-x^2+2=0$	6. $y^5-5y^2+2y=0$
7. $x^5-6x^3+3x-2=0$	8. $y^8-4y^4+y^2-2=0$	9. $2x^7-x^5=5-2x$
10. $t^9-6t^3=0$	11. $4a^6-3a+2=0$	12. $6x^5-5x=\sqrt{2}$

9 10 Chapter Summary

A polynomial equation is always equivalent to a monic polynomial equation.
Formulae in terms of the coefficients of a polynomial equation exist for the roots of equations of degree less than five but not for equations of degree ≥ 5.
Complex numbers $a+bi$; a, b real numbers, $i^2=-1$.

(1) $a+bi=c+di$ if and only if $a=c$ and $b=d$.

(2) The complex number $x+yi$ is represented by the point (x, y) in the Argand plane.

(3) Addition of complex numbers.

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

(i) The set of complex numbers is closed under addition
(ii) Addition is commutative and associative
(iii) $0+0i$ is the zero complex number, the additive identity
(iv) $-a-bi$ is the additive inverse of $a+bi$.

(4) Multiplication of complex numbers.

$$(a+bi)\times(c+di)=(ac-bd)+(ad+bc)i$$

(i) The set of complex numbers is closed under multiplication
(ii) Multiplication is commutative and associative
(iii) $1+0i$ is the multiplicative identity or unit
(iv) $\dfrac{a}{a^2+b^2}-\dfrac{b}{a^2+b^2}i$ is the multiplicative inverse of $a+bi$.

(5) Multiplication is distributive over addition.

(6) $a+bi$ and $a-bi$ are conjugate complex numbers.

(7) $\dfrac{a+bi}{c+di}$ is found by writing $\dfrac{a+bi}{c+di}=\dfrac{a+bi}{c+di}\cdot\dfrac{c-di}{c-di}$.

Synthetic division. The Remainder Theorem.
The number of roots of a polynomial equation.

REVIEW EXERCISE 9—10

1. Show that the polynomials $P(x)=3x^2-2x-1$ and $Q(x)=x^2-\frac{2}{3}x-\frac{1}{3}$ are not identical but have the same zeros.

2. Sketch on the same graph the parabolas whose equations are $y=3x^2-2x-1$ and $y=x^2-\frac{2}{3}x-\frac{1}{3}$.

Find (a) the sum (b) the difference (c) the product (d) the quotient (first divided by second) of the following pairs of complex numbers:

3. $5-2i,\ 3-i$ 4. $6-\frac{1}{2}i,\ \frac{1}{2}-6i$ 5. $\sqrt{2}-i,\ \sqrt{2}+i$

6. $4-\sqrt{3}i,\ 4+\sqrt{3}i$ 7. $2+3i,\ -2-3i$ 8. $\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ 3-2i$

Verify, by finding the roots of the following equations $ax^2+bx+c=0$, that the sum of the roots is $-\frac{b}{a}$ and their product is $\frac{c}{a}$:

9. $x^2-4x+6=0$ 10. $2x^2-\frac{1}{2}x+4=0$ 11. $-y^2+2y-4=0$

12. Plot points on the Argand plane representing the following complex numbers. What is the connection between the points?
(a) $4+3i$ (b) $(0+i)\times(4+3i)$
(c) $(-1+0i)\times(4+3i)$ (d) $(0-i)\times(4+3i)$

Solve the following equations for x:

13. $\mathrm{Re}(2x+yi)=4$ 14. $\mathrm{Re}(6-2i)=3x$ 15. $\mathrm{Im}(6-\frac{1}{2}i)=4x$

Find the multiplicative inverses of the following complex numbers. Check each result.

16. $2-3i$ 17. $-\frac{1}{2}+5i$ 18. $\sqrt{2}+\frac{3}{2}i$

Solve the following for z:

19. $(3-i)z=(2-i)\times(-1\times i)$ 20. $(6-2i)z=\dfrac{3+i}{1-i}$

21. Find a quadratic equation whose roots are $-6+4i$ and $-6-4i$.

22. Is it possible to find a quadratic equation with real coefficients whose roots are $-6+4i$ and $6-4i$? Explain.

Use synthetic division to express the following polynomials $\mathrm{P}(x)$ in the form $(x-a)\mathrm{Q}(x)+\mathrm{R}$ for the given $x-a$:

23. $\mathrm{P}(x)=5x^2-4x+2$ (a) $x-1$ (b) $x+3$ (c) $x-\frac{1}{2}$

24. $\mathrm{P}(x)=4x^3+x^2+x-6$ (a) $x+1$ (b) $x-2$ (c) $x+3$

25. $\mathrm{P}(x)=x^5-x^4-6x^2-1$ (a) $x+1$ (b) $x+2$ (c) $x-1$

26. $\mathrm{P}(x)=2x^2-5\sqrt{2}x+6$ (a) $x+1$ (b) $x-\sqrt{2}$ (c) $x+\sqrt{2}$

Use the Remainder Theorem to find the remainder when the following polynomials $P(x)$ are divided by the indicated $x-a$:

27. $P(x)=6x^2-3x+2$ (a) $x-1$ (b) $x+2$ (c) $x-\frac{1}{2}$

28. $P(x)=2x^3-6x+5$ (a) $x+5$ (b) $x-5$ (c) $x+\frac{1}{2}$

29. $P(x)=3x^4-5x^2+6x$ (a) x (b) $x-2$ (c) $x+3$

30. $P(x)=4x^2-2\sqrt{3}x-5$ (a) $x+2$ (b) $x-\sqrt{3}$ (c) $x+\sqrt{3}$

31. Find k so that $kx^4-k^2x^2+3x+6$ leaves zero remainder on division by $x-2$.

32. For what values of m will $2x^2-3x+4$ leave a remainder of 3 on division by $x-m$?

33. For what k will -3 be a root of $x^4-k^2x^2+2x-k=0$?

State the possible number of real roots of the following equations (do not attempt to solve the equations):

34. $6x^2+3x-4=0$

35. $x^3-5x+2=0$

36. $x^4-6x^2+5x=0$

37. $-x^8+6x^2-2=0$

38. $5x^{10}-6x^8+x^6-x^4=0$

39. $3x^7-x^6+5x^4-3x^2-2=0$

40. Sketch graphs to illustrate that a real quintic equation can have either one, three, or five real roots.

SYMBOLS AND ABBREVIATIONS IN GEOMETRY

$=$	is equal to	∥ gm.	parallelogram
$\neq$	is not equal to	rect.	rectangle
$\equiv$	is congruent to	int.	interior
$\not\equiv$	is not congruent to	co-int.	co-interior
$\angle$	angle	ext.	exterior
$\triangle$	triangle	sq.	square
$\therefore$	therefore	vert. opp.	ertically opposite
$\because$	because or since	str.	straight
$>$	is greater than	rt. $\angle$	right angle
$<$	is less than	supp.	supplementary
$\odot$	circle	comp.	complementary
$\perp$	(is) perpendicular to	reqd.	required
$\parallel$	(is) parallel to	hyp.	hypotenuse
$\nparallel$	is not parallel to	pt.	point
alt.	alternate	diag.	diagonal
corr.	corresponding	isos.	isosceles
adj.	adjacent	$\|\|\|$	is similar to
const.	construction	incl.	included
quad.	quadrilateral	defn.	definition

Chapter 10

The Circle

10 1 Definitions

A *circle* is the set of points in a plane which are a constant distance from a given point in the plane. The given point is the *centre.*

A *radius* of a circle is a line segment joining the centre to any point on the circle.

A *chord* of a circle is a line segment joining any two points on the circle.

A *diameter* of a circle is a chord which passes through the centre.

A *secant* is any line intersecting a circle at two distinct points.

In Figure 10-1, O is the centre of the circle, OA a radius, BC a diameter, DE a chord, and PS a secant.

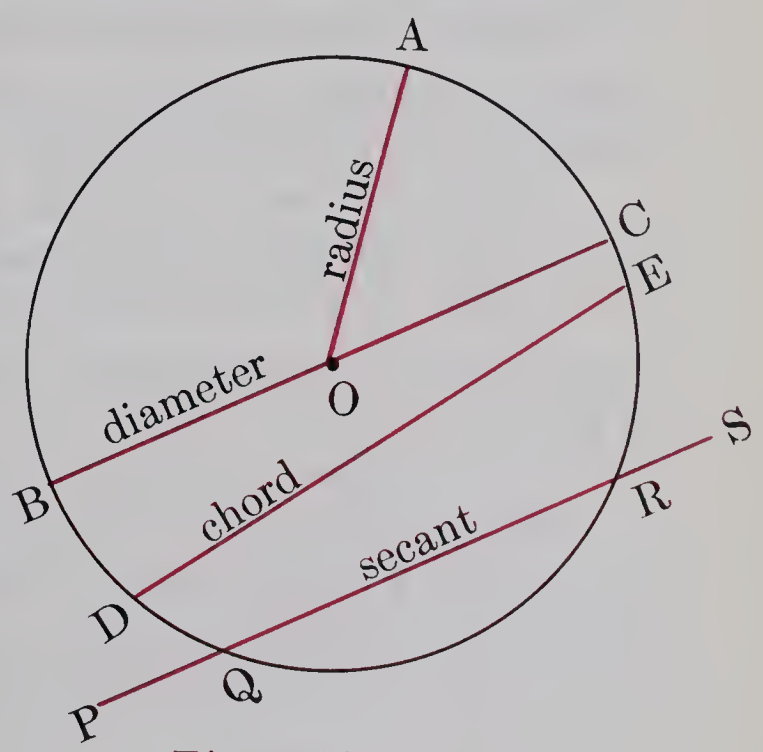


Figure 10-1

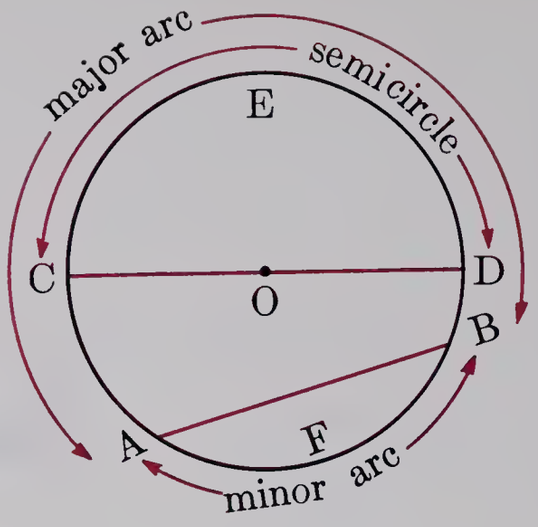
A circle may be described in terms of its centre and the length of its radius, or any three points on the circle. (We will see later that three non-collinear points determine a unique circle.) In Figure 10-1 we have a circle, centre O, or circle ABC. This is frequently abbreviated to ⊙, centre O, or ⊙ ABC where the sign ⊙ is used as an abbreviation for circle.

In Figure 10-2, C and D are the endpoints of a diameter. Each of the two sets of points on the circle from C to D constitutes a *semicircle.*

If A and B are any two points on a circle, then the set of all points on the circle from A to B is the *arc* of the circle (symbol $\overset{\frown}{AB}$).

If the set of points of an arc AB is a subset of the points of a semicircle, then $\overset{\frown}{AB}$ is a *minor* arc.

If the set of points of a semicircle is a subset of the set of points of an arc AB, then $\overset{\frown}{AB}$ is a *major* arc.



To avoid ambiguity in describing the two arcs AB (minor and major), we can use a point on the particular arc. In Figure 10-2 the minor arc AB can be referred to as $\overset{\frown}{AFB}$ and the major arc AB as $\overset{\frown}{AEB}$.

Certain sets of points form regions associated with the circle.

The *interior* of a circle is the set consisting of all points in the plane of the circle whose distances from the centre are *less than* the radius of the circle.

The *exterior* of a circle is the set consisting of all points in the plane of the circle whose distances from the centre are *greater than* the radius of the circle.

Notice that in these two statements we have used radius instead of length of radius. This is a common abbreviation. Similarly the word diameter is used in both senses.

Usually we use the words *inside* and *outside* as abbreviations for "in the interior of" and "in the exterior of". We note that a circle divides the set of points in the plane of the circle into three subsets. If O is the centre of a circle with radius r and P is any point in the plane of the circle, then P is inside the circle if and only if $OP < r$; P is on the circle if and only if $OP = r$; P is outside the circle if and only if $OP > r$.

The set of points forming the interior of a circle may be further divided into various subsets.

A *segment* of a circle is that set of points in the plane of the circle bounded by a chord of the circle and the arc intercepted by that chord. A *major segment* is bounded by a chord and its major arc; a *minor segment* is bounded by a chord and its minor arc (Figure 10-3).

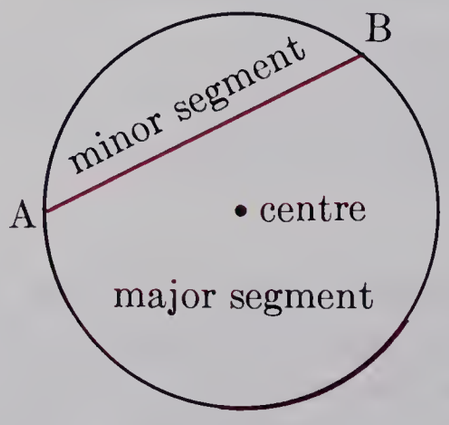


Figure 10-3

We note that if the centre of the circle is inside the set of points forming the segment, then that segment is a major segment.

A *sector* of a circle is that set of points in the plane of the circle bounded by an arc, called the *sector arc*, and the radii joining the endpoints of the arc to the centre.

The angle in the sector at the centre of the circle is called the *sector angle*.

Corresponding to major and minor arcs we have major and minor sectors as in Figure 10-4.

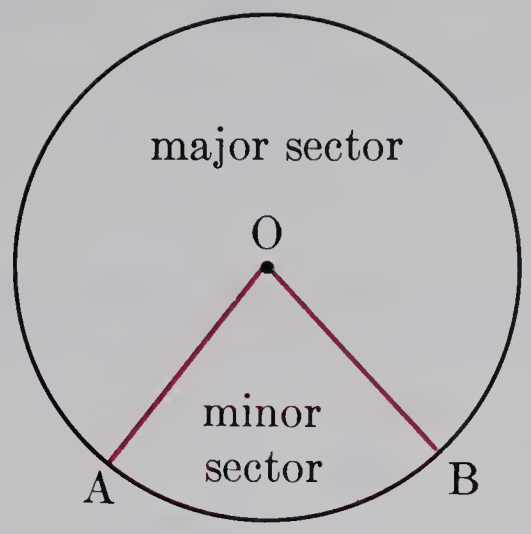


Figure 10-4

EXERCISE 10—1

1. Draw a circle of radius 2 in. and make an angle of 120° at the centre O. The sides of the angle cut the circle at A and B.
   Mark (a) the minor arc AB, (b) the major arc AB, (c) the diameter through A, (d) the minor sector AOB, (e) the major sector AOB.
2. Draw a circle of radius 5 cm. and a chord of length 6 cm. with endpoints A and B.
   Mark (a) the minor segment, (b) the major segment, (c) the secant of which chord AB is a part.
3. Draw a circle of radius $1\frac{1}{2}$ in. Draw two chords, one 2 in. long and the other $2\frac{1}{2}$ in. long. Which is nearer the centre?
4. Draw a circle of radius 5 cm. Construct a chord 3 cm. from the centre. Measure the length of the chord.
5. Draw a circle of radius $1\frac{1}{2}$ in. and two nonparallel chords, one 1 in. long and the other 2 in. long. Construct the perpendicular bisectors of the chords. Where do they intersect? Where should they intersect if your construction is accurate? Why did we have to specify nonparallel chords?

10 2 Circumference, Arc Length, and Area

The *circumference* of a circle is the length of the circle. While this definition suffices for our purposes, there are some logical objections to it. Those interested in a more rigorous development of circumference and area of a circle should refer to Appendix A (page 363).

If C is the measure of the circumference and d the measure of the diameter in the same unit, we know from our work in earlier years that $C = \pi d$ or $C = 2\pi r$, where r is the measure of the radius.

A circle encloses a part of the plane and this part is a simple closed figure or circular region. As such it has associated with it a unique real number called the measure of its area or simply its area. Normally, for brevity, we talk about the area of a circle instead of the area of the circular region. However, we should keep in mind that the circle has length, or circumference, and that its area is the area of the region. Again we know that if A represents the measure of the area, then $A = \pi r^2$.

A *sector angle* is the angle formed by the two radii of the sector.

In Figure 10-5 the sector angle of the minor sector AOB is $\angle AOB$.

If the length of AB is a units, the radius of the circle r units, and the measure of $\angle AOB$ in *radians* is θ, then from our work in Grade 11 we know that

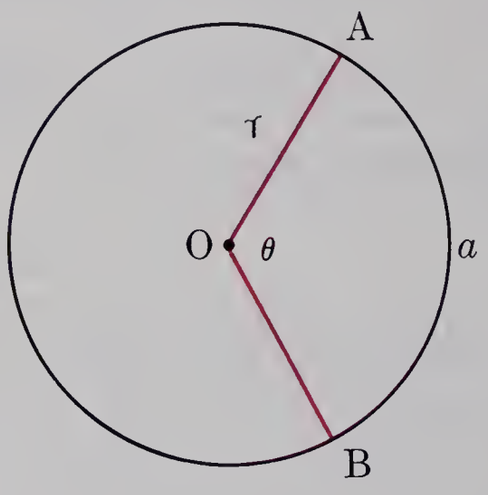


Figure 10-5

$$\theta = \frac{a}{r}.$$

$$\therefore\ a = r\theta. \qquad (\theta \text{ in radians})$$

If θ is given in sexagesimal measure, then

since $180° = \pi$ radians,

$$\theta° = \frac{\pi\theta}{180} \text{ radians.}$$

$$\therefore\ a = \frac{\pi r \theta}{180}. \qquad (\theta \text{ in degrees})$$

The comparative simplicity of the first formula for arc length is immediately obvious.

In the same way the area of a sector is easily calculated if the measure of the sector angle, in radians, is known.

If the sector angle in Figure 10-5 were 2π radians, the sector would be the interior of the circle.

$$\therefore\ \frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\theta}{2\pi}.$$

$$\therefore\ \text{Area of sector} = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \tfrac{1}{2} r^2 \theta. \qquad (\theta \text{ in radians})$$

Again, if θ is the measure of the sector angle in degrees and A the measure of the area of the sector, this would become

$$A = \frac{\pi r^2 \theta}{360}, \text{ since } \theta^\circ = \frac{\pi\theta}{180} \text{ radians.}$$

Also, since $A = \frac{1}{2}r^2\theta$ and $a = r\theta$,

$$\therefore A = \tfrac{1}{2}r \times r\theta$$
$$= \tfrac{1}{2}ar.$$

This enables us to calculate the area of the sector given the length of the sector arc and the radius.

Example

Calculate the length of the sector arc and the area of the sector in a circle of radius 5 in. if the sector angle is 50°.

Solution:

$\theta = 50^\circ \simeq .873$ (radians) (from the tables)

$r = 5$ (in.)

$a = r\theta$

$\simeq 5 \times .873$

$= 4.365$

$\therefore$ length of arc $\simeq 4.4$ in.

$A = \frac{1}{2}r^2\theta$

$\simeq \dfrac{25 \times .873}{2}$

$\simeq 10.9$

$\therefore$ Area of sector $\simeq 11$ sq. in.

EXERCISE 10—2

(Use $\frac{22}{7}$ as an approximation for π unless otherwise stated and give results correct to 2 figures where necessary.)

1. Prove that the length of a string stretched around the equator of the earth, at a constant height of 3 ft. above the ground, is 6π ft. greater than the circumference of the earth. (Assume that the equator is a circle.)

2. A circle is inscribed in a square. If the length of side of the square is 7 in., find the area of that part of the square outside the circle.

3. If, in (2), the length of side of the square is $2r$ in., find the ratio of the area of the circle to that of the square.

4. If A is the measure of the area of a circle and C the measure of the circumference, prove that $A = \frac{C^2}{4\pi}$.

5. Find the length of the sector arc and the area of each of the following sectors:

   (i) sector angle = 75°, radius = 8 in.

   (ii) sector angle = 2 radians, radius = $3\frac{1}{2}$ in.

   (iii) sector angle = $\frac{\pi}{4}$ radians, radius = $2\frac{1}{2}$ ft.

   (iv) sector angle = 60°, radius = 6 ft.

   (v) sector angle = $1\frac{3}{5}$ radians, radius = $2\frac{1}{2}$ ft.

6. Find, in radians and degrees, the size of the sector angle in each of the following cases:

   (i) sector arc = 5 in., radius = 4 in.

   (ii) sector arc = $\frac{\pi}{6}$ in., radius = 2 in.

   (iii) sector arc = $\frac{\pi}{8}$ ft., diameter = 8 ft.

7. Find the area of a sector of a circle of diameter 3.6 in. if the length of the sector arc is 1.2 in.

8. A flywheel turns through an angle of 120 radians per second and is driven by a belt travelling at 60 ft. per second. Find the radius of the flywheel.

9. A sector of a circle, radius 35 in., has an area of 588 sq. in. Find the size of the sector angle in radians and to the nearest degree.

10. A sector of a circle, radius 16 in., has an area of 44 sq. ft. Find the length of the sector arc.

In the remaining examples use 3.142 as an approximation for π and evaluate to 3 figures using logarithm tables.

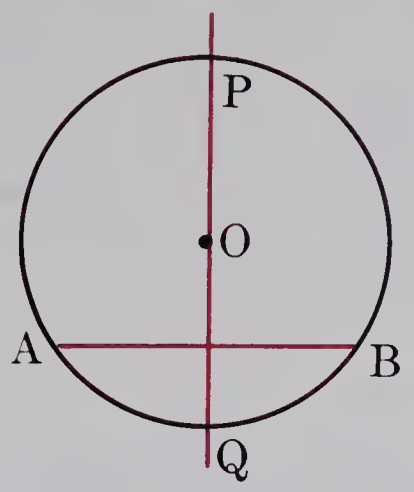
11. A sector of a circle has an area of 6.55 sq. ft. and the sector angle is 30°. Find the radius of the circle.

12. A sector of a circle has a radius of 6.40 in. and the sector angle is 50°. Find the length of the sector arc and the area of the sector.

13. Find, to the nearest degree, the size of the sector angle if the area of the sector is 6.54 sq. in. and the radius is 4.14 in.

10 3 Properties of Chords

THEOREM: The centre of a circle lies on the perpendicular bisector of any chord. (⊥ bisector of chord)



Given: ⊙ centre O, with PQ the perpendicular bisector of any chord AB.

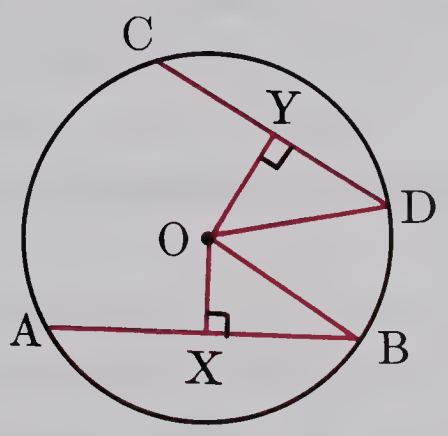
Required: To prove that O lies on PQ.

Proof: PQ is the locus of points equidistant from A and B. (Why?)
But AO = OB. (Radii of ⊙)
∴ O lies on PQ.

Corollaries:
1. The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. (Bisector of chord)
2. The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord. (⊥ from centre)
3. There is a unique circle passing through three noncollinear points.
   This circle is called the circumcircle of the triangle determined by the points.

Example

Prove that if AB and CD are two chords of a circle such that AB > CD, then AB is nearer to the centre.



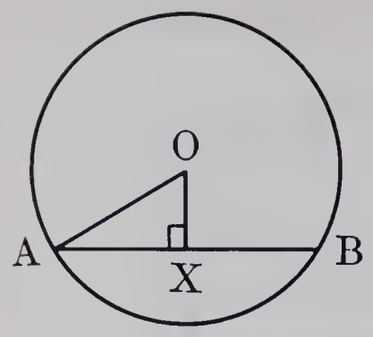
Given: ⊙, centre O, with chords AB and CD such that AB > CD, OX ⊥ AB, and OY ⊥ CD.

Required: To prove OY > OX.

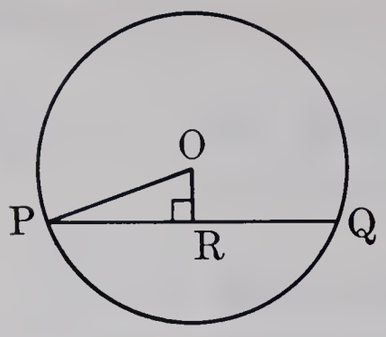
Proof: Join OB and OD.

$XB = \frac{1}{2}AB$ and $YD = \frac{1}{2}CD$	(⊥ from centre)
But $AB > CD$.	(given)
∴ $XB > YD$.	
$OD^2 = OY^2 + YD^2$	(Pythagoras)
$OB^2 = OX^2 + XB^2$	(Pythagoras)
But $OD = OB$.	(Radii of same ⊙)
∴ $OY^2 + YD^2 = OX^2 + XB^2$.	
∴ $OY^2 - OX^2 = XB^2 - YD^2$.	
But $XB^2 > YD^2$.	($XB > YD$)
∴ $OY^2 > OX^2$.	
∴ $OY > OX$.	

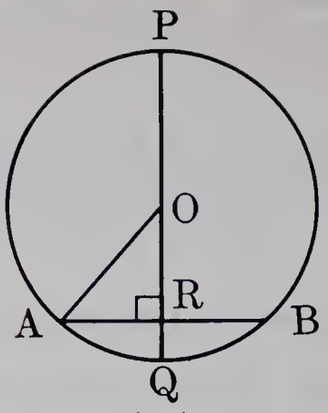
EXERCISE 10—3



(*i*)



(*ii*)



(*iii*)

1. In diagram (*i*), the chord AB is 8 cm. long and OX⊥AB. If OX = 3 cm. and O is the centre of the circle, calculate the length of the radius.
2. In diagram (*ii*), O is the centre of the circle. If PQ = 24 cm. and the radius is 13 cm., calculate the distance of PQ from O.
3. In diagram (*iii*), the chord AB = 16 cm. and the radius OA = 17 cm. PQ⊥AB. Calculate the lengths of PR and RQ.
4. Prove that the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. (Corollary 1)
5. Prove that the line drawn from the centre of a circle perpendicular to a chord, bisects the chord. (Corollary 2)
6. Construct three circles to pass through two given points. How many such circles can be constructed?
7. Prove that the locus of the centres of circles passing through two fixed points is the perpendicular bisector of the line segment joining the two points.
8. AB and BC are two chords of a circle. If the perpendicular bisectors of AB and BC intersect at O, prove that O is the centre of the circle. (See (5), Ex. 10-1.)
9. Prove that there is a unique circle passing through three noncollinear points (Corollary 3). Construct the circle.
10. Prove that two equal chords of a circle are equidistant from the centre. (= chords)
11. State and prove the converse of (10). (Equidistant from centre)
12. If AB and CD are two chords of a circle such that AB is nearer to the centre, prove that AB > CD.
13. Two circles intersect at A and B. PAQ and RBS are two parallel lines meeting the circles at P, Q and R, S respectively. Prove PQ = RS.

10 4 Sector Angle and Inscribed Angle

DEFINITION: An angle is *inscribed* in a circle when its vertex is on the circle and each side of the angle intersects the circle at one other point.

The points at which the sides of the angle intersect the circle determine two arcs. The angle is said to be *subtended* in the circle by the arc not containing the vertex of the angle. In Figure 10-6

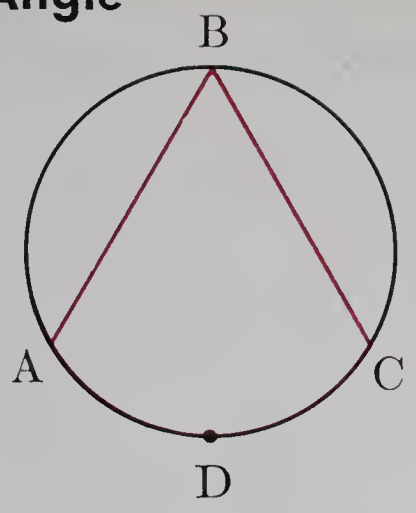
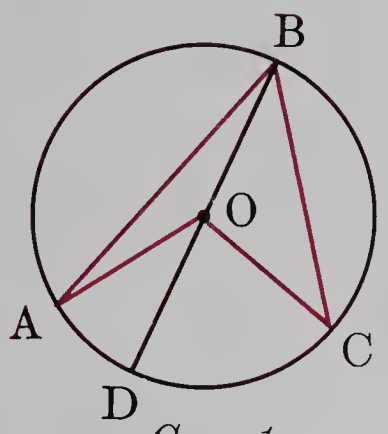


Figure 10-6

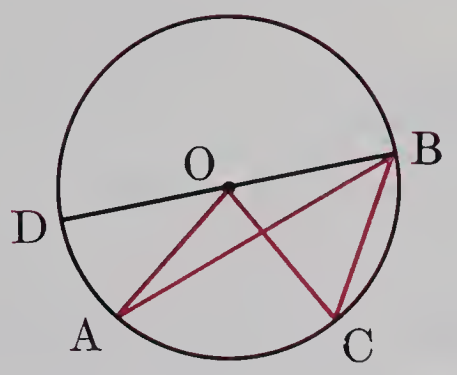
$\angle ABC$ is an *inscribed angle*; $\angle ABC$ is *subtended* by $\overset{\frown}{ADC}$.

If AD and CD are joined, $\angle ADC$ is also an inscribed angle but is subtended by $\overset{\frown}{ABC}$. What are the subtending arcs for $\angle BAD$ and $\angle BCD$?

THEOREM: A sector angle in a circle is twice the inscribed angle subtended by the same arc. (sector $\angle = 2$ inscribed $\angle$)



Case 1



Case 2

Given: $\odot$, centre O, with A, B, and C three points on the circle such that $\angle ABC$ is an inscribed angle.

Required: To prove $\angle AOC = 2\angle ABC$.

Proof: Join BO and produce to D.

$OA = OB$	(radii of same $\odot$)
$\therefore \angle OAB = \angle OBA.$	(base $\angle$'s of isos. $\triangle$)
Also $\angle ACD = \angle OAB + \angle OBA.$	(ext. $\angle$ of $\triangle$ = sum of int. opp. $\angle$'s)
$\therefore \angle AOD = 2\angle OBA.$	
Similarly $\angle COD = 2\angle OBC.$	

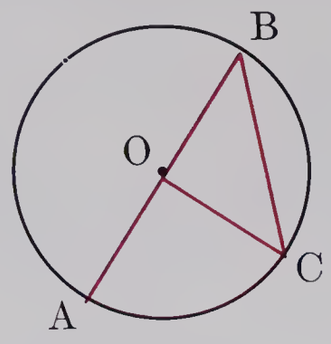
Case 1: $\angle AOD + \angle COD = 2(\angle OBA + \angle OBC)$

$\therefore \angle AOC = 2\angle ABC.$

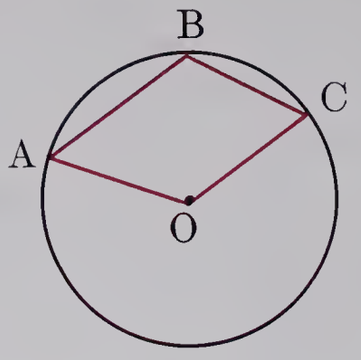
Case 2: $\angle COD - \angle AOD = 2(\angle OBC - \angle OBA)$

$\therefore \angle AOC = 2\angle ABC.$

Two other cases are possible as shown in the diagrams below. The proofs are left to the student.



Case 3



Case 4

If the subtending arc is a semicircle, the inscribed angle is said to be inscribed in a semicircle.

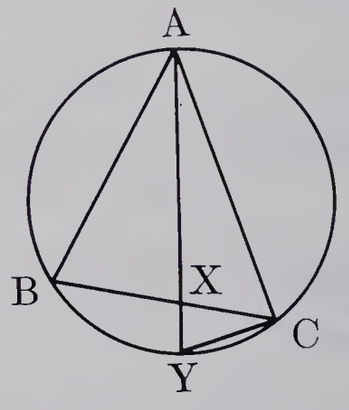
Corollaries: 1. The angle inscribed in a semicircle is a right angle. ($\angle$ in semi $\odot$)

2. Angles inscribed in a circle that are subtended by the same arc are equal. (on same arc)

Example

$\triangle ABC$ is inscribed in a circle and the bisector of $\angle A$ intersects BC at X and the circle at Y. Prove $\triangle ABX \,|||\, \triangle AYC$.

Given: $\triangle ABC$ inscribed in a $\odot$. The bisector of $\angle BAC$ intersects BC at X and the circle at Y.



Required: To prove $\triangle ABX ||| \triangle AYC$.

Proof: In $\triangle$'s ABX, AYC,

$\angle BAX = \angle CAY$,	(given)
$\angle ABX = \angle AYC$.	(on same arc)
$\therefore \triangle ABX \,|||\, \triangle AYC$.	(equiangular)

EXERCISE 10—4

1. Prove the corollaries of the theorem in this section.

2. The sector angle subtended by an arc AC is
   (a) 60°, (b) 90°, (c) 140°, (d) $\frac{\pi}{6}$, (e) 192°, (f) 180°.
   What is the size of the inscribed angle subtended by the same arc AC?
   What is the size of the inscribed angle subtended by the other arc AC?

3. An arc AB subtends an inscribed angle of
(a) 30°, (b) 48°, (c) $\frac{\pi}{4}$, (d) 126°, (e) 150°.
What is the size of the sector angle subtended by the same arc?
What is the size of the inscribed angle subtended by the other arc AB?

4. A, B, C, and D are four points in order on a circle. If $\angle BAD = 72°$, $\angle ADC = 67°$, and $\angle ABD = 58°$, calculate the sizes of $\angle ACD$, $\angle BDA$, $\angle ACB$, $\angle DCB$.

5. A, B, and C are three points on a circle, centre O. If $\angle AOB = 70°$ and $\angle CAB = 32°$, calculate the sizes of $\angle ACB$ and $\angle ABC$.

6. Two circles intersect at A and B. Through A two lines PAQ and XAY are drawn intersecting the first circle at P and X and the second at Q and Y. Prove that $\angle XBP = \angle YBQ$.

7. In $\triangle ABC$, $AB = AC$. A circle is drawn on AB as diameter and intersects BC at D. Prove that D is the midpoint of BC.

8. AB and CD are two chords of a circle intersecting at a point P inside the circle. Prove that $AP \cdot PB = CP \cdot PD$. (intersecting chords)

9. AB and CD are two chords of a circle intersecting at a point P inside the circle. If AP = 6 in., AB = 8 in., and CP = 3 in., calculate the length of CD.

10. In a circle, centre O, AB and CD are perpendicular chords intersecting at a point P inside the circle. Prove that $\angle AOD + \angle BOC = 180°$.

11. AB is a diameter of a circle and C is any other point on the circle. $CN \perp AB$. Prove that $CN^2 = AN \cdot NB$.

12. $\triangle ABC$ is inscribed in a circle. The bisectors of $\angle ABC$ and $\angle ACB$ meet the circle at D and E respectively. If BE is parallel to CD, prove that $\angle BAC = 60°$.

13. AB is a diameter of a circle, centre O, and X is any other point on the circle. AX is produced to Y so that XY = OX.
Prove that $\angle YOB = 3\angle XOY$.

10.5 Concyclic Points

DEFINITION: A set of points $\{P_1, P_2, \ldots, P_n\}$ is concyclic if all members of the set lie on the same circle.

The polygon $P_1 P_2 \ldots P_n$ is called a cyclic polygon.

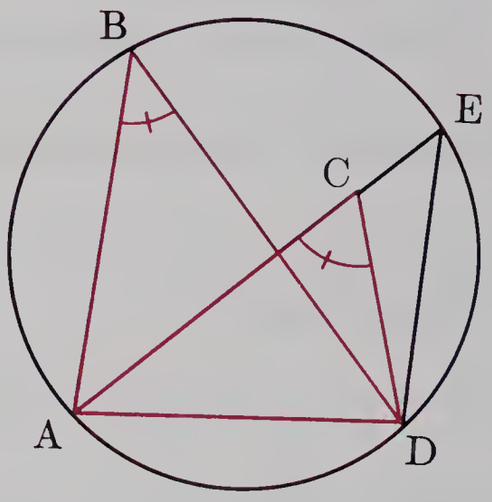
Notice that the circle itself is a concyclic set and the vertices of any triangle are concyclic, i.e. a triangle is a cyclic polygon; thus any three non-collinear points are concyclic.

The question now arises, when are four points concyclic? Obviously not all sets of four points are concyclic; we merely need to consider the three vertices of a triangle and any point not on the circumcircle of the triangle; these four points are *not* concyclic. However, four points can be concyclic; any four points on a given circle are concyclic. Presumably such points will have certain necessary special properties associated with them and some of these properties may be sufficient to ensure that the points are concyclic.

We have already seen that inscribed angles subtended by the same arc are equal.

The converse theorem, if true, would be one method of proving four points concyclic.

THEOREM: If two points on a line subtend equal angles at two other points on the same side of the line, the four points are concyclic. (subtended $\angle$'s =)



Given: Line AD with B and C two points on the same side of AD such that $\angle ABD = \angle ACD$.

Required: To prove that A, B, C, and D are concyclic.

Proof: Construct the circle through A, B, and D and let AC intersect the circle at E.

Then *either* E is between A and C,
or C is between A and E,
or E and C coincide.

$\angle ABD = \angle AED$	(on same arc)
But $\angle ABD = \angle ACD$.	(given)
$\therefore \angle AED = \angle ACD$.	(1)

If C is between A and E,

$\angle ACD > \angle AED$.	(Ext. $\angle$ of $\triangle >$ int. opp. $\angle$)

This is impossible since it contradicts (1).

If E is between A and C,

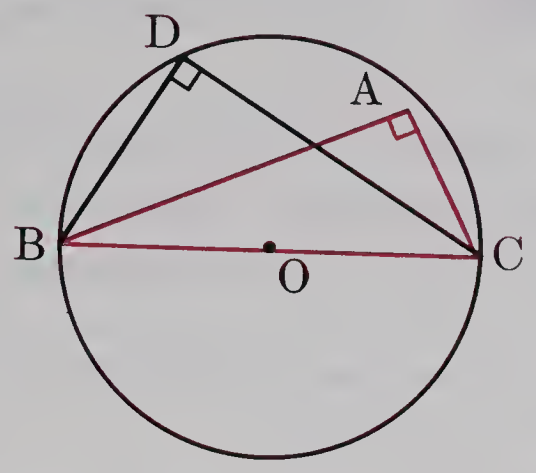
$\angle AED > \angle ACD$.	(Ext. $\angle$ of $\triangle >$ int. opp. $\angle$)

This is impossible since it contradicts (1).

$\therefore$ C and E must coincide.
$\therefore$ C is on the circle through A, B, and D.
$\therefore$ A, B, C, and D are concyclic points.

Corollary: If a given line subtends a constant angle on one side of the line, the locus of the vertex of the angle is an arc of the circle of which the given line is a chord.

THEOREM: The circle constructed on the hypotenuse of a right-angled triangle as diameter passes through the vertex of the right angle. (hyp. as diam.)



Given: $\triangle ABC$ with $\angle BAC = 90°$.

Required: To prove that the circle on BC as diameter passes through A.

Proof: Let D be any point on the circle on the same side of BC as A.

$\therefore \angle BDC = 90°$. ($\angle$ in semi-$\odot$)

But $\angle BAC = 90°$. (given)

$\therefore \angle BDC = \angle BAC$.

$\therefore$ D, B, C, and A are concyclic. (subtended $\angle$'s $=$)

$\therefore$ Circle on BC as diameter passes through A.

Corollary: The midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle. (midpt. of hyp.)

EXERCISE 10—5

1. Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle (midpt. of hyp.).

2. In $\triangle ABC$, $AB = AC$. D and E are the midpoints of AB and AC respectively. Prove that D, B, E, and C are concyclic points.

3. In $\triangle XYZ$, $XY = XZ$. The bisectors of $\angle XYZ$ and $\angle XZY$ meet XZ and XY at P and Q respectively. Prove that Q, Y, Z, and P are concyclic points.

4. PQ and RS are two parallel chords of a circle, centre O. PS and QR intersect at X. Prove that P, X, O, and R are concyclic points.

5. In $\triangle ABC$, $AB = AC$. AB and AC are produced to D and E respectively so that $BD = CE$. Prove that B, D, E, and C are concyclic points.

6. In $\triangle ABC$, BC is fixed and $\angle BAC$ is a constant acute angle. $CD \perp AC$ and meets the line determined by AB at D. Find the locus of D.

7. $\triangle ABC$ is acute-angled. Squares ABDE and BCFG are drawn external to the triangle. If AG and DC intersect at X, prove
(i) B, G, X, and C are concyclic points;
(ii) $AG \perp DC$; (iii) BX bisects $\angle DXG$.

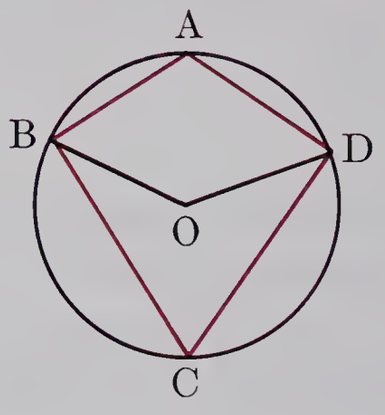
10 6 Cyclic Quadrilaterals

We have seen that if two points on a line subtend equal angles at two other points on the same side of the line, then the four points are concyclic. We also know that it is possible to inscribe quadrilaterals in any circle but obviously not all quadrilaterals are cyclic. One special property of cyclic quadrilaterals leads us to another way of proving four points concyclic. This can be expressed in the form of a dual theorem (a theorem and its converse) as follows:

THEOREM: The opposite angles of a quadrilateral are supplementary if and only if the quadrilateral is cyclic.

In this we state that, if a quadrilateral is cyclic, then its opposite angles are supplementary and also that, if the opposite angles are supplementary, then the quadrilateral is cyclic. We shall prove both statements.

THEOREM: The opposite angles of a cyclic quadrilateral are supplementary. (Opp. $\angle$'s of cyclic quad.)



Given: Quad. ABCD inscribed in a $\odot$, centre O.

Required: To prove
(i) $\angle BAD + \angle BCD = 180°$
(ii) $\angle ABC + \angle ADC = 180°$.

Proof: Join BO and DO.

Assume that A lies on the minor arc BD and C on the major arc.

$\angle BOD = 2\angle BCD$ (sector $\angle$ = 2 inscribed $\angle$)

Reflex $\angle BOD = 2\angle BAD$ (same reason)

$\therefore \angle BOD + \text{reflex } \angle BOD = 2(\angle BCD + \angle BAD)$.

But $\angle BOD + \text{reflex } \angle BOD = 360°$.

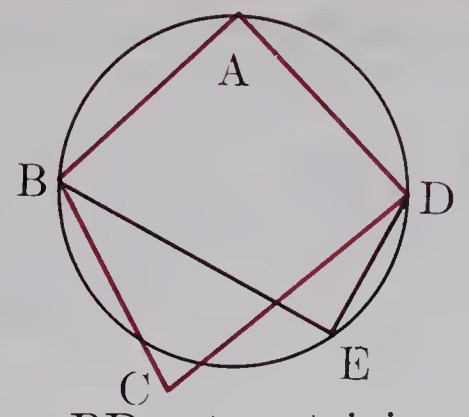
$\therefore \angle BAD + \angle BCD = 180°$.

Similarly, by joining AO and CO, it may be proved that

$\angle ABC + \angle ADC = 180°$.

Corollary: The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (Ext. $\angle$ of cyclic quad.)

THEOREM: If the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic. (Opp. ∠'s supp.)



Given: Quad. ABCD with $\angle A + \angle C = 180°$.

Required: To prove that ABCD is a cyclic quadrilateral.

Proof: Construct the circumcircle of $\triangle ABD$.
Let E be a point of the circumcircle on the arc BD not containing A.

$\therefore \angle A + \angle E = 180°$. (Opp. ∠'s of cyclic quad.)
But $\angle A + \angle C = 180°$. (given)
$\therefore \angle C = \angle E$.
$\therefore$ C lies on the $\odot$ BED. (=subtended ∠'s)
But $\odot$ BED is $\odot$ ABD.
$\therefore$ ABCD is a cyclic quadrilateral.

Corollary: If the exterior angle of a quadrilateral is equal to the interior opposite angle, the quadrilateral is cyclic. (Ext. ∠ = Int. opp. ∠).

EXERCISE 10—6

1. What are the angles of cyclic quadrilateral ABCD if
   (a) $\angle A = 60°$, $\angle B = 75°$, (b) $\angle B = 35°$, $\angle C = 120°$,
   (c) $\angle C = 17°$, $\angle D = 95°$, (d) $\angle A = 42°$, $\angle D = 111°$?

2. What are the angles A and C of cyclic quadrilateral ABCD if exterior angle A is (a) 30°, (b) 65°, (c) 90°, (d) 135°?

3. If A, B, and D are fixed in each part of (1), draw diagrams to show that C is not uniquely determined.

4. Prove each of the corollaries stated in this section.

5. Prove that a parallelogram inscribed in a circle must be a rectangle.

6. ABCD is a parallelogram, and a circle, centre A and radius AB, intersects the line determined by BC at E. Prove AECD is a cyclic quadrilateral.

7. ABCD is a cyclic quadrilateral and E is any point on AC. EX ∥ CB, meets AB at X, and EY ∥ CD, meets AD at Y. Prove that AXEY is a cyclic quadrilateral.

8. Two circles intersect at A and B. CAD and EBF are two lines meeting the first circle at C and E and the second at D and F. Prove that CE ∥ DF.

9. △ABC is inscribed in a circle and P, Q, and R are points on the arcs AB, BC, and AC. Prove that $\angle APB + \angle BQC + \angle ARC = 360°$.

10. △ABC is acute-angled and the altitudes BE and CF intersect at O. Prove that (i) $\angle FAO = \angle FEO$, (i) $\angle FEB = \angle FCB$.

11. Using the result of (10) and by joining AO and producing it to meet BC at D, prove that AD⊥BC and hence that the altitudes of an acute-angled triangle are concurrent.

12. Prove that the altitudes of an obtuse-angled triangle are concurrent.

13. ABCD is a cyclic quadrilateral with BC > CD. DC is produced to E and the bisector of $\angle BCE$ intersects the circle at F. Prove that AF bisects $\angle BAD$.

10 7 Tangents and Secants

In Section 10-1 of this chapter, we defined a secant as a line which intersects a circle at two distinct points. However, with certain curves it is possible for a line to intersect the curve at more than two points and a more general definition of a secant would be a line which intersects a curve at two or more distinct points.

Strictly speaking we should prove that a line cannot intersect a circle in more than two points. This can be proved by assuming that a circle passes through three collinear points and then showing that the centre of this circle must lie on the point of intersection of two parallel lines, and therefore, cannot exist. The proof is left to the reader.

We use this general definition of a secant to define a tangent to any curve.

DEFINITION: The tangent to a curve at a point A on the curve is the limiting position of a secant AB_n where $\{B_n\}$ is a set of points on the curve such that the length of the chord AB_n approaches zero.

In Figure 10-7 we have a set of secants AB_1, AB_2, AB_3 . . . or $\{AB_n\}$, such that the length of the chord approaches zero. As the length of AB_n approaches zero, the secant AB_n approaches the position AC which we define to be the tangent at A.

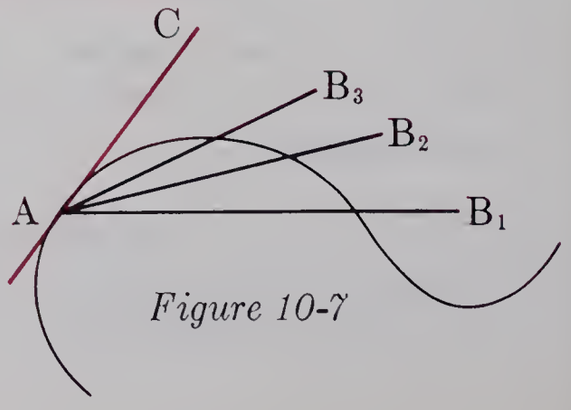


Figure 10-7

As with a locus, we may regard $\{B_n\}$ as positions of a moving point. As B_1 approaches A along the curve by passing successively through the positions B_1, B_2, B_3, etc., the secant AB_1 will approach the position AC which is called the tangent at A.

In the diagram B_n approaches A from above. If B_n approaches A from below, that is from the other side, the same tangent AC is reached. At a point such as A on the curve there is only one tangent.

Figure 10-8 shows the same idea applied in the case of the circle.

A tangent to a curve may intersect the curve again at some other point but, in the case of a circle, this is impossible. All points on the tangent, other than the point of contact, are outside the circle. The proof of this is fairly difficult and we will not attempt it here. We will assume here that a tangent is a line which touches a circle at only one point.

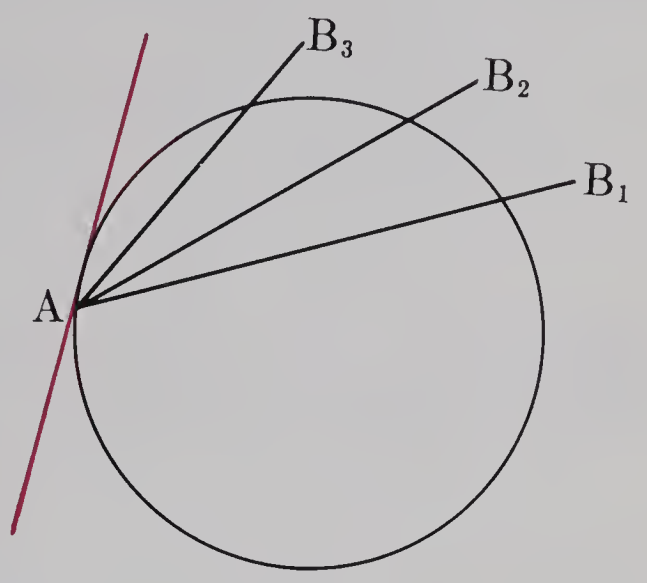
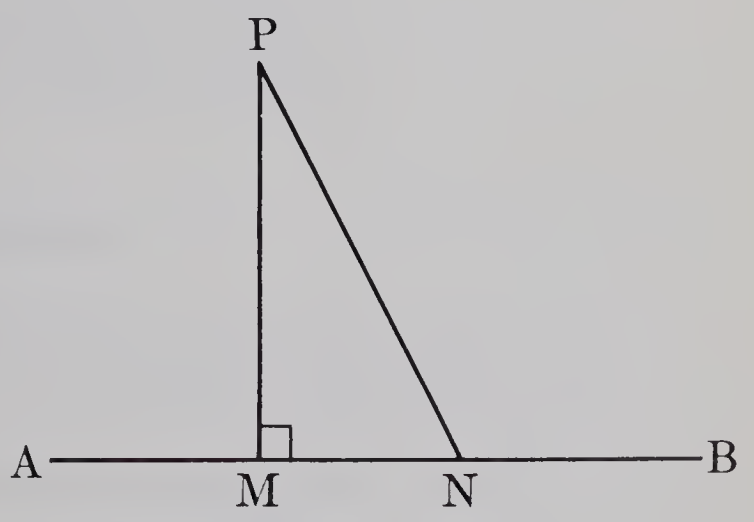


Figure 10-8

Before discussing our first theorem on tangents, we should prove one result which appears self-evident and which we may indeed have used or even proved in a previous grade. This states that the shortest line segment which may be drawn from an external point to a line is the perpendicular from the external point to the line.

Given: A line AB with P any point not on AB and PM the shortest line segment from P to AB.

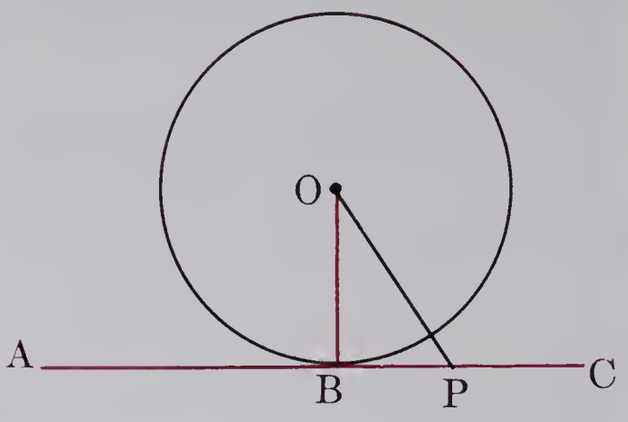


Required: To prove $PM \perp AB$.

Proof: *Either* (a) $PM \perp AB$
or (b) PM is not perpendicular to AB.

Assume that PM is not perpendicular to AB.
Construct $PN \perp AB$.
$\therefore$ In rt. $\angle$'d $\triangle PMN$,
$PM^2 = PN^2 + MN^2$. (Pythagoras)
$\therefore PM^2 > PN^2$.
$\therefore PM > PN$.
But PM is the shortest line segment from P to AB. (given)
$\therefore$ (b) is false.
$\therefore PM \perp AB$.

THEOREM: A tangent to a circle is perpendicular to the radius at the point of contact. (tgt. ⊥ radius)



Given: Tgt. ABC meeting ⊙, centre O, at B. OB is a radius.

Required: To prove OB⊥AC.

Proof: Let P be any point on ABC other than B.
Then P is outside the circle.
∴ OP > OB.
∴ OB is the shortest distance from O to AC.
∴ OB⊥AC.

THEOREM: A line perpendicular to a radius at its point of contact with a circle is a tangent to the circle. (⊥ to radius).

The proof of this is left to the reader.

Corollaries: 1. At any point on a circle, there is one and only one tangent.

2. A line perpendicular to a tangent at its point of contact with a circle passes through the centre of the circle. (⊥ to tgt.)

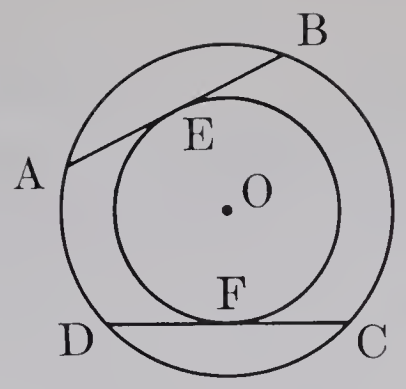
3. The locus of the centres of circles which touch a given line at a given point is the line perpendicular to the given line through the given point.

4. A line is a tangent to a circle if and only if the distance from the centre of the circle to the line is equal to the radius of the circle.

EXERCISE 10—7

1. Prove that a line perpendicular to a tangent at its point of contact with the circle passes through the centre of the circle. (⊥ to tgt.)
2. Construct a tangent to a circle at a point on the circle.
3. AOB is a diameter of a circle, centre O. Prove that the tangents at A and B are parallel.
4. OA and OB are radii of a circle, centre O, and ∠AOB = 115°. If the tangents at A and B intersect at P, calculate the size of ∠APB.
5. AB is a diameter of a circle and C is any other point on the circle. BD is constructed so that D is on the same side of AB as C and ∠CBD = ∠BAC. Prove that BD is a tangent to the circle.

6. In the diagram, AEB and DFC are tangents to the inner, and chords of the outer, of two concentric circles (circles with the same centre). Prove that AB = CD.

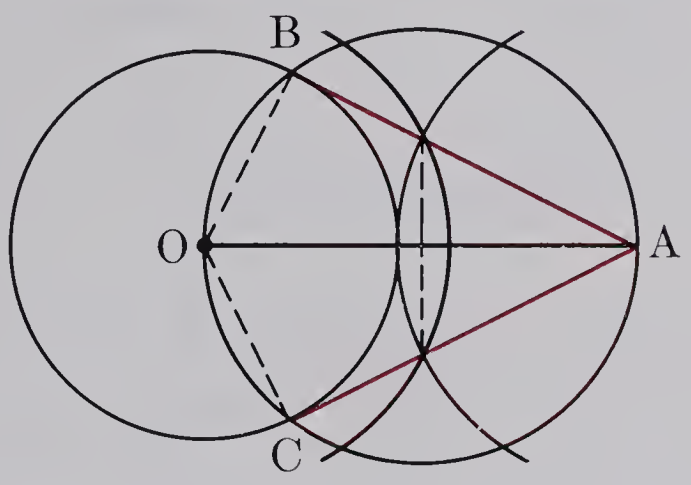


7. ∠ABC is an obtuse angle. Construct a circle of radius 1 in. so that AB and BC are tangents to the circle.

8. POQ is a diameter of a circle, centre O, and R is any other point on the circle. The tangents at R and Q intersect at S. Prove OS ∥ PR.

10 8 Tangents from an External Point

CONSTRUCTION: To construct tangents to a circle from a point outside the circle.



Given: ⊙, centre O, with A any point outside the circle.

Required: To construct tangents from A to the circle.

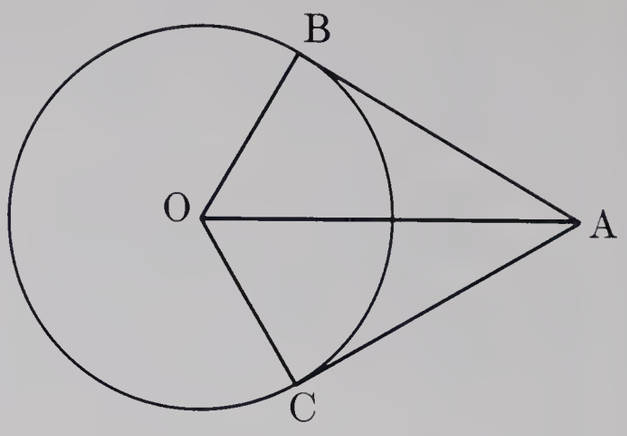
Construction: Join OA.
On OA as diameter, construct a circle to intersect the given circle at B and C.
Then AB and AC are the required tangents.

Proof: Join OB and OC.
∠OBA = 90°. (∠ in semi ⊙)
∴ AB is a tangent. (⊥ to radius)
Similarly AC is a tangent.

A tangent is a line and therefore, by our axioms of geometry, is of infinite length but we often speak of the length of a tangent from an external point. In the diagram for the above construction, A is an external point and the tangents from A meet the circle at B and C. When we speak of the length of the tangent AB, we mean the length of the line segment between A and B.

Example

Prove that the two tangents drawn to a circle from an external point are equal. (Tgts. from ext. pt.)



Given: ⊙, centre O, with AB and AC tangents from an external point A.

Required: To prove AB = AC.

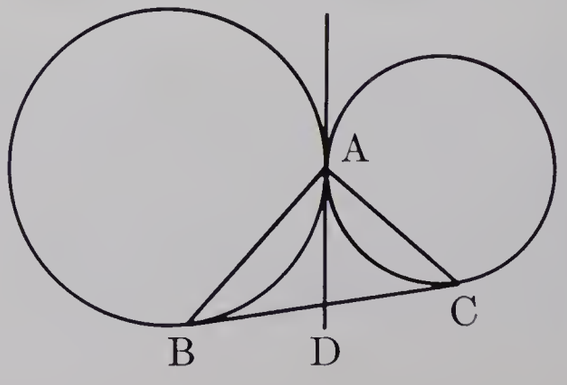
Proof: Join OB and OC.

In △'s ABO, ACO,

AO = AO	
OB = OC	(radii of same ⊙)
∠OBA = ∠OCA = 90°	(radius ⊥ tgt.)
∴ △ABO ≡ △ACO.	(hyp., side)
∴ AB = AC.	

EXERCISE 10—8

1. A is a point outside a circle, centre O, and tangent AB = 3 in. If OB = 4 in., calculate the length of OA.
2. Using the same diagram as in (1) but with AO = 13 cm., OB = 5 cm., calculate the length of the tangent AB.
3. A tangent 12 cm. long is drawn to a circle from a point 18 cm. from the centre of the circle. Calculate the radius of the circle. (Give the result in simplest surd form.)
4. Prove that the tangents drawn to a circle from an external point subtend equal angles at the centre of the circle. (tgt. from ext. pt.)
5. Prove that the tangents drawn to a circle from an external point make equal angles with the line joining the point to the centre.
6. Construct the inscribed circle of a given triangle. (The sides of the triangle are tangents to the circle.)
7. In the diagram, the circles are externally tangent (touch each other externally) at A. AD and BC are common tangents. Prove that D is the midpoint of BC and that ∠BAC = 90°.



10 9 Angle between Chord and Tangent

THEOREM: If a chord and a tangent are drawn through a point on a circle, each of the angles determined by the tangent and the chord is equal to the angle inscribed in the circle on the opposite side of the chord. ($\angle$ bet. chord and tgt.)

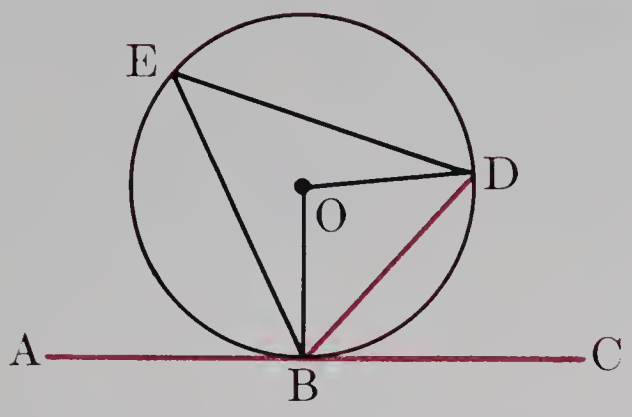


Figure 10-9

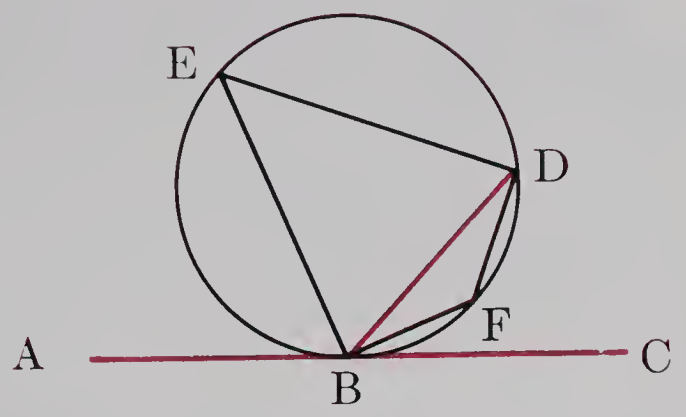


Figure 10-10

Given: $\odot$, centre O, with ABC a tangent meeting the circle at B. BD is any chord and $\angle$DEB is the inscribed angle on the opposite side of BD from $\angle$DBC. $\angle$DFB is the inscribed angle on the opposite side of BD from $\angle$DBA.

Required: To prove (i) $\angle DBC = \angle DEB$

(ii) $\angle DBA = \angle DFB$.

Proof: Assume $\angle$DBC is acute.

In Figure 10-9, join BO and DO.

Let $\angle DBC = x°$.

$\angle OBC = 90°$ (radius $\perp$ tgt.)

$\therefore \angle OBD = (90-x)°$.

$OB = OD$ (radii of same $\odot$)

$\therefore \angle OBD = \angle ODB$. (base $\angle$'s of isos. $\triangle$)

$\therefore \angle ODB = (90-x)°$.

$\therefore \angle BOD = 180° - 2(90-x)°$ (sum of $\angle$'s of $\triangle$)

$= 180° - 180° + 2x°$

$= 2x°$.

$\therefore \angle DEB = x°$. (sector $\angle$ = 2 inscribed $\angle$)

$\therefore \angle DBC = \angle DEB$.

In Figure 10-10,

$\angle DEB + \angle DFB = 180°$. (opp. $\angle$'s of cyclic quad.)

$\therefore \angle DFB = (180-x)°$.

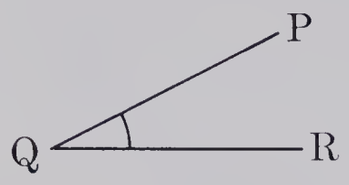
But $\angle DBA = (180-x)°$. ($\angle$ABC a str. $\angle$)

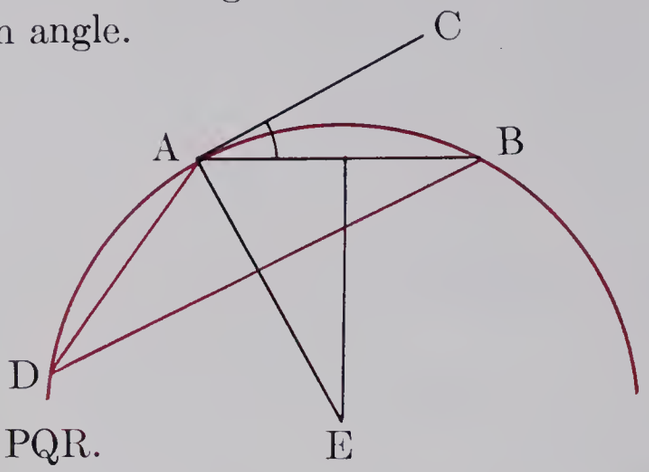
$\therefore \angle DBA = \angle DFB$.

Note: In this proof we have assumed that the chord BD makes one acute angle and one obtuse angle with the tangent ABC. If the chord is perpendicular to the tangent, the proof is simplified since the angles involved are right angles. The proof is, therefore, left to the reader.

Example

On a given line segment, construct a segment of a circle so that the inscribed angle is equal to a given angle.





Given: Line segment AB and $\angle PQR$.

Required: To construct on AB a segment of a circle so that the inscribed angle $= \angle PQR$.

Construction: Construct $\angle BAC = \angle PQR$.
Construct $AE \perp AC$.
Construct the perpendicular bisector of AB to meet AE at E.
With centre E and radius EA, draw a circle.
Let D be any point on the circle on the opposite side of AB from C.
Then $\angle ADB = \angle PQR$.

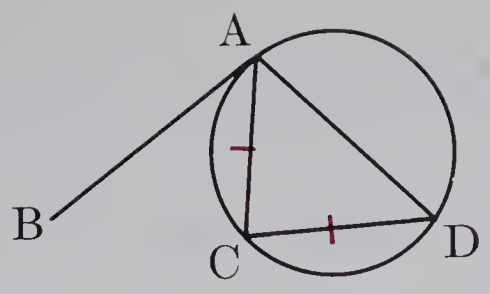
Proof:

$EA = EB$	(E on perp. bisector of AB)
$\therefore \odot$, centre E and radius EA, passes through B.	
AC is a tangent.	($\perp$ to radius)
$\therefore \angle BAC = \angle ADB$.	($\angle$ bet. chord and tgt.)
But $\angle BAC = \angle PQR$.	(const.)
$\therefore \angle ADB = \angle PQR$.	

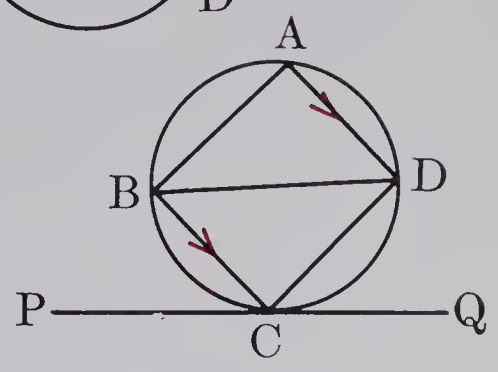
EXERCISE 10—9

1. AOB is a diameter of a circle, centre O. C and D are two points on the circle such that AC bisects $\angle DAB$. The tangent at C intersects AD produced at E. Prove that $\angle AEC = 90°$.
2. PAQ is a tangent to a circle with A the point of contact. BC is a chord of the circle parallel to PQ. Prove that $\triangle ABC$ is isosceles.

3. In the diagram, BA is a tangent to the circle. $CA = CD$ and $\angle BAC = 40°$. Calculate the size of $\angle C$.



4. PCQ is a tangent to the circle and $AD \parallel BC$. If $\angle DCQ = 43°$ and $\angle BCP = 48°$, calculate the size of $\angle ADB$ and of $\angle ABD$.



5. $\triangle ABC$ is inscribed in a circle and DE, parallel to the tangent at A, intersects AB at D and AC at E. Prove that DBCE is a cyclic quadrilateral.

6. AB and AC are tangents to a circle with B and C the points of contact. The bisector of $\angle ABC$ meets the circle at D. Prove that $\triangle DBC$ is isosceles.

7. Two circles intersect at A and D. The tangent to the second circle at A intersects the first circle at B. C is any point on the minor arc BD and CD is produced to meet the second circle at E. Prove $AE \parallel BC$.

8. AB is a tangent to a circle with B the point of contact and secant ACD intersects the circle at C and D. Prove that $AB^2 = AD \cdot AC$. (tgt. and secant.)

9. Two circles intersect at A and B. BA is produced to C, CD is a tangent to the first circle, and CE is a tangent to the second. Prove that $CD = CE$. (Use the result of (8).)

10. On a given line segment AB, construct a segment of a circle so that the inscribed angle is 60°.

11. Construct $\triangle ABC$ given that $BC = 2\frac{1}{2}$ in., $\angle BAC = 60°$, and the median $AD = 1\frac{1}{2}$ in.

12. Construct $\triangle ABC$ given that $BC = 3$ in., $\angle BAC = 75°$, and altitude $AD = 1\frac{1}{2}$ in.

13. In a given circle, inscribe a triangle equiangular to a given triangle.

14. AB and AC are equal chords of a circle. The tangent at A intersects the bisector of $\angle ABC$ at D. Prove $AD \parallel BC$ and $\angle BAC = 2\angle BDC$.

15. Two circles intersect at B and C. The line segment ABD intersects one circle at A and the other at D. The tangents at A and D intersect at P. Prove that PACD is a cyclic quadrilateral.

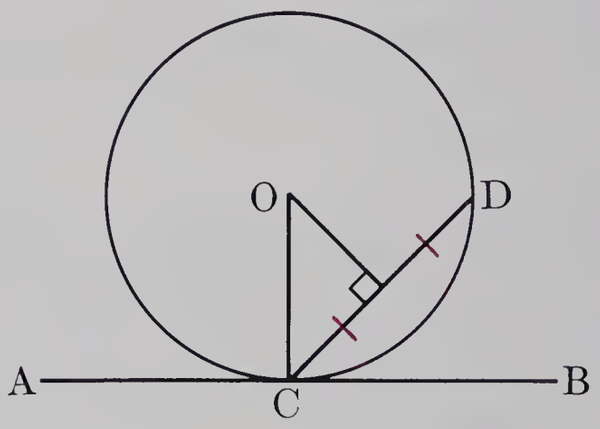
10 10 Construction of Circles (Supplementary)

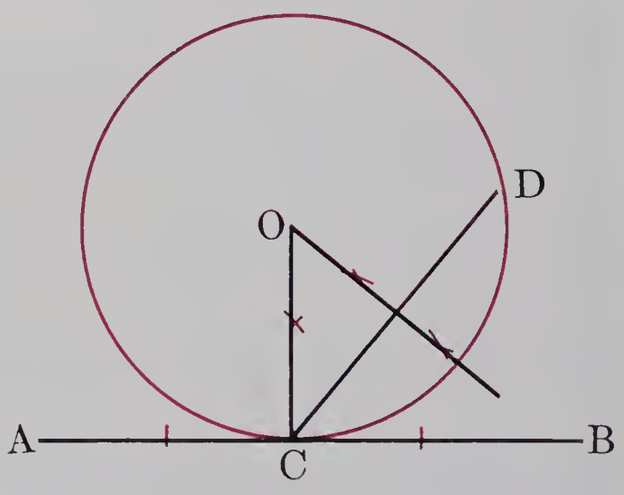
To construct a circle, we must be able to find its centre and its radius. When a set of conditions is given, we usually find the centre as the point of intersection of two loci each of which is obtained from one of the given conditions. We must then be given the radius or some point through which the circle passes. Three independent conditions are, therefore, necessary if we wish to construct a circle. In some cases more than one solution may be possible. We shall show some examples but shall leave the proofs to the student.

Example 1

Construct a circle to pass through a given point and to be tangent to a given line at a given point on the line.

As with any construction problem, it is useful to construct a trial diagram in order to analyze the problem. Since ACB must be a tangent of the circle with C the point of contact, the centre must lie on a line through C perpendicular to AB. Also since C and D must be on the circle, the centre must lie on the perpendicular bisector of CD. The intersection of these two loci will give us the centre of the required circle. It is then obvious that the radius is either OC or OD.





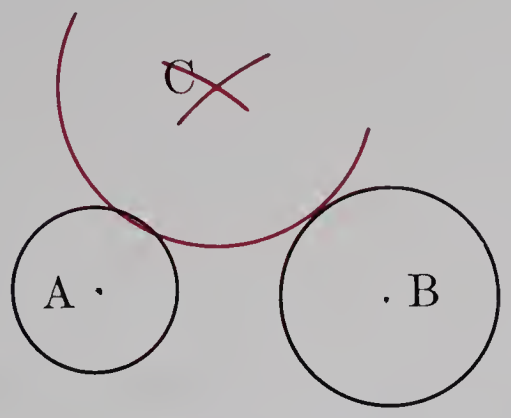
Given: Line ACB and D, any point not on AB.

Required: To construct a circle to pass through D and to be tangent to AB at C.

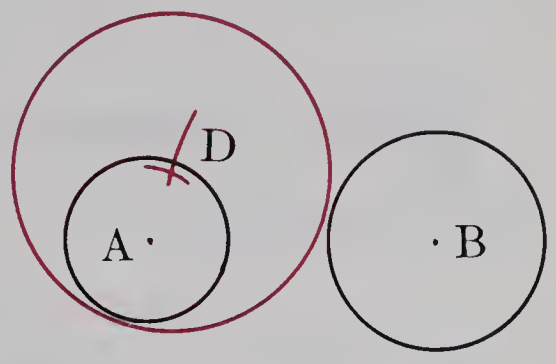
Construction: Construct OC $\perp$ AB.
Construct the perpendicular bisector of CD to intersect OC at O.
With centre O and radius OC, draw a circle.
This is the required circle.

Example 2

Construct a circle of radius 1 in. to touch two given circles of radii $\frac{1}{2}$ in. and $\frac{5}{8}$ in. and whose centres are $1\frac{1}{2}$ in. apart. (In this case there are six possible solutions. We shall give two of them and leave it to the reader to find the other six.)



(*i*)



(*ii*)

Given: Two circles, centres A and B and radii $\frac{1}{2}$ in. and $\frac{5}{8}$ in. respectively and whose centres are $1\frac{1}{2}$ in. apart.

Required: To construct a circle of radius 1 in. to touch the given circles.

Construction: (i) With centre A and radius $1\frac{1}{2}$ in., draw an arc.
With centre B and radius $1\frac{5}{8}$ in., draw an arc to intersect the previous arc at C.
With centre C and radius 1 in., draw a circle.
This is one of the required circles.

(ii) With centre B and radius $\frac{3}{8}$ in., draw an arc.
With centre B and radius $1\frac{1}{2}$ in., draw an arc to intersect the previous arc at D.
With centre D and radius 1 in., draw a circle.
This is one of the required circles.

EXERCISE 10—10

1. Construct a circle to touch two parallel lines and a transversal.
2. C is a point on a given line AB and DE is another line not parallel to AB. Construct a circle to touch AB at C and to touch DE.
3. A is a point outside a given circle. Construct a circle to pass through A and to touch the given circle at a given point.
4. Repeat (3) with A inside the given circle.

5. Construct a circle of radius 1 in. to pass through two given points A and B if $AB = 1\frac{1}{4}$ in. (What is your conclusion if $AB = 2\frac{1}{2}$ in.?)
6. Construct a circle of radius 1 in. to touch the sides of a given $\angle ABC$.
7. Construct a circle of given radius to touch a given circle and a given line.
8. Construct a circle to touch a given circle at a given point and also to touch a given line.

10 11 Chapter Summary

DEFINITIONS

A *circle* is the set of points in a plane which are a constant distance from a given point. The given point in the plane is the *centre.*

A *radius* of a circle is a line segment joining the centre to any point on the circle.

A *chord* of a circle is a line segment joining any two points on the circle.

A *diameter* of a circle is a chord which passes through the centre.

A *secant* is any line intersecting a circle at two distinct points.

A *tangent* to a circle (or any curve) at a point A on the circle is the limiting position of a secant AB_n where $\{B_n\}$ is a set of points on the curve such that the length of the secant AB_n approaches zero.

If A and B are the endpoints of a diameter, each of the two sets of points on the circle from A to B constitutes a *semicircle.*

An *arc* of a circle is the set of points on the circle between any two points on the circle.

A *major* arc is an arc of a circle which has the set of points of a semicircle as a subset.

A *minor* arc is an arc of a circle which is a subset of the set of points of a semicircle.

The *interior* of a circle is the set of points in the plane of the circle whose distances from the centre are *less than* the radius of the circle.

The *exterior* of a circle is the set of points in the plane of the circle whose distances from the centre are *greater than* the radius of a circle.

A *segment* of a circle is that set of points in the plane of the circle bounded by a chord and the arc intercepted by that chord.

A *major segment* is bounded by a chord and its major arc.

A *minor segment* is bounded by a chord and its minor arc.

A *sector* of a circle is that set of points in the plane of the circle bounded by an arc and the radii joining the endpoints of the arc to the centre.

The *circumference* of a circle is the length of the circle.

A *sector angle* is the angle subtended at the centre of a circle by an arc of the circle.

An angle is *inscribed* in a circle when its vertex is on the circle and each side of the angle intersects the circle at one other point.

A set of points is *concyclic* if all members of the set lie on a circle.

If $P_1, P_2, \ldots, P_n$ are concyclic points the polygon $P_1P_2 \ldots P_n$ is a *cyclic polygon*.

IMPORTANT RESULTS

The centre of a circle lies on the perpendicular bisector of any chord. (⊥ bisector of chord)

The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. (bisector of chord)

The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord. (⊥ from centre)

Equal chords of a circle are equidistant from the centre. (= chords)

Chords equidistant from the centre of a circle are equal. (equidistant from centre)

A sector angle in a circle is twice the inscribed angle subtended by the same arc. (sector $\angle = 2$ inscribed $\angle$)

An angle inscribed in a semicircle is a right angle. ($\angle$ in semi $\odot$)

Angles inscribed in a circle that are subtended by the same arc are equal. (on same arc)

If two chords of a circle intersect, the product of the lengths of the segments of one is equal to the product of the lengths of the segments of the other. (intersecting chords)

If two points on a line subtend equal angles at two other points on the same side of the line, the four points are concyclic. (subtended $\angle$'s =)

The circle constructed on the hypotenuse of a right-angled triangle as diameter passes through the vertex of the right angle. (hyp. as diam.)

The opposite angles of a cyclic quadrilateral are supplementary. (opp. $\angle$'s of cyclic quad.)

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (ext. ∠ of cyclic quad.)

If the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic. (opp. ∠'s supp.)

If an exterior angle of a quadrilateral is equal to the interior opposite angle, the quadrilateral is cyclic. (ext. ∠ = int. opp. ∠)

A tangent to a circle is perpendicular to the radius at the point of contact. (tgt. ⊥ radius)

A line perpendicular to a radius at its point of contact with a circle is a tangent to the circle (⊥ to radius)

Two tangents drawn to a circle from an external point are equal. (tgt. from ext. pt.)

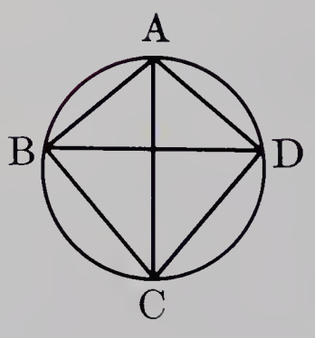
Tangents drawn to a circle from an external point subtend equal angles at the centre of the circle. (tgt. from ext. pt.)

If a chord and tangent are drawn through a point on a circle, each of the angles determined by the tangent and the chord is equal to the angle inscribed in the circle on the opposite side of the chord (∠ bet. chord and tgt.)

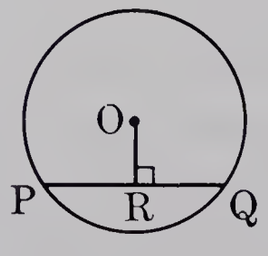
If a tangent and a secant be drawn to a circle from an external point, the square of the length of the tangent is equal to the product of the distances from the external point to the points of intersection of the secant and the circle. (tgt. and secant)

REVIEW EXERCISE 10—11

1. In the diagram, $\angle BCD = 60°$, $\angle ABD = 24°$. Calculate the sizes of $\angle ACB$ and $\angle ADB$.

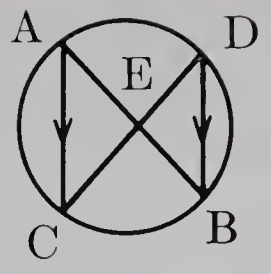


2. In the diagram, O is the centre of a circle, radius 5 cm. Chord PQ = 6 cm. and OR ⊥ PQ. Calculate the length of OR.



3. AB is a diameter of a circle, centre O. CD is a chord parallel to AB. Prove that BC bisects $\angle DCO$.

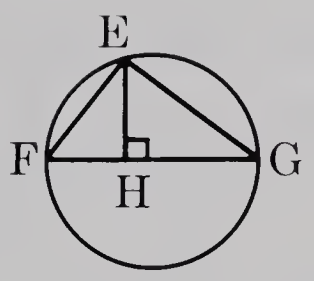
4. In the diagram, AC ∥ DB. If ∠DEB = 104°, calculate the size of ∠EAC.



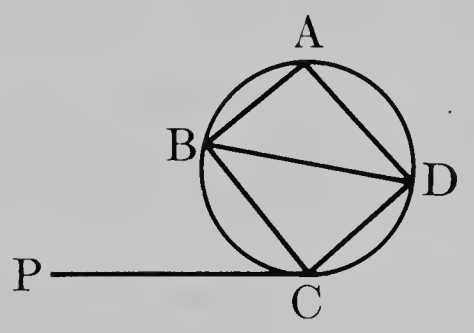
5. AB and CD are two chords of a circle intersecting at E. If AE = 10 cm., CE = 6 cm., and DE = 8 cm., calculate the length of BE.

6. If a chord, 5 cm. from the centre of a circle, is 24 cm. long, calculate the radius of the circle.

7. In the diagram, FG is a diameter of the circle. EH⊥FG and ∠G = 35°. Calculate the size of ∠FEH.



8. In the diagram, PC is a tangent and ∠PCB = 72°. If ∠CBD = 33°, calculate the size of ∠A.



9. AB is a diameter of a circle and AC and AD are two chords such that ∠CAB = ∠DAB. Prove that AC = AD.

10. ABCD is a cyclic quadrilateral. If AD is produced to E and CD to F, prove that ∠B = ∠DEF + ∠DFE.

11. Two concentric circles (circles with the same centre) with centre O are such that a line PQRS intersects the outer circle at P and S and the inner at Q and R. OR is produced to meet the outer circle at A and OQ is produced to meet the outer circle at B. Prove △PQB ≡ △SRA.

12. ABCD is a cyclic quadrilateral and AD is a diameter of the circle. AE is perpendicular to the line determined by BC. Prove that ∠EAB = ∠CAD.

13. △ABC with AB > AC is inscribed in a circle. AC is produced to D so that AB = AD. AE ∥ DB and meets the circle at E. Prove that EB = EC.

14. AOB is a diameter of a circle, centre O, and D is any other point on the circle. AE is perpendicular to the tangent at D. Prove that AD bisects ∠EAB.

15. AB and AC are tangents to a circle from an external point A. (B and C are the points of contact.) P is any point on the minor arc BC. The tangent at P intersects AB at D and AC at E. Prove that the perimeter of $\triangle ADE = 2AB$.

16. AOB is a diameter of a circle, centre O, and C is any other point on the circle. The tangent at C intersects the tangents at A and B at D and E respectively. Prove that $\angle DOE = 90°$.

17. AB and AC are tangents to a circle, centre O, from an external point A. $AD \perp AB$ and meets OC produced at D. BC produced meets AD produced at E. Prove $\triangle DEC$ isosceles.

18. Two circles intersect at X and Y. Two lines AXB and CXD intersect one circle at A and C and the other at B and D. Prove that $\angle AYC = \angle BYD$.

19. In a circle, centre O, AB and CD are two chords which intersect at right angles at E. Prove that $\angle BOC + \angle AOD = 180°$.

20. Two circles intersect at X and Y. Two lines AXB and CXD intersect one circle at A and C and the other at B and D. Prove that $\angle AYB = \angle CYD$.

21. AB and CD are two chords of a circle which intersect at E so that $AE = CE$. Prove that $BE = DE$.

22. $\triangle ABC$ is an acute-angled triangle inscrided in a circle. CE and BD are diameters of the circle. Prove that $\angle BAE = \angle CAD$.

23. ABCD is a parallelogram and a circle through A and B intersects AD at P and BC at Q. Prove that P, Q, C, and D are concyclic points.

24. AB and AC are two chords of a circle, centre O. D and E are the midpoints of AB and AC respectively. Prove that A, D, O, and E are concyclic points.

25. Construct a quadrilateral ABCD with $AB = 2$ in., $BC = 3$ in., $AD = 2$ in., $\angle ABC = 60°$, $\angle ADC = 90°$.

26. Construct $\triangle ABC$ with $BC = 3$ in., $\angle BAC = 75°$, and given a point D on BC where the bisector of $\angle BAC$ intersects BC.

27. Given a circle and a line, construct a line which will be tangent to the circle and perpendicular to the given line.

28. Find a point P outside a given circle such that the tangents drawn to the circle from P shall contain an angle of 60°.

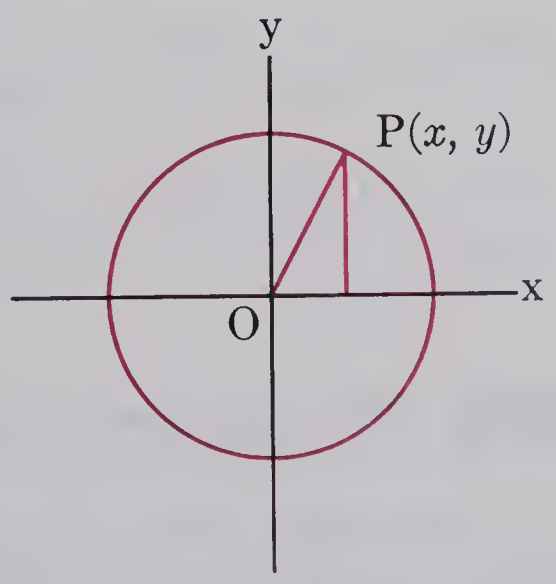
Chapter 11

Analytic Geometry of the Circle

11 1 Equation of a Circle, Centre the Origin

We have already defined a circle as the set of points in a plane which are a constant distance from a given point in the plane.

From this definition, and using the rectangular coördinate system, we can combine the powerful methods of algebra with those of geometry to investigate the properties of the circle. By taking the origin of our coördinate system as the centre of the circle, we can easily develop an equation for the circle.



Given: A circle, centre the origin and radius r units.

Required: To find the equation of the circle.

Solution: P(x, y) is any point on the circle iff

$$PO = r.$$

$$\therefore \sqrt{x^2+y^2} = r.$$

$$\therefore \quad x^2+y^2 = r^2.$$

$\therefore$ equation of the circle is $x^2+y^2=r^2$.

For the remainder of this chapter we will consider only circles that have their centres at the origin.

Example 1

Find the equation of a circle of radius 6 units.

Solution: Equation of circle is $x^2+y^2=36$.

Example 2

Find the radius of the circle represented by

$$4x^2+4y^2=25.$$

Solution: An equivalent equation is $x^2+y^2=\frac{25}{4}$.

$\therefore$ radius of circle is $\sqrt{\frac{25}{4}}$ units $=2\frac{1}{2}$ units.

EXERCISE 11—1

1. Find equations of the circles with the following radii:

   (a) 3 (b) $1\frac{1}{2}$ (c) $\sqrt{2}$ (d) $3\sqrt{5}$ (e) $\frac{2\sqrt{3}}{5}$

2. Find the radii of the circles represented by the following equations:

   (a) $x^2+y^2=36$ (b) $x^2+y^2=5$

   (c) $3x^2+3y^2=25$ (d) $9x^2+9y^2=49$

   (e) $4x^2+4y^2=21$ (f) $5x^2+5y^2=21$

3. Find equations of the circles which pass through the following points:

   (a) (3, 4) (b) (−3, 4) (c) (5, −12)

   (d) (1, 3) (e) (−5, 0) (f) (2, −5)

4. Show that the points P(3, 4) and Q(−3, 4) are on the circle represented by $x^2+y^2=25$. How far are P and Q from the y-axis?

5. Show that the points A(−1, 2) and B(−1, −2) are on the circle represented by $x^2+y^2=5$. How far are A and B from the x-axis?

11.2 The Relation Defined by $x^2+y^2=r^2$

We have seen that a circle, centre the origin and radius r units, can be represented by the equation $x^2+y^2=r^2$.

Conversely the graph of $\{(x, y)|x^2+y^2=r^2, x, y\in \mathrm{Re}\}$ is a circle.

Let us examine this a little more closely without assuming that its graph is a circle.

Domain:

If $x^2+y^2=r^2$,

then $y^2=r^2-x^2$.

$y=\pm\sqrt{r^2-x^2}$

First we notice that, since $y\in \mathrm{Re}$, r^2-x^2 must be positive or zero.

$\therefore r^2-x^2\geq 0.$

$\therefore x^2\leq r^2.$

$\therefore |x|\leq r \qquad (r>0)$

or $-r\leq x\leq r.$

Hence, the domain of the relation is $\{x\in \mathrm{Re}\,|\,-r\leq x\leq r\}$.

This will mean that the graph of the relation will be confined to that region of the plane between the dotted lines in Figure 11-1. We also notice that if $y=\pm\sqrt{r^2-x^2}$, then, for every value of x in the domain, there will be two values of y in the range of the relation. Therefore, the relation is *not* a function.

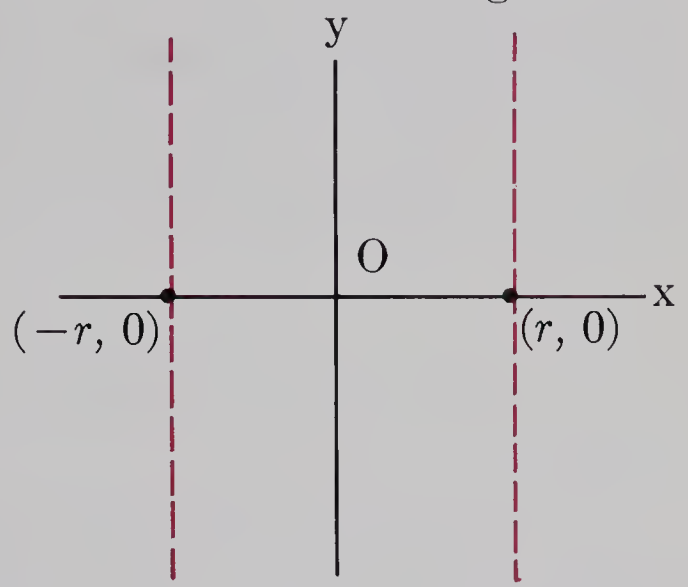


Figure 11-1

Range: Since

$x^2+y^2=r^2,$

$\therefore x^2=r^2-y^2.$

$\therefore x=\pm\sqrt{r^2-y^2}.$

$\therefore$ since $x\in \mathrm{Re}$, $r^2-y^2\geq 0.$

$\therefore -r\leq y\leq r.$

$\therefore$ the range of the relation is $\{y\in \mathrm{Re}\,|\,-r\leq y\leq r\}$.

The graphical effect of this restriction on the range of the relation is illustrated in Figure 11-2. The graph of the relation is confined to that region of the plane between the dotted lines.

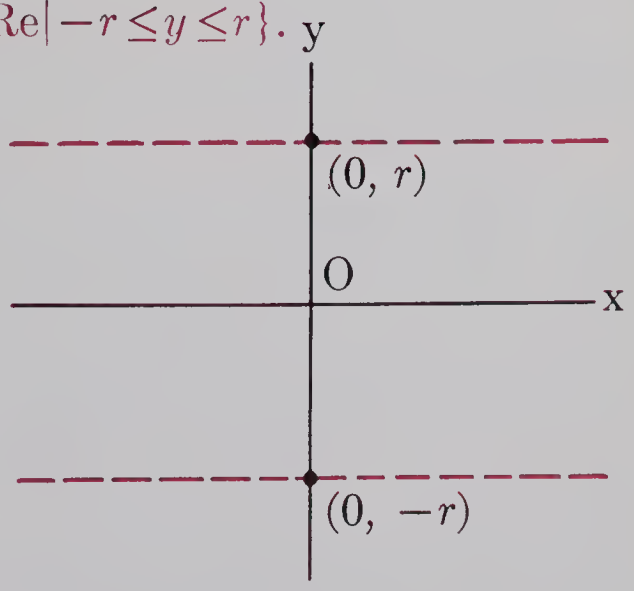


Figure 11-2

Intercepts: If $y=0$ and $x^2+y^2=r^2$, then $x^2=r^2$.

$$\therefore x=\pm r.$$

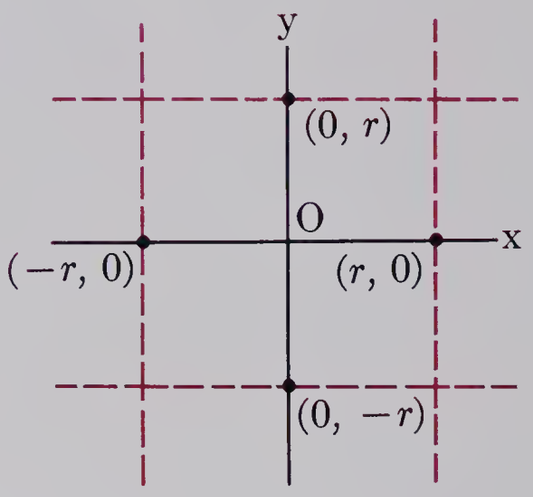
The graph intersects the x-axis at the points $(r, 0)$ and $(-r, 0)$.

If $x=0$ and $x^2+y^2=r^2$, then $y^2=r^2$.

$$\therefore y=\pm r.$$

The graph intersects the y-axis at the points $(0, r)$ and $(0, -r)$.

Figure 11-3 summarizes the position reached at present. The graph of the relation is confined to that region of the plane between the dotted lines and must pass through the four points $(\pm r, 0)$ and $(0, \pm r)$.



Symmetry

In Chapter 6 we saw that the y-axis was an axis of symmetry of the parabola $y=ax^2$. We will now look a little more closely at the idea of symmetry.

DEFINITION: A figure is symmetrical with respect to a line as an axis of symmetry if, for every point of the figure, there is another point of the figure such that the axis is the perpendicular bisector of the line segment joining the two points.

At the moment, we are mainly concerned with the possibility that one or both of the coördinate axes may be axes of symmetry.

If, in Figure 11-4, P is *any* point on a graph and Oy is an axis of symmetry, Oy must be the perpendicular bisector of the line joining PQ where Q is a second point on the graph. Since PQ$\perp$Oy, the ordinate of Q must be the same as the ordinate of P. Since PA=QA, the abscissa of Q must be the negative of the abscissa of P.

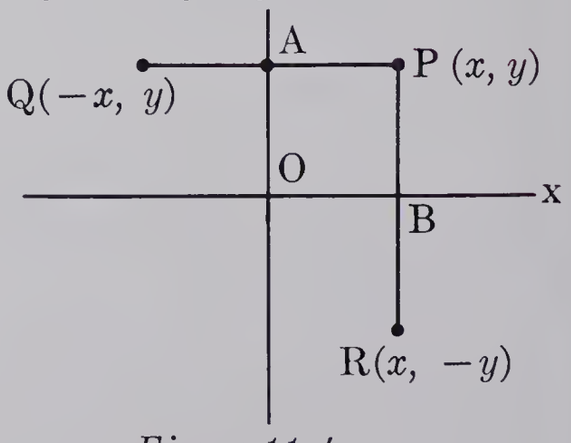


Figure 11-4

Hence, if P (x, y) is *any* point on the graph and the graph is symmetrical with respect to the y-axis, Q$(-x, y)$ must also be a point on the graph. The graph of a relation will be symmetrical with respect to the y-axis if the equation defining the relation remains unaltered when x is replaced by $-x$.

Again, in Figure 11-4, if P(x, y) is any point on a graph and the graph is symmetrical with respect to the x-axis, the point R$(x, -y)$ must also be a point on the graph.

The graph of a relation will be symmetrical with respect to the x-axis if the equation defining the relation remains unaltered when y is replaced by $-y$.

If we replace x by $-x$ in $y=ax^2$, as in Chapter 6, we obtain the equation

$$y=a(-x)^2.$$
$$\therefore y=ax^2.$$

The equation is unaltered and so we know that the graph of the function is symmetrical with respect to the y-axis.

If we replace y by $-y$ we obtain $-y=ax^2$. This is *not* the same as $y=ax^2$ and, hence, the graph of the function is *not* symmetrical with respect to the x-axis.

In the relation defined by $x^2+y^2=r^2$, replacing x by $-x$ *or* y by $-y$ leaves the equation unaltered and so the graph of this relation is symmetrical with respect to both axes.

DEFINITION: A figure is symmetrical with respect to a point called the centre of symmetry if, for every point of the figure, it is possible to find a second point of the figure such that the centre bisects the line segment joining the two points.

We are again mainly interested in whether or not our graph has the origin as a centre of symmetry. In other words, is the graph of the relation symmetrical about the origin?

If, in Figure 11-5, P is *any* point on the graph of a relation and PO is produced to Q so that PO = OQ, then PQ is bisected at O. If Q is also a point on the graph, the graph is, by definition, symmetrical about the origin. By constructing PM$\perp$Ox and QN$\perp$Ox, we can easily prove that $\triangle$POM $\equiv$ $\triangle$QON and that OM and ON are equal in length but opposite in direction as are OP and OQ. Hence, if P has coördinates (x, y), the coördinates of Q will be $(-x, -y)$.

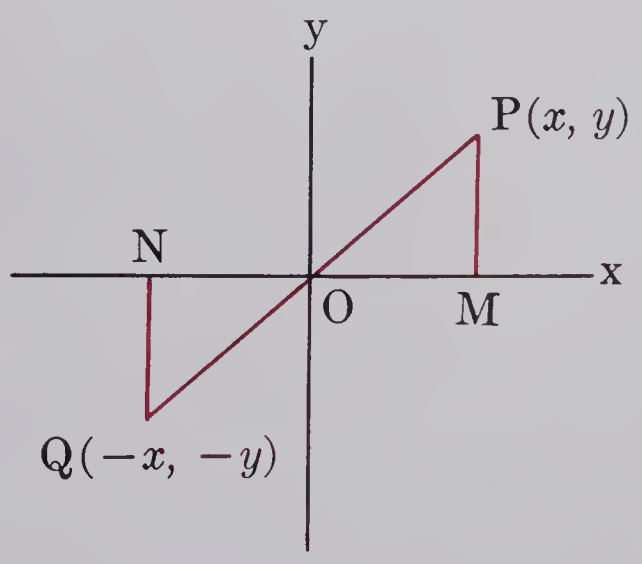


Figure 11-5

Therefore, the graph of a relation is symmetrical about the orgin if the equation defining the relation remains unaltered when x is replaced by $-x$ *and* y is replaced by $-y$.

In the equation $x^2+y^2=r^2$, replacing x by $-x$ and y by $-y$ gives

$$(-x)^2+(-y)^2=r^2.$$
$$\therefore x^2+y^2=r^2.$$

The equation is unaltered and so the graph is symmetrical with respect to the origin.

It should be noted that if the graph of a relation is symmetrical about both the x-axis and the y-axis, it is symmetrical about the origin. The converse of this statement is not necessarily true. For example the graph of the function defined by $y=x^3$ is symmetrical about the origin but is not symmetrical about either axis. (Check by applying the tests given above.)

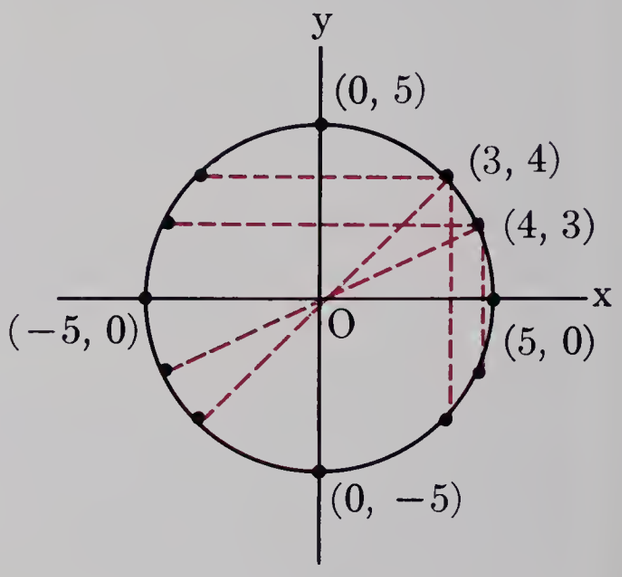
Checking for symmetry enables us to sketch the graph fairly quickly. If a graph is symmetrical about the axes or about the origin, then for every point plotted we can immediately plot its *reflection* through the axis of symmetry or through the origin.

Example

State the domain, range, and intercepts of the graph of $x^2+y^2=25$, and sketch the graph.

Solution: Domain is $-5\leq x\leq 5$.
Range is $-5\leq y\leq 5$.
x-intercepts are ± 5.
y-intercepts are ± 5.

x	3	4
y	4	3



Using the known symmetry of the graph and the two points (3, 4) and (4, 3) on the graph, we can immediately plot six other points as shown by the dotted lines. The points $(-3, 4)$ and $(-4, 3)$ will be on the graph because of symmetry about the y-axis, the points $(3, -4)$ and $(4, -3)$ because of symmetry about the x-axis, and the points $(-3, -4)$ and $(-4, -3)$ because of symmetry about the origin.

Naturally, since in this case we know that the equation represents a circle, centre the origin and radius 5 units, the simplest way to draw the graph is with a pair of compasses. However, for other relations, the use of symmetry as above will enable us to make a quick sketch of the graph.

EXERCISE 11—2

$(x, y \in \mathrm{Re})$

1. State the domain, range, and intercepts for each of the following:

(a) $x^2+y^2=36$ (b) $x^2+y^2=64$ (c) $4x^2+4y^2=25$

(d) $x^2+y^2=18$ (e) $3x^2+3y^2=49$ (f) $2x^2+2y^2=11$

2. State any properties of symmetry of the graphs of the relations defined by the following equations:

(a) $y=4x^2+1$ (b) $3x^2+10y^2=8$ (c) $y=x^3$

(d) $2x+3y=7$ (e) $2x+3y^2=7$ (f) $y^2=x^3$

(g) $y=|x|$ (h) $y=|x+1|$ (i) $|y|=x$

11 3 Interior and Exterior Regions

We know that the graph of the relation defined by $x^2+y^2=r^2$ is a circle, centre the origin and radius r units. What about the graph of the relations defined by the inequalities $x^2+y^2<r^2$ and $x^2+y^2>r^2$?

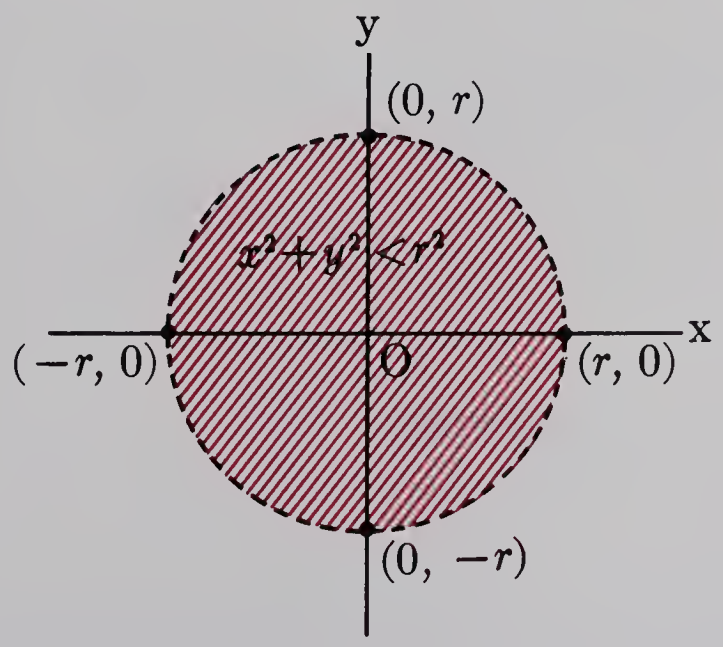
If $x^2+y^2<r^2$,

then $\sqrt{x^2+y^2}<r$. $(r>0)$

This states that the distance of a point $\mathrm{P}(x, y)$ from the origin is less than r units.

$\therefore$ $\mathrm{P}(x, y)$ is a point on the graph if and only if $\mathrm{PO}<r$.

$\therefore$ the graph of the relation is the set of all points inside the circle represented by $x^2+y^2=r^2$.

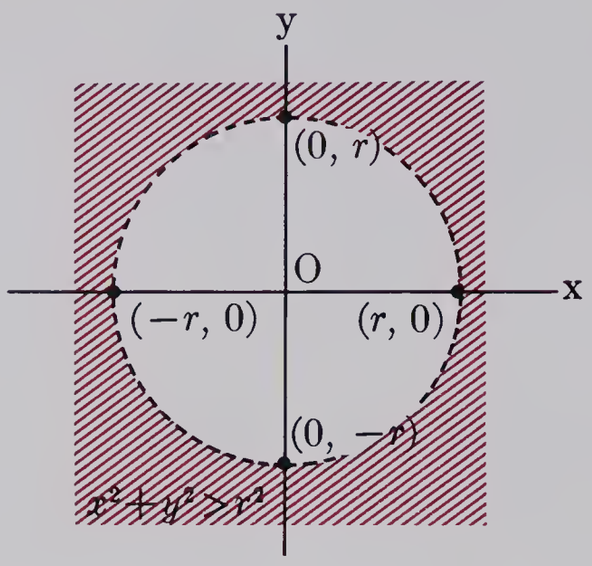


The dotted boundary indicates that the circle is not part of the graph.

If $x^2+y^2>r^2$,
then $\sqrt{x^2+y^2}>r$. $(r>0)$

As before, P(x, y) is a point on the graph if and only if PO $>r$.

$\therefore$ the graph of the relation is the set of all points outside the circle represented by $x^2+y^2=r^2$.

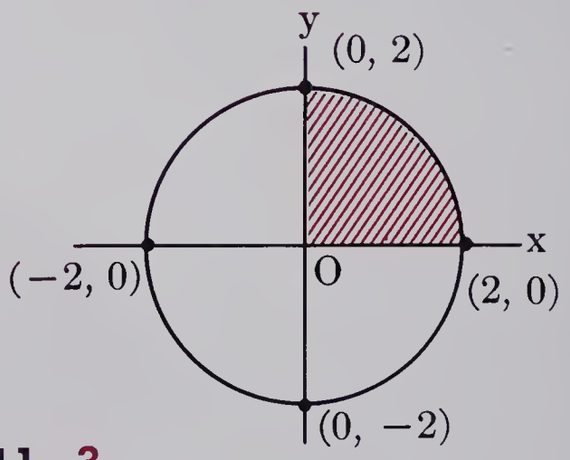


If $x^2+y^2\geq r^2$, then the circle is part of the graph and is drawn as a solid boundary.

Example

Sketch the graph $\{(x, y)|x^2+y^2\leq 4; x\geq 0, y\geq 0\}$.

Solution: Since $x\geq 0$ and $y\geq 0$, the graph is the set of all points in the first quadrant on or inside the circle represented by $x^2+y^2=4$ and including the axes.



EXERCISE 11—3

$(x, y\in \mathrm{Re})$

1. Sketch the graphs of the relations defined by the following inequalities:
   (a) $x^2+y^2\leq 9$
   (b) $x^2+y^2>16$
   (c) $x^2+y^2<25, x\geq 0, y\leq 0$
   (d) $x^2+y^2\geq 4, x<0, y<0$
   (e) $x^2+y^2\leq 16, x<0, y\geq 0$
   (f) $4x^2+4y^2<25, x\geq 1, y>1$

2. Sketch the graphs of the following relations:

(a) $\{(x, y)|x^2+y^2<25\}\cap\{(x, y)|x+y>5\}$

(b) $\{(x, y)|x^2+y^2\leq 16\}\cap\{(x, y)|2x-y<4\}$

(c) $\{(x, y)|x^2+y^2<9\}\cup\{(x, y)|x^2+y^2>25\}$

(d) $\{(x, y)|x^2+y^2\geq 1\}\cap\{(x, y)|x^2+y^2\leq 16\}\cap\{(x, y)|x\geq 0\}$

11.4 Analytic Proofs

Example 1

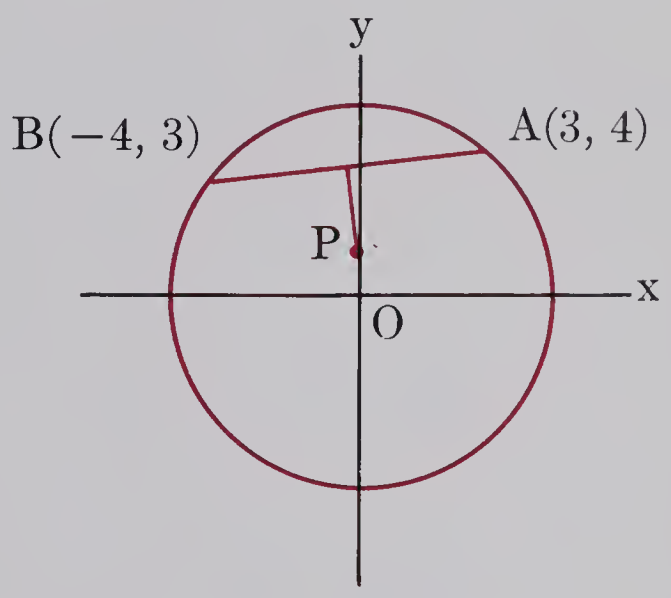
A circle, centre the origin, passes through the points A(3, 4) and B(−4, 3). Find the equation of the circle and prove that the perpendicular bisector of AB passes through the centre of the circle.

Solution: Let the equation of the circle be $x^2+y^2=r^2$. Since A(3, 4) is a point on the circle,

$$3^2+4^2=r^2.$$
$$\therefore \quad 25=r^2.$$
$$\therefore \quad r=5. \qquad (r>0)$$

$\therefore$ equation of circle is

$$x^2+y^2=25.$$



Check: For the point B(−4, 3),

$$x^2+y^2=16+9$$
$$=25.$$

$\therefore$ B is on the circle.

P(x, y) is *any* point on the perpendicular bisector of AB iff

$$PA=PB.$$

$$\therefore \quad \sqrt{(x-3)^2+(y-4)^2}=\sqrt{(x+4)^2+(y-3)^2}.$$
$$\therefore \quad x^2-6x+9+y^2-8y+16=x^2+8x+16+y^2-6y+9.$$
$$\therefore \quad -14x-2y=0.$$
$$\therefore \quad 7x+y=0.$$

$\therefore$ equation of perpendicular bisector of AB is

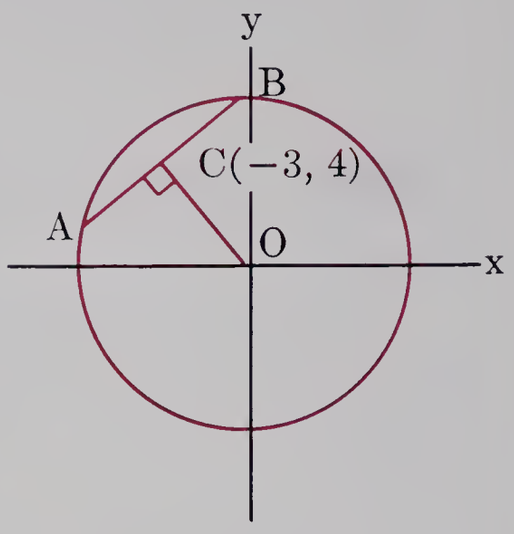
$$7x+y=0.$$

Since this equation is satisfied by $x=0$, $y=0$, it is the equation of a line through the origin.

$\therefore$ the perpendicular bisector of AB passes through the centre of the circle.

Example 2

A line, drawn perpendicular to a chord of the circle represented by $x^2+y^2=36$, passes through the centre of the circle and meets the chord at the point $(-3, 4)$. Find the length of the chord and the equation of the chord.



Solution: Let the chord be AB and let C be the point $(-3, 4)$.

$$AO = 6 \text{ units}$$

$$OC^2 = (4-0)^2+(-3-0)^2$$
$$= 16+9$$
$$= 25$$

$$AC^2 = AO^2 - OC^2 \qquad \text{(Pythagoras)}$$
$$= 36-25$$
$$= 11$$

$$\therefore \quad AC = \sqrt{11} \text{ units.}$$

$$AB = 2\,AC \qquad (\perp \text{ from centre})$$
$$= 2\sqrt{11} \text{ units}$$

$$\text{Slope of } OC = \frac{4-0}{-3-0}$$
$$= -\frac{4}{3}$$

$\therefore$ slope of $AB = \frac{3}{4}$.

$\therefore$ equation of AB is

$$y-4 = \tfrac{3}{4}(x+3).$$
$$\therefore \; 4y-16 = 3x+9.$$

Length of chord is $2\sqrt{11}$ units.

Equation of chord is $3x-4y+25=0$.

EXERCISE 11—4

1. For each of the circles represented by the following equations, the coördinates of the midpoint of a chord are given. Calculate the length of the chord and find the equation of the chord in each case.

(a) $x^2+y^2=25$ $(1, 2)$ (b) $x^2+y^2=35$ $(-3, -1)$

(c) $4x^2+4y^2=57$ $(2, 2\frac{1}{2})$ (d) $x^2+y^2=9$ $(\frac{1}{3}, -\frac{8}{3})$

(e) $2x^2+2y^2=43$ $(3, 2\frac{1}{2})$ (f) $x^2+y^2=4$ $(3, 2)$

2. In (1f) there is no solution. Explain why this is so and how this could be checked before starting to work on the problem.

3. Find the length of the perpendicular from the centre of the circle represented by $x^2+y^2=16$, to the chord joining a point of intersection of the circle and x-axis to a point of intersection of the circle and y-axis.

4. A circle, centre the origin, passes through the point A(5, 12). Find the equation of the circle. Prove that the perpendicular bisector of the chord joining A to the point of intersection of the circle and the positive x-axis, passes through the centre of the circle.

5. A(a, b) is a point on the circle represented by $x^2+y^2=r^2$. If B is the point $(r, 0)$, prove that the perpendicular bisector of the chord AB passes through the centre of the circle.

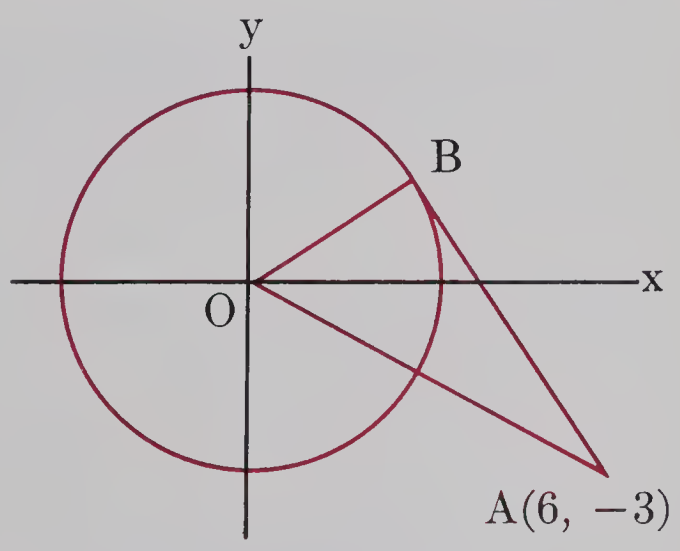
6. P(a, b) is any point on the circle represented by $x^2+y^2=r^2$. If A and B are the points of intersection of the circle and x-axis, prove that $\angle \text{APB}=90°$.

7. A(a, b) is any point on the circle represented by $x^2+y^2=r^2$. If B is the point $(r, 0)$, prove that the line joining the centre of the circle to the midpoint of AB is perpendicular to AB.

11 5 Length of Tangent

Example

Find the length of the tangent drawn from the point $(6, -3)$ to the circle whose equation is $x^2+y^2=25$.



Solution: Let A be the point $(6, -3)$ and let B be the point of contact of tangent and circle.

$$\begin{aligned} \text{AO}^2 &= (6-0)^2+(-3-0)^2 \\ &= 36+9 \\ &= 45 \\ \text{OB}^2 &= 25 \\ \angle \text{OBA} &= 90° \quad \text{(tgt. } \perp \text{ radius)} \\ \therefore\ \text{AB}^2 &= \text{OA}^2-\text{OB}^2 \quad \text{(Pythagoras)} \\ &= 45-25 \\ &= 20. \end{aligned}$$

$\therefore$ length of tgt. AB $= 2\sqrt{5}$ units.

EXERCISE 11—5

1. In each of the following, the equation of a circle and the coördinates of a point not on the circle are given. Find the length of the tangent drawn to the circle from the given point.

(a)	$x^2+y^2=4$	$(8, 2)$
(b)	$x^2+y^2=49$	$(8, -2)$
(c)	$x^2+y^2=3$	$(-3, 7)$
(d)	$2x^2+2y^2=21$	$(4\frac{1}{2}, 2\frac{1}{2})$
(e)	$16x^2+16y^2=225$	$(-2\frac{1}{4}, -5)$
(f)	$x^2+y^2=25$	$(2, -3)$

2. In (1f) there is no solution. Explain the reason for this and state how this possibility could be checked before starting to work on the problem.

3. Find the length of the tangent from the point $A(a, b)$ to the circle represented by $x^2+y^2=r^2$. Indicate any restrictions on a and b.

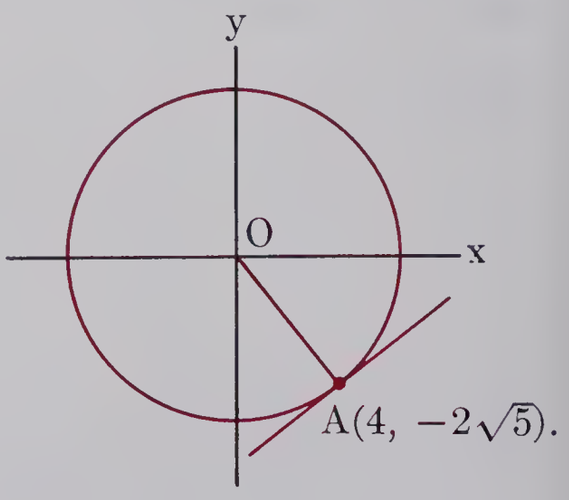
4. Using the result of (3), write down the lengths of the tangents in (1). What is the significance of the result in (1f)?

11 6 Equation of a Tangent

Example

A tangent is drawn to a circle represented by $x^2+y^2=36$ at the point $(4, -2\sqrt{5})$ on the circle. Find the equation of the tangent.

Solution: Let A be the point $(4, -2\sqrt{5})$.



$$\text{Slope of OA} = \frac{-2\sqrt{5}}{4} = -\frac{\sqrt{5}}{2}.$$

Radius OA $\perp$ tgt.

$\therefore$ slope of tgt. is $\dfrac{2}{\sqrt{5}}$.

$\therefore$ equation of tgt. is

$$(y+2\sqrt{5}) = \frac{2}{\sqrt{5}}(x-4).$$

$\therefore \sqrt{5}(y+2\sqrt{5}) = 2(x-4).$

$\therefore \sqrt{5}y+10 = 2x-8.$

$\therefore$ equation of tgt. is $2x-\sqrt{5}y=18.$

Note: For the purpose of (2) in Exercise 11-6, this equation may be written as $4x-2\sqrt{5}y=36$.

EXERCISE 11—6

1. Find the equation of the tangent to each of the circles represented by the following equations at the given point on the circle:

(a)	$x^2+y^2=25$	$(3, 4)$	(b)	$x^2+y^2=25$	$(3, -4)$
(c)	$x^2+y^2=25$	$(-3, 4)$	(d)	$x^2+y^2=25$	$(-3, -4)$
(e)	$x^2+y^2=169$	$(-12, 5)$	(f)	$x^2+y^2=289$	$(15, -8)$
(g)	$x^2+y^2=41$	$(-5, -4)$	(h)	$2x^2+2y^2=17$	$(2\frac{1}{2}, 1\frac{1}{2})$

2. Rewrite each of the results in (1) in the form $Ax+By=r^2$, where r is the radius of the circle. From your results can you devise a method of writing down immediately the equation of a tangent drawn to a circle at a point on the circle?

3. Prove your conjecture in (2) by finding the equation of the tangent at the point (a, b) on the circle represented by $x^2+y^2=r^2$.

4. If tangents are drawn to the circle defined by $x^2+y^2=25$ at points with abscissa 3, show that these tangents intersect on the x-axis.

5. If tangents are drawn to the circle defined by $x^2+y^2=13$ at points with ordinate -3, show that these tangents intersect on the y-axis.

6. Substitute $2+mx$ for y in $x^2+y^2=4$ and find the roots of the resulting equation to show that lines through the point $(0, 2)$ intersect the circle $x^2+y^2=4$ in no more than two points. What value of m corresponds to the tangent at $(0, 2)$?

7. Repeat (6) for the lines $y-2=m(x-2)$ through the point $(2, 2)$ on the circle $x^2+y^2=8$. What value of m corresponds to the tangent at $(2, 2)$?

11 7 Chapter Summary

The equation of a circle, centre the origin and radius r units, is $x^2+y^2=r^2$.
For the relation $x^2+y^2=r^2$ and its graph,

(i) The domain is $\{x \in \text{Re} \mid -r \leq x \leq r\}$.

(ii) The range is $\{y \in \text{Re} \mid -r \leq y \leq r\}$.

(iii) The x-intercepts are $\pm r$.
The y-intercepts are $\pm r$.

(iv) The graph of the relation is symmetrical about both axes and the origin.

(v) If, for any point $P(x, y)$, $x^2+y^2<r^2$, the point P is inside the circle represented by $x^2+y^2=r^2$.

(vi) If, for any point $P(x, y)$, $x^2+y^2>r^2$, the point P is outside the circle represented by $x^2+y^2=r^2$.

REVIEW EXERCISE 11—7

1. Write equations for circles with radii as follows:

   (a) 7 (b) $2\sqrt{6}$ (c) $\frac{2\sqrt{3}}{3}$

2. State the radii of the circles represented by the following equations:

   (a) $x^2+y^2=144$ (b) $4x^2+4y^2=25$ (c) $9x^2+9y^2=28$

3. Find the equations of circles which pass through the following points:

   (a) $(5, -12)$ (b) $(6, 2)$ (c) $(-1\frac{1}{2}, -3\frac{1}{2})$

4. State the domain, range, and intercepts of the circles defined by each of the following:

   (a) $x^2+y^2=64$ (b) $x^2+y^2=100$ (c) $x^2+y^2=27$

   (d) $4x^2+4y^2=49$ (e) $9x^2+9y^2=17$ (f) $5x^2+5y^2=17$

5. State whether the following points are inside, outside, or on the graphs of the circle represented by $x^2+y^2=49$:

   (a) $(3, \sqrt{2})$ (b) $(-7, 0)$ (c) $(-6, -6)$

   (d) $(-5, 2\sqrt{6})$ (e) $(6, 2\sqrt{3})$ (f) $(4\frac{1}{2}, -5\frac{1}{2})$

6. Draw the graph of the relations defined by each of the following:

   (a) $x^2+y^2>4$ (b) $x^2+y^2\geq 4$ (c) $x^2+y^2<4$

   (d) $x^2+y^2<4,\ x\geq 0,\ y\geq 0$ (e) $4x^2+4y^2=49,\ x<0,\ y\geq 0$

7. The point $(3, -2)$ is the midpoint of a chord of the circle represented by $x^2+y^2=38$. Find the length of the chord and the equation of the chord.

8. Find the length of the tangent drawn from the point $(8, -5)$ to the circle represented by $x^2+y^2=40$.

9. Find the equation of the tangent at the point $(4, -5)$ on the circle represented by $x^2+y^2=41$.

10. A tangent meets a circle, centre the origin, at the point $(-3, -2\frac{1}{2})$. Find the equation of the circle and the tangent.

11. Find the equation of a circle, centre $(2, 1)$ and radius 4 units.

12. The tangent at the point B$(3, 4)$ on the circle $x^2+y^2=25$ meets the y-axis at A and the x-axis at C. Find the value of AB:BC.

13. If the tangent at the point B(a, b) on the circle $x^2+y^2=r^2$ meets the y-axis at A and the x-axis at C, prove that $\text{AB:BC}=a^2:b^2$.

Chapter 12

The Trigonometric Functions

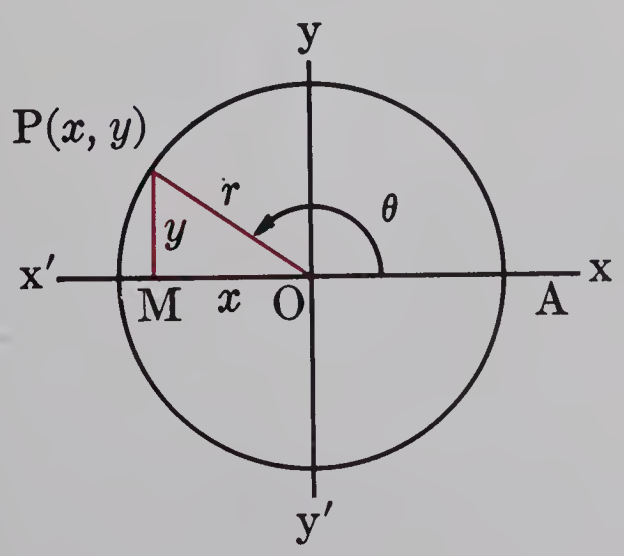
12 1 The Trigonometric Functions of θ, $0 \leq \theta \leq 2\pi$

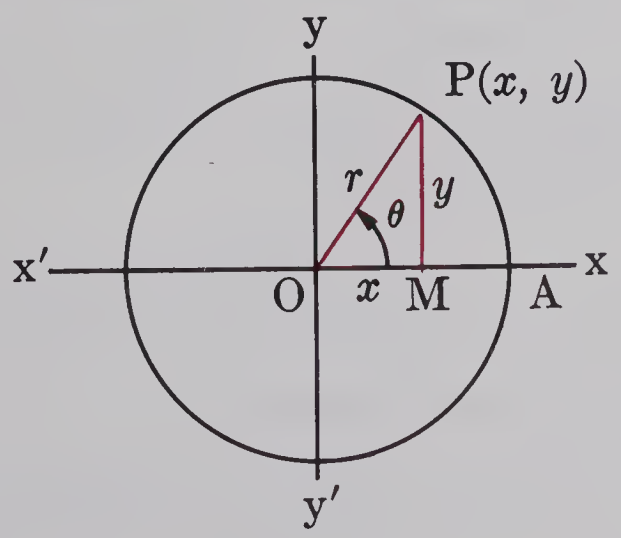
A *radian* is the measure of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

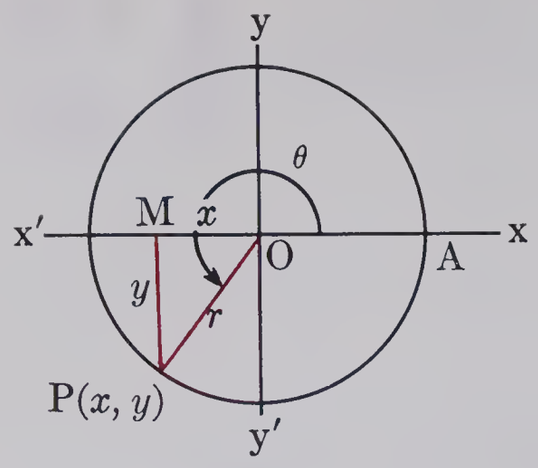
A *degree* is the measure of $\frac{1}{180}$ of a straight angle, or 180 is the measure of a straight angle in degrees.

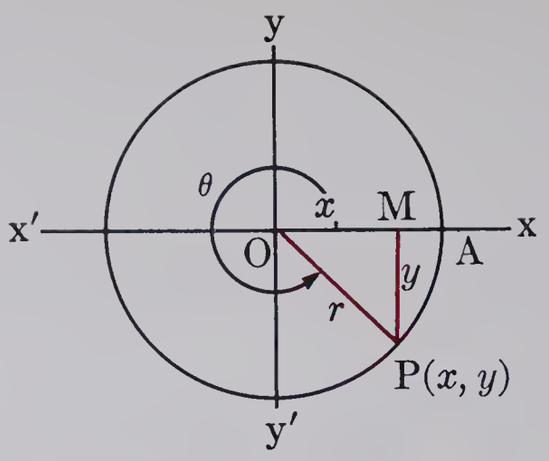
$$\pi \text{ radians} = 180°,$$
$$1 \text{ radian} = \frac{180°}{\pi} \simeq 57.288°,$$
$$1° \simeq 0.0175 \text{ radians}$$

Take a point $P(x, y)$ in a Cartesian coördinate plane with origin O and a fixed initial ray Ox. By a counterclockwise rotation from Ox, the final ray OP determines a unique angle θ, $0° \leq \theta \leq 360°$ if the unit of measurement is the degree and $0 \leq \theta \leq 2\pi$ if the unit of measurement is the radian. Note that in either case θ is a *real number, the measure of the angle.*









The values of the trigonometric functions of $(0 \leq \theta \leq 2\pi)$ are defined as follows:

$$\text{sine of } \theta = \sin\theta = \frac{y}{r} \qquad \text{cosecant of } \theta = \csc\theta = \frac{r}{y} \quad (y \neq 0)$$

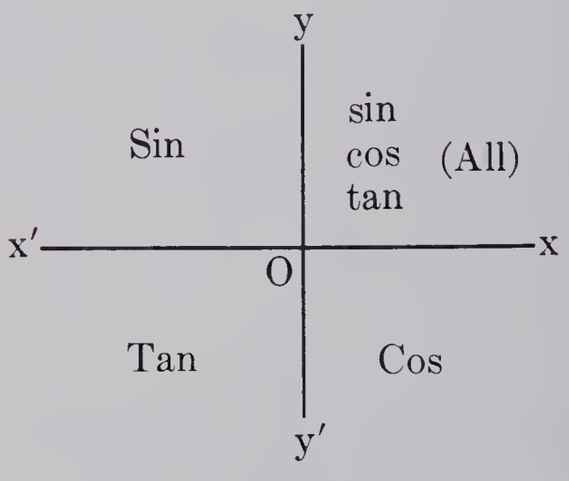
$$\text{cosine of } \theta = \cos\theta = \frac{x}{r} \qquad \text{secant of } \theta = \sec\theta = \frac{r}{x} \quad (x \neq 0)$$

$$\text{tangent of } \theta = \tan\theta = \frac{y}{x} \ (x \neq 0) \qquad \text{cotangent of } \theta = \cot\theta = \frac{x}{y} \ (y \neq 0)$$

where $r^2 = OP^2 = x^2 + y^2$.

We use the phrase *trigonometric functions* here but we have not yet shown that the sine, cosine, tangent, cotangent, cosecant, and secant of θ are in fact functions of θ according to the definition of Chapter 2. We will show this in Section 12-4 of this chapter, but meanwhile we will continue to call them trigonometric functions for convenience.

It follows from these definitions that the quadrants in which the sine, cosine, and tangent functions are *positive* are those shown in the diagram (the CAST diagram).



In order that we can use the tables, we should recall the following results which we obtained in Grade 11:

The Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

The Supplementary Identities

$\sin\ (180°-\theta)=\sin\ \theta$	*or*	$\sin\ (\pi-\theta)=\sin\ \theta$
$\cos\ (180°-\theta)=-\cos\ \theta$		$\cos\ (\pi-\theta)=-\cos\ \theta$
$\tan\ (180°-\theta)=-\tan\ \theta$		$\tan\ (\pi-\theta)=-\tan\ \theta$
$\csc\ (180°-\theta)=\csc\ \theta$		$\csc\ (\pi-\theta)=\csc\ \theta$
$\sec\ (180°-\theta)=-\sec\ \theta$		$\sec\ (\pi-\theta)=-\sec\ \theta$
$\cot\ (180°-\theta)=-\cot\ \theta$		$\cot\ (\pi-\theta)=-\cot\ \theta$

The Complementary Identities

$\sin\ (90°-\mathrm{A})=\cos\ \mathrm{A}$	*or*	$\sin\ (\frac{\pi}{2}-\mathrm{A})=\cos\ \mathrm{A}$
$\cos\ (90°-\mathrm{A})=\sin\ \mathrm{A}$		$\cos\ (\frac{\pi}{2}-\mathrm{A})=\sin\ \mathrm{A}$
$\tan\ (90°-\mathrm{A})=\cot\ \mathrm{A}$		$\tan\ (\frac{\pi}{2}-\mathrm{A})=\cot\ \mathrm{A}$
$\csc\ (90°-\mathrm{A})=\sec\ \mathrm{A}$		$\csc\ (\frac{\pi}{2}-\mathrm{A})=\sec\ \mathrm{A}$
$\sec\ (90°-\mathrm{A})=\csc\ \mathrm{A}$		$\sec\ (\frac{\pi}{2}-\mathrm{A})=\csc\ \mathrm{A}$
$\cot\ (90°-\mathrm{A})=\tan\ \mathrm{A}$		$\cot\ (\frac{\pi}{2}-\mathrm{A})=\tan\ \mathrm{A}$

Example 1

Without using tables, find the value of $\sin 60°$ and $\cos \frac{\pi}{3}$.

Solution: An angle of 60° and an angle of $\frac{\pi}{3}$ radians are equal in size. If all the angles of a triangle are 60°, the triangle is equilateral.

Construct $\triangle OPQ$ to be equilateral as shown and let PN be perpendicular to OQ. Let (x, y) be the coördinates of P, where x and y are both positive.

Then the coördinates of Q are $(2x, 0)$.

But $OQ = OP = r = 2x$. (why?)

In $\triangle OPN$,

$\angle ONP = 1$ rt. $\angle = 90°$

$\therefore\ OP^2 = ON^2 + NP^2$. (Pythagoras)

$4x^2 = x^2 + y^2$

$y^2 = 3x^2$

$y = \sqrt{3}x$

$\therefore\ \sin 60° = \frac{y}{r} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$ and $\cos \frac{\pi}{3} = \frac{x}{r} = \frac{x}{2x} = \frac{1}{2}$.

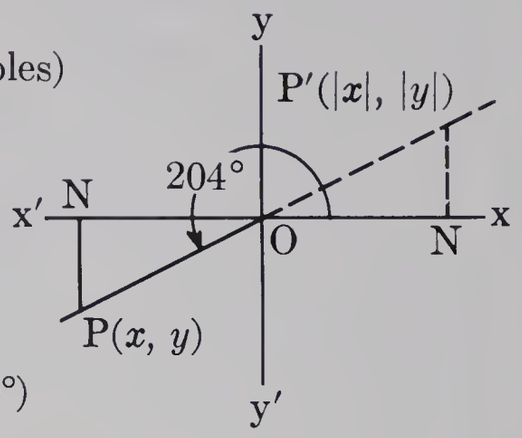
Example 2

Find the values of (i) sin 36° (ii) tan 147° (iii) cos 204° (iv) tan 321°.

Solution: (i) From the tables $\sin 36° = .5878$.

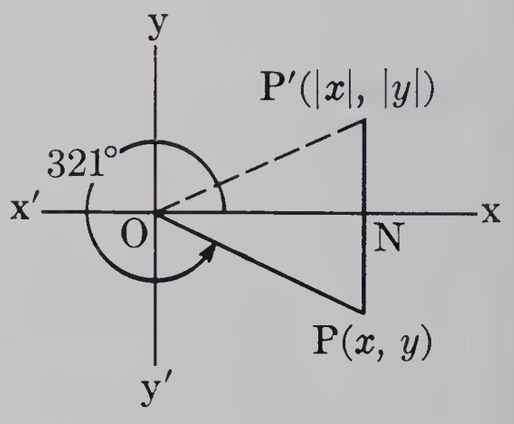
(ii) $\tan 147° = -\tan (180° - 147°)$
$= -\tan 33°$
$\simeq -.6494$ (from the tables)

(iii) From the diagram
204° is the third quadrant.
$\therefore$ cos 204° is negative.



Also $|\cos 204°| = \frac{|y|}{r}$
$= \cos (204° - 180°)$
$= \cos 24°.$
$\therefore \cos 204° = -\cos 24°$
$\simeq -.9135.$ (from the tables)

(iv) From the diagram
321° is in the fourth quadrant.
$\therefore$ tan 321° is negative.



$\angle P'ON = \angle PON = 360° - 321°$
$= 39°$
$|\tan 321°| = \frac{|y|}{|x|}$
$= \tan 39°$
$\therefore \tan 321° = -\tan 39°$
$\simeq -.8098.$ (from the tables)

EXERCISE 12—1

1. Without using tables, find the values of the sine, cosine, and tangent functions of the following angles:

(a) 45° (b) $\frac{5\pi}{3}$ (c) $\frac{\pi}{6}$ (d) 240° (e) 300° (f) π
(g) $\frac{2\pi}{3}$ (h) $\frac{5\pi}{4}$ (i) 270° (j) 135° (k) 210° (l) $\frac{7\pi}{6}$

2. Using tables, find the values of the sine, cosine, and tangent functions of the following angles:

(a) 17° (b) 123° (c) 236° (d) 52° (e) 76° (f) 161°
(g) 337° (h) 287° (i) 195° (j) 36° (k) 143° (l) 262°

3. Use the defining diagram to establish the following identities when $0 \leq \theta \leq \frac{\pi}{2}$:

(a) $\sin(\pi+\theta) = -\sin\theta$ (b) $\tan(2\pi-\theta) = -\tan\theta$

(c) $\cos(180°-\theta) = -\cos\theta$ (d) $\sin(90°-\theta) = \cos\theta$

(e) $\cos(270°-\theta) = -\sin\theta$ (f) $\sin(2\pi-\theta) = -\sin\theta$

(g) $\cos(\frac{3\pi}{2}+\theta) = \sin\theta$ (h) $\tan(180°+\theta) = \tan\theta$

(i) $\sin(\frac{3\pi}{2}-\theta) = -\cos\theta$ (j) $\cos(360°-\theta) = \cos\theta$

(k) $\cos(\pi+\theta) = -\cos\theta$ (l) $\tan(\theta-180°) = \tan\theta$

4. Find two angles θ, $0° \leq \theta \leq 360°$, for the following values of each trigonometric function:

(a) $\sin\theta = .3267$ (b) $\cos\theta = .5348$ (c) $\tan\theta = .7156$

(d) $\cos\theta = -.6743$ (e) $\tan\theta = -2.5146$ (f) $\sin\theta = -.4537$

(g) $\sin\theta = -.7641$ (h) $\sin\theta = .6372$ (i) $\cos\theta = .2836$

(j) $\tan\theta = 1.5732$ (k) $\cos\theta = -.4182$ (l) $\tan\theta = -.4873$

12 2 The Trigonometric Functions of Any Angle

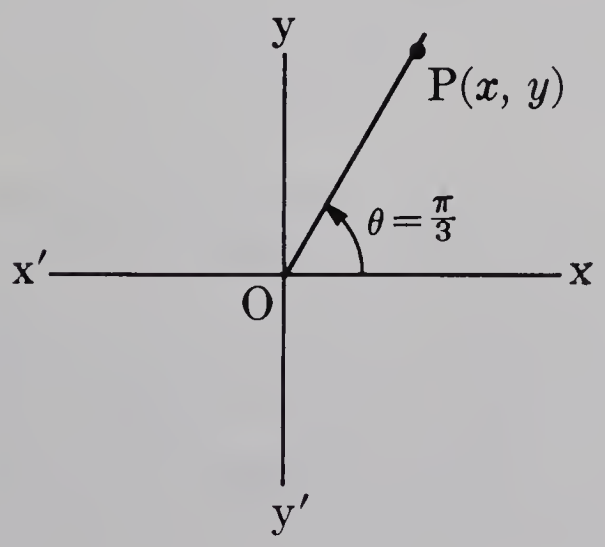
In our definitions above, we restricted the values of the angle θ to be from 0 to 2π (0° to 360°). For the geometrical uses of trigonometry this is usually sufficient; in fact for many cases in geometry a domain of 0° to 180° is enough. However two questions arise:

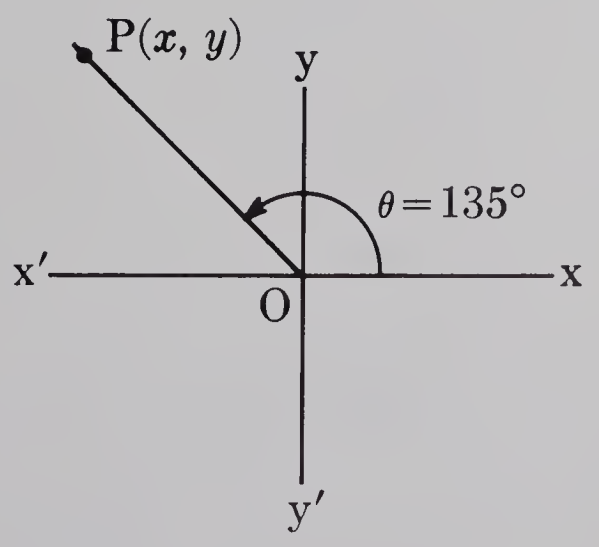
(1) Do we have to restrict θ in this way?

(2) Are the *trigonometric functions* really functions in accordance with the definition of a function in Chapter 2?

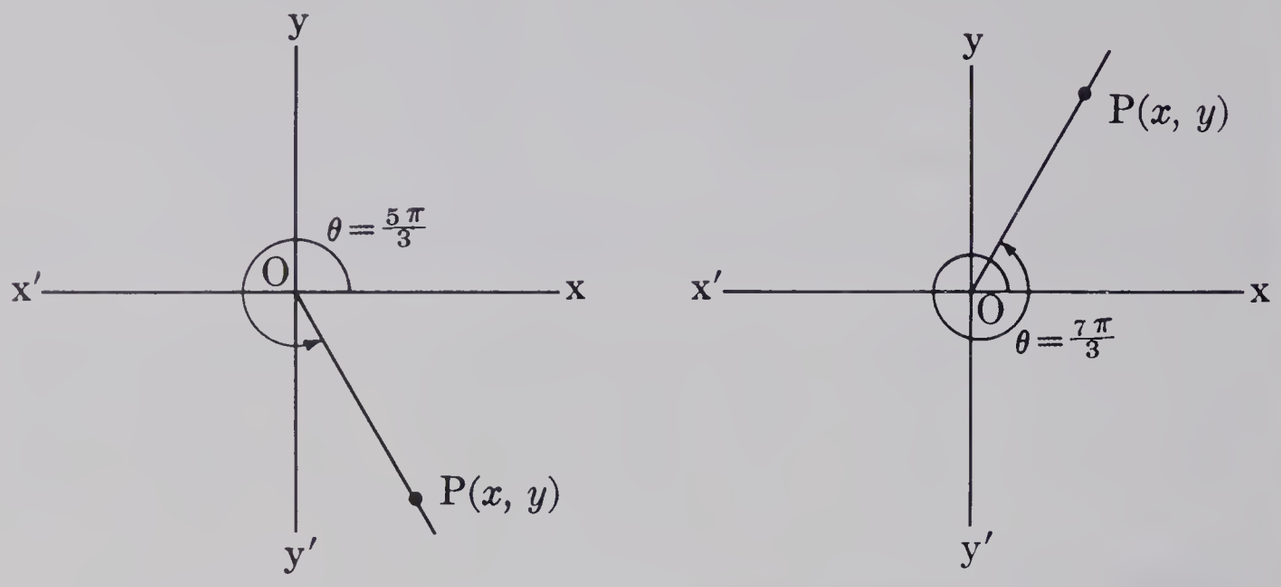
Let us consider these questions in turn, and in this section we will consider only the first question.

What is meant by an angle of $\frac{\pi}{3}$? 135°? The diagrams illustrate the answers we know already.





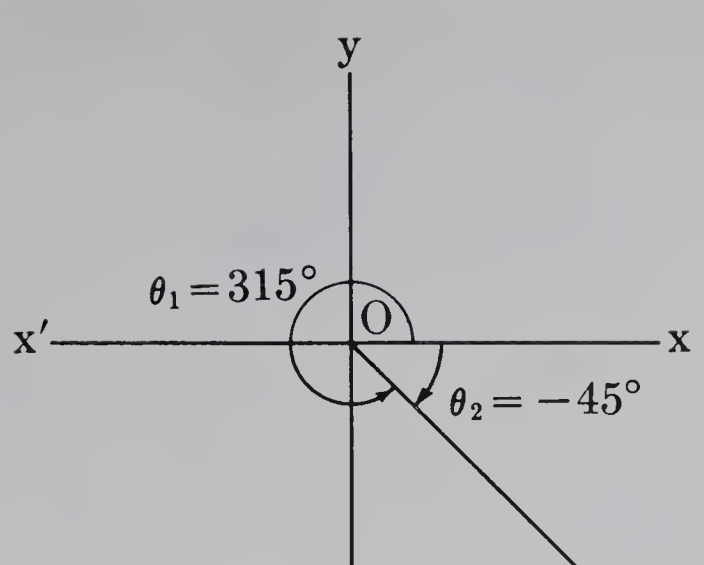
What do we mean by angles of $\frac{5\pi}{3}$ and $\frac{7\pi}{3}$? $\frac{5\pi}{3}$ is less than 2π, and we know that to form and angle of $\frac{5\pi}{3}$, we turn OP counterclockwise from Ox through an angle of $\frac{5\pi}{3}$. Presumably then for an angle $\frac{7\pi}{3}$, we will turn OP counterclockwise through an angle $\frac{2\pi}{3}$ more than for $\frac{5\pi}{3}$, that is, $\frac{\pi}{3}$ *more* than 2π. These results are shown in the diagrams.

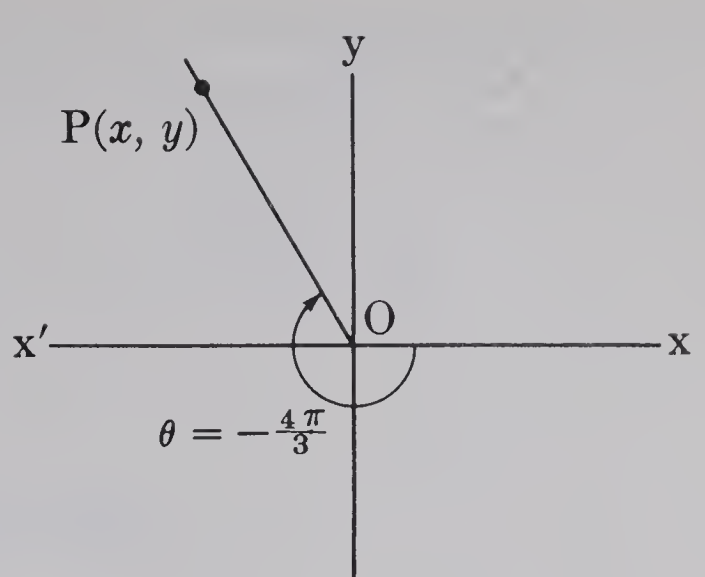


We notice immediately therefore that we are determining an angle greater than 2π (360°) by the same rules which we used for an angle less than 2π. We see also that the *geometrical angle* between the rays Ox and OP in this example is an angle of $\frac{\pi}{3} = \frac{7\pi}{3} - 2\pi$.

We might then ask why we bother with angles θ which are not such that $0 \leq \theta \leq 2\pi$ ($0° \leq \theta \leq 360°$). We do so because such angles frequently occur in practice. In fact any rotating object moves through angles which increase steadily with time; for example the earth rotates through 360° each day and a merry-go-round turns through 360° at each revolution. Also we frequently use the trigonometric functions in physics and engineering where θ is actually a product of a time (in seconds, say) and a frequency of a vibration, that is, the number of vibrations per second. Such a product of a time and a frequency is a real number just as the angle measure is a real number. In such a case, the product, frequency $\times$ time, may increase indefinitely, and so we cannot restrict θ to lie between 0 and 2π.

Let us now consider a negative angle, say $-45°$. What will this mean? We recall that negative integers are measured in the opposite direction to positive integers on the number line. Later in this chapter we will be representing the measures of angles on one of the axes of a Cartesian graph. If we apply this idea of opposite direction to angles, then $-45°$ will be the same absolute size as an angle of 45°, but it will be *measured clockwise instead of counterclockwise.* Similarly an angle of $-\frac{4\pi}{3}$ will be an angle of *magnitude* $\frac{4\pi}{3}$ measured clockwise from Ox.





Note that the position of OP in the case of $\theta = -45°$ is the same as that of OP when $\theta = 315° = 360° - 45°$. We call such angles coterminal angles; the ray OP is the common coterminal ray.

It will be noted that in each of the diagrams we have marked a point P with coördinates (x, y). We did this because in every case the *trigonometric functions* are defined in terms of x and y exactly as they were defined for

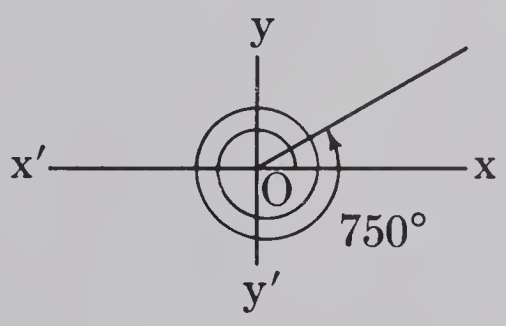
$$0 \leq \theta \leq 2\pi.$$

We can now see that, to find the *trigonometric functions* of any angle θ, we must add or subtract integral multiples of 2π (360°) until we obtain an angle ψ between 0 and 2π (0° and 360°). Then the *trigonometric function of* θ has the same value as the *trigonometric function of* ψ, and so we calculate, or find from the tables, the required value of the trigonometric function for the angle ψ. This is the value of the trigonometric function for the original angle θ.

Examples

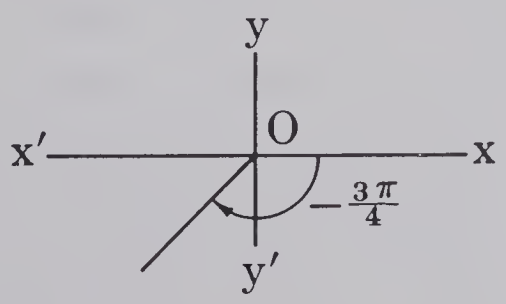
(i) Find the sine of 750°.

$750° - 360° = 390,$
$390° - 360° = 30°,$
or directly,
$750° - 2 \times 360° = 30°.$
$\therefore \sin 750° = \sin 30° = \frac{1}{2}.$



(ii) Find the cosine of $-\frac{3\pi}{4}$.

$-\frac{3\pi}{4} + 2\pi = (2 - \frac{3}{4})\pi = \frac{5\pi}{4}$
$\therefore \cos(-\frac{3\pi}{4}) = \cos(\frac{5\pi}{4})$
$= -\cos\frac{\pi}{4}$
$= -\frac{1}{\sqrt{2}}.$



(iii) Find the tangent of 486°.

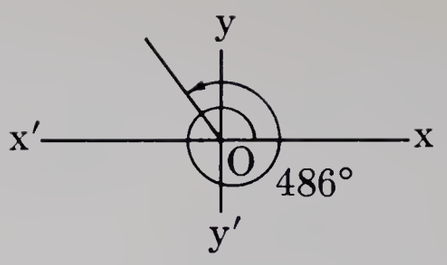
$486° - 360° = 126°$

$\therefore \tan 486° = \tan 126°$

$= -\tan 54°$

$\simeq -1.3764$

(from the tables).



EXERCISE 12—2

1. State which of the sine, cosine, and tangent functions are positive for each of the following angles:

(a) $\frac{2\pi}{3}$	(b) $\frac{11\pi}{4}$	(c) $-\frac{5\pi}{6}$	(d) $\frac{15\pi}{4}$
(e) $-\frac{9\pi}{4}$	(f) $-\frac{22\pi}{3}$	(g) $\frac{26\pi}{3}$	(h) $\frac{8\pi}{3}$
(i) $-\frac{\pi}{6}$	(j) $\frac{13\pi}{6}$	(k) $-\frac{3\pi}{4}$	(l) $-\frac{7\pi}{2}$
(m) $\frac{3\pi}{4}$	(n) $-\frac{4\pi}{3}$	(o) $\frac{7\pi}{6}$	(p) $\frac{5\pi}{2}$
(q) 6π	(r) -3π	(s) 5π	(t) $-\pi$
(u) $\frac{5\pi}{3}$	(v) $-\frac{16\pi}{3}$	(w) $\frac{13\pi}{4}$	(x) $\frac{15\pi}{6}$

2. Give the values of the sine, cosine, and tangent functions for each of the angles in (1).

3. Use the tables to find the value of the sine function for each of the following angles:

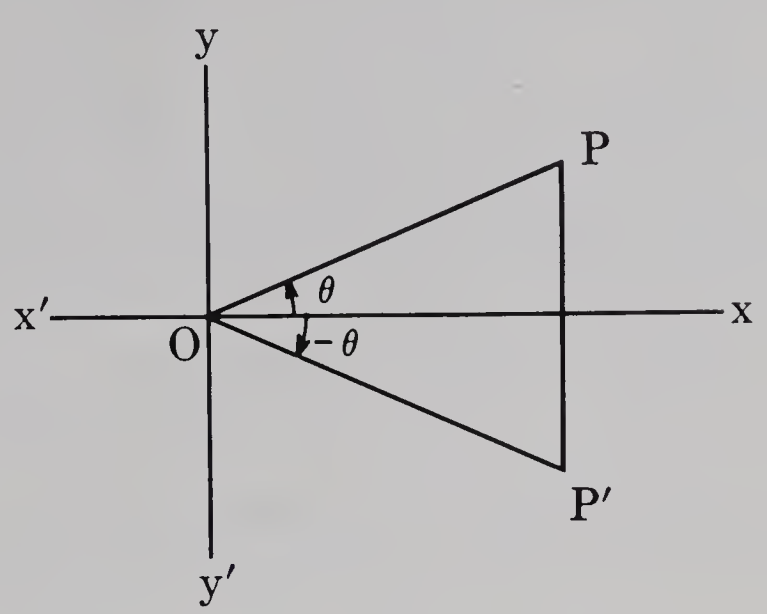
(a) 31°	(b) 326°	(c) 784°	(d) −67°
(e) 523°	(f) −207°	(g) −125°	(h) 225°
(i) −624°	(j) 432°	(k) 384°	(l) −246°
(m) 389°	(n) −493°	(o) −287°	(p) −57°
(q) −819°	(r) 924°	(s) −735°	(t) −127°
(u) 255°	(v) −537°	(w) 675°	(x) −439°

4. Find the value of the cosine function for each of the following angles:

(a) 208°	(b) −156°	(c) 269°	(d) 761°
(e) 386°	(f) −251°	(g) −652°	(h) 368°
(i) −439°	(j) −451°	(k) 837°	(l) 143°
(m) −513°	(n) 247°	(o) −546°	(p) −395°
(q) −161°	(r) 758°	(s) −401°	(t) 344°
(u) −634°	(v) 704°	(w) 521°	(x) −793°

12 3 The Trigonometric Functions of Positive and Negative Angles of Equal Magnitude

Let θ be the positive acute angle xOP.
Then $-\theta$ is the negative angle xOP′ equal to $\angle$xOP in magnitude.
Let PNP′ be perpendicular to Ox.
Then $\triangle OPN \equiv \triangle OP'N$. (a.s.a.)
If P is the point (x, y),



$$\sin\theta = \frac{y}{r} \text{ and } \sin(-\theta) = -\frac{y}{r}.$$

$$\therefore \sin(-\theta) = -\sin\theta.$$

Similarly $\cos\theta = \frac{x}{r} = \cos(-\theta)$.

$$\therefore \cos(-\theta) = \cos\theta$$

and

$$\therefore \tan(-\theta) = -\tan\theta.$$

The same results can be proved for any angle θ situated in any of the four quadrants, i.e., $0 \leq \theta \leq 2\pi$, and therefore for any angle θ of any magnitude.

EXERCISE 12—3

1. Prove $\cos(-\theta) = \cos\theta$ if $\frac{\pi}{2} < \theta < \pi$.

2. Prove $\sin(-\theta) = -\sin\theta$ if $\frac{\pi}{2} < \theta < \pi$.

3. Find the values of the sine function for the following negative angles from the values for the corresponding positive angles:

(a)	$-23°$	(b)	$-197°$	(c)	$-51°$	(d)	$-234°$
(e)	$-295°$	(f)	$-126°$	(g)	$-322°$	(h)	$-418°$
(i)	$-53°$	(j)	$-25°$	(k)	$-385°$	(l)	$-603°$
(m)	$-949°$	(n)	$-509°$	(o)	$-862°$	(p)	$-757°$
(q)	$-473°$	(r)	$-574°$	(s)	$-338°$	(t)	$-234°$
(u)	$-352°$	(v)	$-406°$	(w)	$-508°$	(x)	$-459°$

4. Find the values of the cosine function for the angles in (3).

5. Find the values of the tangent function for the angles in (3) by using the tables. Check the answers obtained by using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ with the results of (3) and (4).

12-4 The Domain and Range of the Sine and Cosine Functions

We can now discuss the second question raised in Section 12-2: Are the *trigonometric functions* considered in this chapter really functions in accordance with the definition of a function in Chapter 2?

Let us consider the sine "function". According to our previous definitions, the sine "function" is defined for all positions of $P(x, y)$ and as we have seen, a point such as P exists for θ equal to any real number. Thus the sine "function" exists for every value of θ belonging to the set of real numbers; that is, the *domain of the sine function is the set of real numbers.*

By the theorem of Pythagoras

$$x^2+y^2=r^2$$

and because $x \in \text{Re}$, therefore $x^2 \geq 0$.

$$\therefore\ y^2 \leq r^2.$$

$$\therefore\ \frac{y^2}{r^2} \leq 1.$$

$$\therefore\ \left|\frac{y}{r}\right| \leq 1.$$

$$\therefore\ -1 \leq \frac{y}{r} \leq 1.$$

$$\therefore\ -1 \leq \sin\theta \leq 1.$$

Hence the range of values of the sine "function" is from -1 to $+1$.

We must now examine the ordered pairs $\left(\theta, \frac{y}{r}\right)$.

By the definition and the construction of the angle θ starting with a fixed initial ray Ox, each angle determines only one final ray OP and so only one $\angle$xOP such that $0° < \angle\text{xOP} < 360°$.

The ratio $\frac{y}{r}$ is uniquely determined by $\angle$xOP; it is independent of the position of P on the ray OP. Therefore each angle θ determines a unique value $\frac{y}{r}$.

Thus the set of ordered pairs $\left(\theta, \frac{y}{r}\right)$ satisfies the requirements for a function: for each angle θ, there is one and only one value of $\frac{y}{r} = \sin\theta$.

Therefore the set of ordered pairs, $\{(\theta, \sin\theta) \mid \theta \in \text{Re}\}$, determines a function, the sine function, in conformity with the definition of a function, and we may omit the quotation marks used in this section.

For the sine function we can make the following statement:

The sine of θ, written sin θ, is a function with the set of real numbers for its domain and with the set of real numbers greater than or equal to -1 and less than or equal to 1 for its range. That is, the domain for θ is the set of real numbers, Re, and the range is $\{\sin\theta | -1 \leq \sin\theta \leq 1\}$.

More concisely:

The sine function is defined by $f(\theta) = \sin\ \theta, \theta \in \text{Re}$, and $-1 \leq f(\theta) \leq 1$. The sine function determines the set of ordered pairs $\{(\theta, \sin\ \theta) \mid \theta \in \text{Re}\}$.

The cosine function is defined by $f(\theta) = \cos\theta$, for $\theta \in \text{Re}$, and $-1 \leq f(\theta) \leq 1$. The cosine function determines the ordered pairs $\{(\theta, \cos\ \theta) \mid \theta \in \text{Re}\}$.

DEFINITION: In the functions sin θ or cos θ, θ is called the argument of the function.

Similarly, tan θ, cot θ, sec θ, and csc θ can be shown to be functions of θ. In these cases also, θ is called the argument of the function.

EXERCISE 12—4

(In all the following questions comment on any pattern you observe in the numbers in the tables.)

1. List the ordered pairs in a table of values of the sine function for the following thirty-three values of the angle θ:
   $-4\pi, -\frac{15\pi}{4}, -\frac{7\pi}{2}, -\frac{13\pi}{4}, \ldots, \frac{13\pi}{4}, \frac{7\pi}{2}, \frac{15\pi}{4}, 4\pi.$
2. List the ordered pairs in a table of values of the cosine function for the values of the angle θ given in (1).
3. List the ordered pairs in a table of values of the sine function for values of θ at 15° intervals between $-180°$ and 180°.
4. Repeat (3) for θ in the interval 0° to 360°.
5. Repeat (3) and (4) for the cosine function.
6. List the ordered pairs for the sine function of θ with $0 \leq \theta \leq 2\pi$ at intervals of $\frac{\pi}{12}$.
7. Repeat (6) for the cosine function.
8. Repeat (3), (4), and (6) for the tangent and cotangent functions.
9. Repeat (3), (4), and (6) for the secant and cosecant functions.

12 5 The Graphs of the Sine and Cosine Functions

In (1) Exercise 12-4, you prepared a table of values for the set of ordered pairs $\{(\theta, \sin\theta)\}$ without using the trigonometric tables, and obtained the following result for $-4\pi \leq \theta \leq 4\pi$:

θ	-4π	$-\frac{15}{4}\pi$	$-\frac{7}{2}\pi$	$-\frac{13}{4}\pi$	-3π	$-\frac{11}{4}\pi$	$-\frac{5}{2}\pi$	$-\frac{9}{4}\pi$	-2π	$-\frac{7}{4}\pi$	$-\frac{3}{2}\pi$	$-\frac{5}{4}\pi$	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{4}\pi$	0
$\sin\theta$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0
θ	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π	$\frac{9}{4}\pi$	$\frac{5}{2}\pi$	$\frac{11}{4}\pi$	3π	$\frac{13}{4}\pi$	$\frac{7}{2}\pi$	$\frac{15}{4}\pi$	4π
$\sin\theta$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0

This table gives us enough data to draw an approximation to the graph of $\sin\theta$ as a function of θ, but even though we have 33 points there are not enough to tell us the shape of the curve in detail. For example, if we plot the points in the table and join them by straight line segments, we get the graph shown. Notice we use θ as the abscissa and $\sin\theta$ as the ordinate in the usual Cartesian plane.

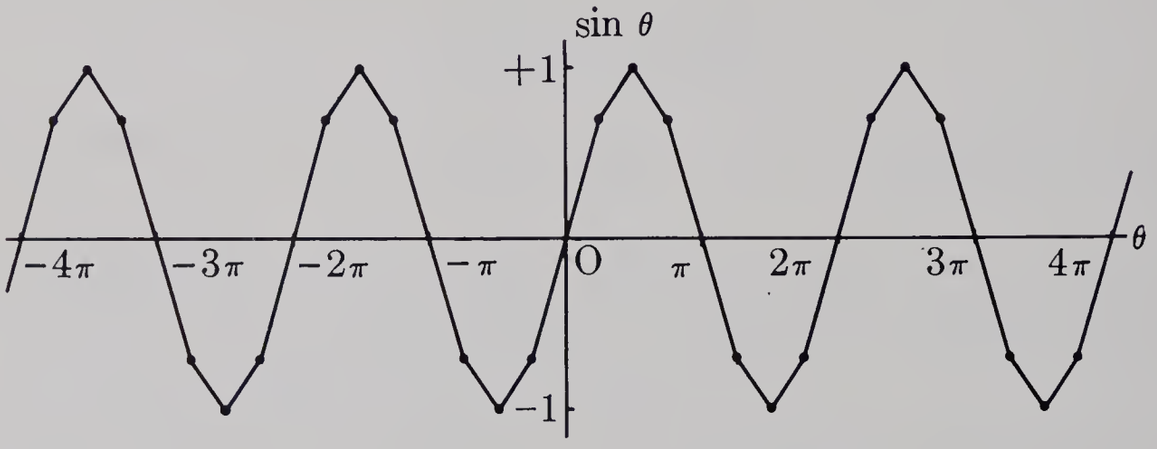


Figure 12-1

Of course such a graph as this is accurate only at the points which we actually plotted from the table of values. In spite of this, we can see that the shape and size of graph between $\theta=0$ and $\theta=2\pi$ is repeated in the intervals $\theta=2\pi$ to $\theta=4\pi$, $\theta=-4\pi$ to $\theta=-2\pi$, and $\theta=-2\pi$ to $\theta=0$. Thus if we can draw the graph between $\theta=0$ and $\theta=2\pi$, then we can draw it for *all* values of θ by repeating the portion already drawn. Therefore we need to calculate only the ordered pairs $(\theta, \sin\theta)$ for $0\leq\theta<2\pi$ (Exercise 12-4(6)). The values of $\sin\theta$ for $2\pi\leq\theta<4\pi$, $-2\pi\leq\theta\leq 0$, and $-4\pi<\theta\leq-2\pi$ merely repeat those for $0\leq\theta<2\pi$. Note that most functions do *not* repeat in this way, for example, $y=x$, $y=x^2$, $y=x^4$. Functions which do repeat regularly in this manner are called periodic functions.

We use the table of values prepared in (6) Exercise 12-4, to draw the graph.

Table of Values of Sin θ for $0 \leq \theta \leq 2\pi$

(Use trigonometric tables to find the values of sin θ.)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin\theta$	0	.2588	.5000	.7071	.8660	.9659	1	.9659	.8660	.7071	.5000	.2588	0

θ	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
$\sin\theta$	0	−.2588	−.5000	−.7071	−.8660	−.9659	−1	−.9659	−.8660	−.7071	−.5000	−.2588	0

From this table of values we may make a more accurate graph by drawing a smooth curve through the points given by the ordered pairs $(\theta, \sin\theta)$. Of course when we plot the points in the diagram, we cannot use more than two significant figures at the most, and so we have to round off the values in the table to two figures.

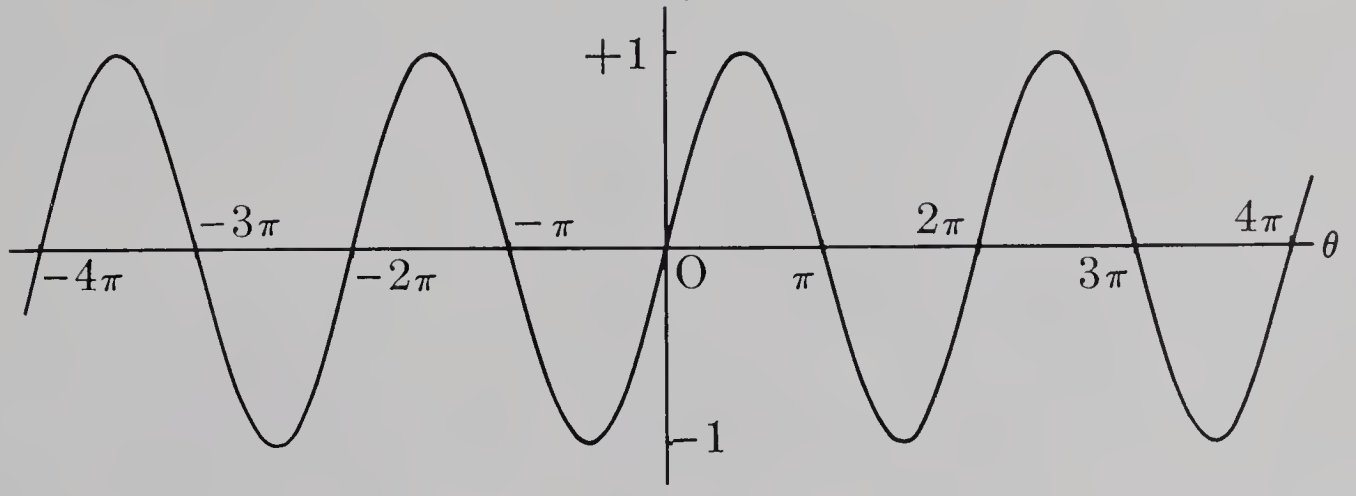


Figure 12-2 The graph of sin θ

EXERCISE 12—5

1. Draw the graph of sin θ for θ in degrees from 0 to 720°. (Use points corresponding to $\theta = 0°, 15°, 30°, \ldots 360°$. See (4) Exercise 12-4.)
2. Repeat (1) for the cosine function.
3. Draw the graph of cos θ for θ in radians from 0 to 4π.
4. What symmetry does the sine function have?
5. What symmetry does the cosine function have?

12 6 The Zeros and Periodicity of the Sine and Cosine Functions

Let us now look carefully at the form of the graph of the sine function. The study of the cosine function is similar and you are asked to work through it in (4), (5), and (6), Exercise 12-6.

We notice that the graph of $\sin\theta$ confirms that $-1 \leq \sin\theta \leq 1$ for all values of θ; in other words, the greatest values or *maximum values* of $\sin\theta$ are all 1 and the least values or *minimum values* of $\sin\theta$ are all -1. $\sin\theta$ attains these maximum values of 1 for

$$\theta = \ldots, -\tfrac{7\pi}{2}, -\tfrac{3\pi}{2}, \tfrac{\pi}{2}, \tfrac{5\pi}{2}, \tfrac{9\pi}{2}, \ldots,$$

and attains its minimum values of -1 for

$$\theta = \ldots, -\tfrac{5\pi}{2}, -\tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{7\pi}{2}, \tfrac{11\pi}{2}, \ldots.$$

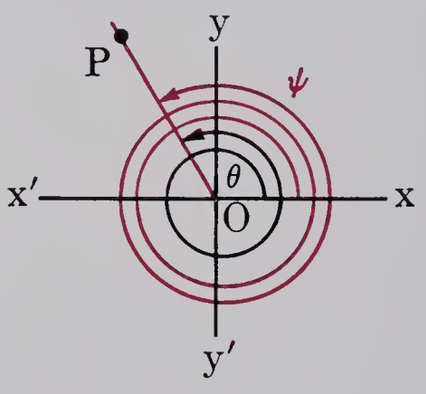
Also $\sin\theta$ is equal to zero for

$$\theta = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots.$$

We see from these results that the maxima occur at values of θ which are 2π apart and similarly the minima occur at values of θ which are 2π apart. If we examine the zeros we see that there are two subsets of zeros; in one subset the value of the sine function is changing from positive to negative at the zero; in the other subset the sine function is changing from negative to positive values. For each subset the zeros are 2π apart.

In fact we can show that this type of relationship is true for $\sin\theta$ for all values of θ, that is, $\sin(\theta+2n\pi) = \sin\theta$, $n \in \mathrm{I}$, $\theta \in \mathrm{Re}$.

Consider any two angles θ and ψ which differ by an integral multiple of 2π radians or 360°, i.e., by an integral number of revolutions. Then the angles have the same coterminal ray OP if we use the same initial ray Ox. The value of the sine function is uniquely determined by the position of the ray OP relative to the coördinate axis Ox. Hence the value of the sine function is the same for θ and for ψ. But $\psi = \theta + 2n\pi$, $n \in \mathrm{I}$; therefore $\sin(\theta+2n\pi) = \sin\theta$.



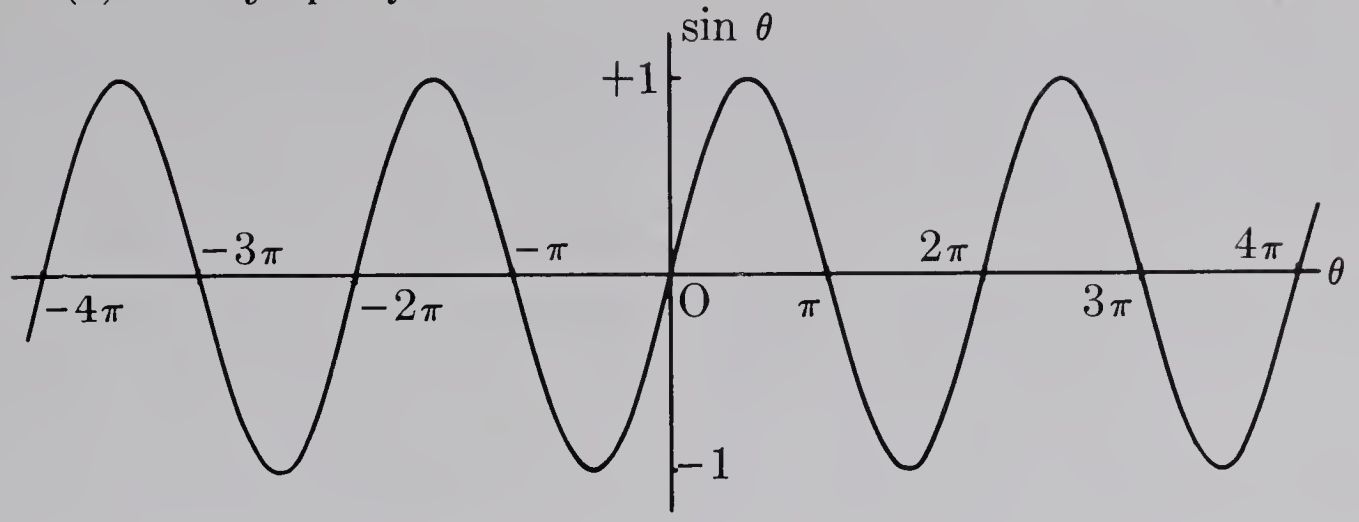
We can prove that $\cos(\theta+2n\pi) = \cos\theta$, $n \in \mathrm{I}$, $\theta \in \mathrm{Re}$ in the same way. ((10) Exercise 12-6)

DEFINITION: The period of the sine or cosine function is 2π, that is $\sin\theta = \sin(\theta+2\pi)$ and $\cos\theta = \cos(\theta+2\pi)$, for all $\theta \in \mathrm{Re}$.

The period is the length of the smallest interval over which the function assumes all its values.

The interrelation of the sine and cosine functions

(*a*) *The graph of sin θ*



(*b*) *The graph of cos θ*

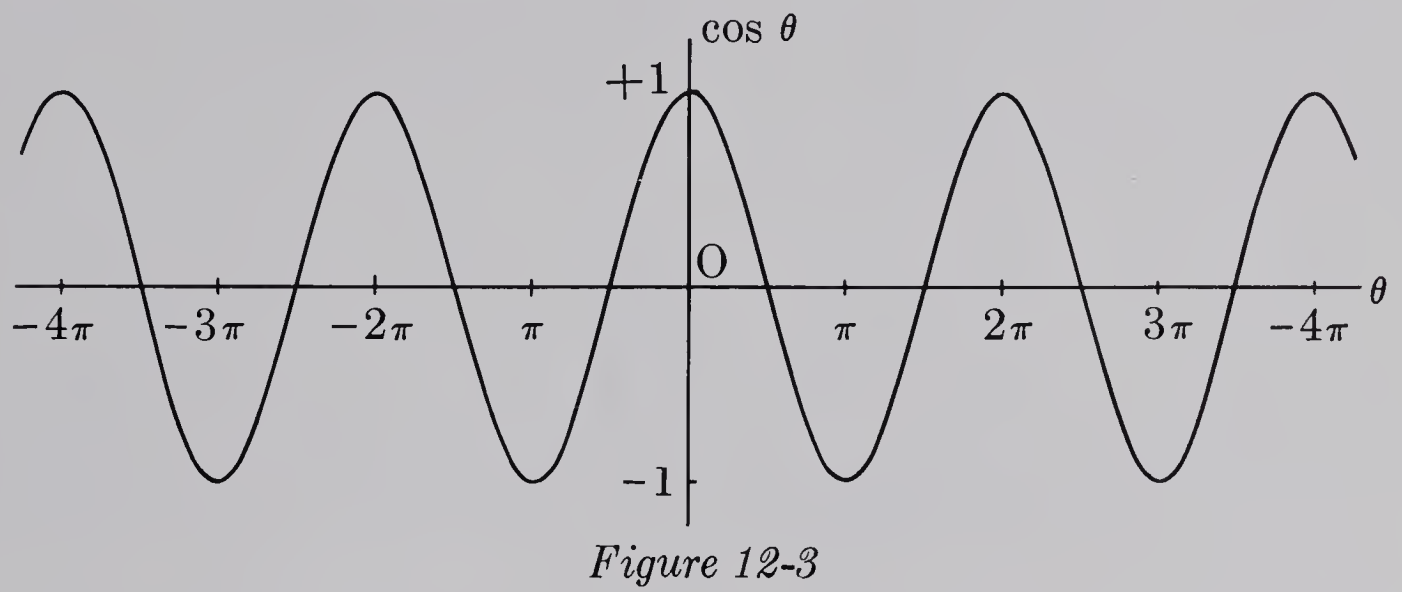


Figure 12-3

If we examine the two graphs in Figure 12-3, we see that the curves appear to be identical in shape and size. We can check that this is the case by tracing (*a*) and superimposing it on (*b*). The difference between the two curves is in their positions with respect to the axes of the coördinate system. Thus:

the zeros of sin θ occur at $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
while those of cos θ occur at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
Similarly the maxima of sin θ are at $\ldots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$
and the maxima of cos θ are at $\ldots, -2\pi, 0, 2\pi, \ldots$
Where are the minima of the two functions?

In fact we can make the cosine curve fit the sine curve drawn to the same axes by "moving" the cosine curve a distance $\frac{\pi}{2}$ to the right parallel to the θ-axis. The relation between the two functions is defined by

$$\sin \theta = \cos (\theta - \tfrac{\pi}{2}), \ \theta \in \text{Re}.$$

You can verify this from the tables of values of the sine and cosine functions you constructed in order to draw the graphs. This relation can also be defined by $\cos \theta = \sin (\theta + \frac{\pi}{2}), \ \theta \in \text{Re}.$

EXERCISE 12—6

1. List the values of ten angles (expressed in radians) for which $\sin\theta = 0$. State the values of these angles in degrees also.

2. List six angles for which $\sin\theta$ has the value 1.

3. List six angles for which $\sin\theta$ has the value -1.

4. Find and list some of the set of values of θ for which $\cos\theta = 0$.

5. Find and list some of the set of values of θ for which $\cos\theta = 1$.

6. Find and list some of the set of values of θ for which $\cos\theta = -1$.

7. Use the ideas of periodicity to give 5 values of θ for which $\sin\theta$ is equal to each of the following:

(a) $\sqrt{\frac{1}{2}}$	(b) $-.8660$	(c) $-.2315$	(d) $-.7071$
(e) 1	(f) .5000	(g) 0	(h) $\frac{\sqrt{3}}{2}$
(i) -1	(j) .7500	(k) $-\frac{1}{2}$	(l) .4638

8. Use the ideas of periodicity to give 5 values of θ for which $\cos\theta$ is equal to each of the following:

(a) .7071	(b) $-\frac{\sqrt{3}}{2}$	(c) 0	(d) $-.3671$
(e) $\frac{1}{2}$	(f) .5471	(g) .8660	(h) $-\sqrt{\frac{1}{2}}$
(i) 1	(j) .8237	(k) $-.6347$	(l) -1

9. (a) Must all consecutive values of θ for which $\sin\theta = \sqrt{\frac{1}{2}}$ be 2π, i.e. one period, apart? Examine the cases of all the other values of $\sin\theta$ in (7) in the same way. How do the cases $\sin\theta = 1$ and $\sin\theta = -1$ differ from the others?

   (b) Repeat part (a) for the values of $\cos\theta$ in (8).

   (c) Are there two disjoint and complementary subsets of values of θ for which $\sin\theta = \frac{\sqrt{3}}{2}$ and such that in each subset the values of θ are periodic? Is a similar result true for $\cos\theta = -\frac{1}{2}$? for other values of $\sin\theta$ and $\cos\theta$?

10. Prove that $\cos(\theta + 2n\pi) = \cos\theta$ for any value of θ.

11. Prove that $\sin(\theta + 2n\pi) = \cos\left(\theta - \frac{\pi}{2}\right)$ and $\cos(\theta + 2n\pi) = \sin\left(\theta + \frac{\pi}{2}\right)$.

12 7 The Amplitude of $a \sin \theta$

Consider the function defined by $y = a \sin \theta$, where a is some given positive real number. We have, in fact, already considered one case and drawn its graph, the case where $a = 1$. If we look back at this graph, we see that $|y| \leq 1$ for all values of θ, and that $|y| = 1$ only at the points where y has a maximum or minimum value.

To see what happens to the graph of $\{(\theta, a \sin \theta) | \theta \in \mathrm{Re}\}$ when $a \neq 1$, consider the case of $a = 2.5$. We calculate the following table of values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$	0	.5000	.7071	.8660	1	.8660	.7071	.5000	0
$2.5 \sin \theta$	0	1.25	1.7677	2.1650	2.5	2.1650	1.7677	1.25	0

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin \theta$	0	−.5000	−.7071	−.8660	−1	−.8660	−.7071	−.5000	0
$2.5 \sin \theta$	0	−1.25	−1.7677	−2.1650	−2.5	−2.1650	−1.7677	−1.25	0

Obviously each value in the last row is 2.5 times the corresponding value in the row above. The graph is shown in Figure 12-4.

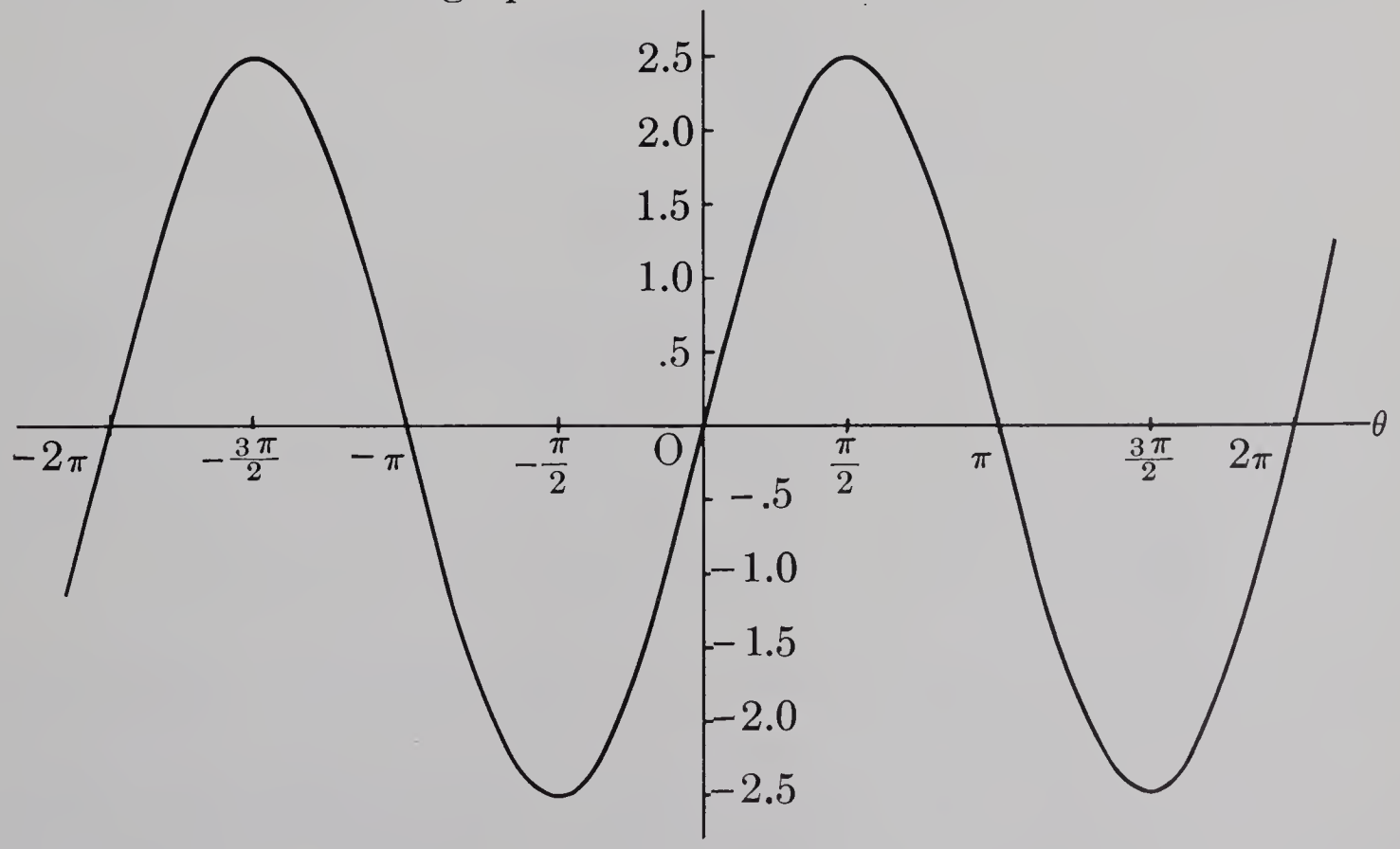


Figure 12-4

We see either from the graph or from the table of values that, for all values of θ, $|y| \leq 2.5$, and $|y| = 2.5$ only at the maximum and minimum points.

If we draw the graphs for other values of $a > 1$, we observe similar results: $|y| \leq a$ for all values of the argument θ and $|y| = a$ only at the maxima and minima.

In these cases we notice also that the graph of $a \sin \theta$ is very much like that of $\sin \theta$. It has zeros for the same values of the argument θ ($\ldots, -2\pi, -\pi, 0, +\pi, +2\pi, \ldots$) and its maxima and minima occur at the same values of the argument θ. The distinction between the two graphs is that the value of y for any given value of θ is greater; in fact, it is a times as great (see Figure 12-4). In particular, the maxima and minima are a times as far from the θ-axis. We might say that the graph of $a \sin \theta$ is the graph of $\sin \theta$ with the ordinate magnified, or amplified, to use a term from sound rather than sight. Thus a is an amplification factor; it is used very frequently in science and is called simply *the amplitude.*

So far we have only discussed the cases when $a = 1$ and when $a > 1$. The case when $a < 1$ is very similar. $|y| = a$ at the maxima and minima as before, but now all the ordinates are *reduced* in the ratio a:1. Figure 12-5 shows the graph of the equation $y = a \sin \theta$ for the case $a = .4$.

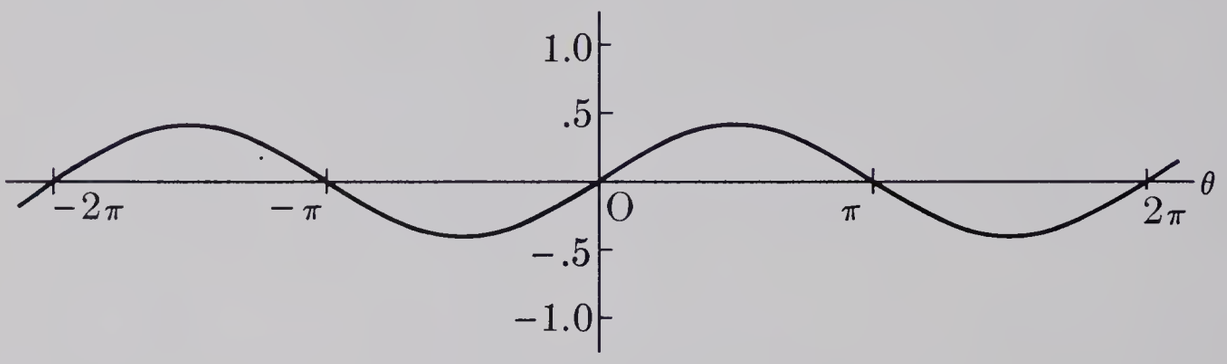


Figure 12-5

When $a < 1$ and $y = a \sin \theta$, we might be tempted to call a the reduction factor or some other name which would indicate a decrease in size. However mathematicians and scientists use the same name, *the amplitude,* for *all* values of a and distinguish between the two cases by noting whether a is greater than, or less than, one. Notice that in the case of $a = 1$, the function is defined by $y = \sin x$ and the amplitude is equal to 1.

DEFINITION: If $y = a \sin \theta$, then $|a|$ is called the amplitude of the function.

Note that *the amplitude is one half of the difference between the maximum and minimum values,* and this is often the practical definition used in physics. The amplitude of a vibration is very important in physics. For example, in sound the changes in amplitude cause the changes in the volume of a sound, and in radio the changes in amplitude of radio waves are used to receive signals on the A.M. (Amplitude Modulation) band of a radio set.

Figure 12-6 illustrates the cases $a = 2.5$, 1.0, and .4 on the same diagram for comparison purposes. Now that we know the shape of the graph of the sine function we can sketch such curves and we plot only the positions of the zeros, the maxima, and the minima.

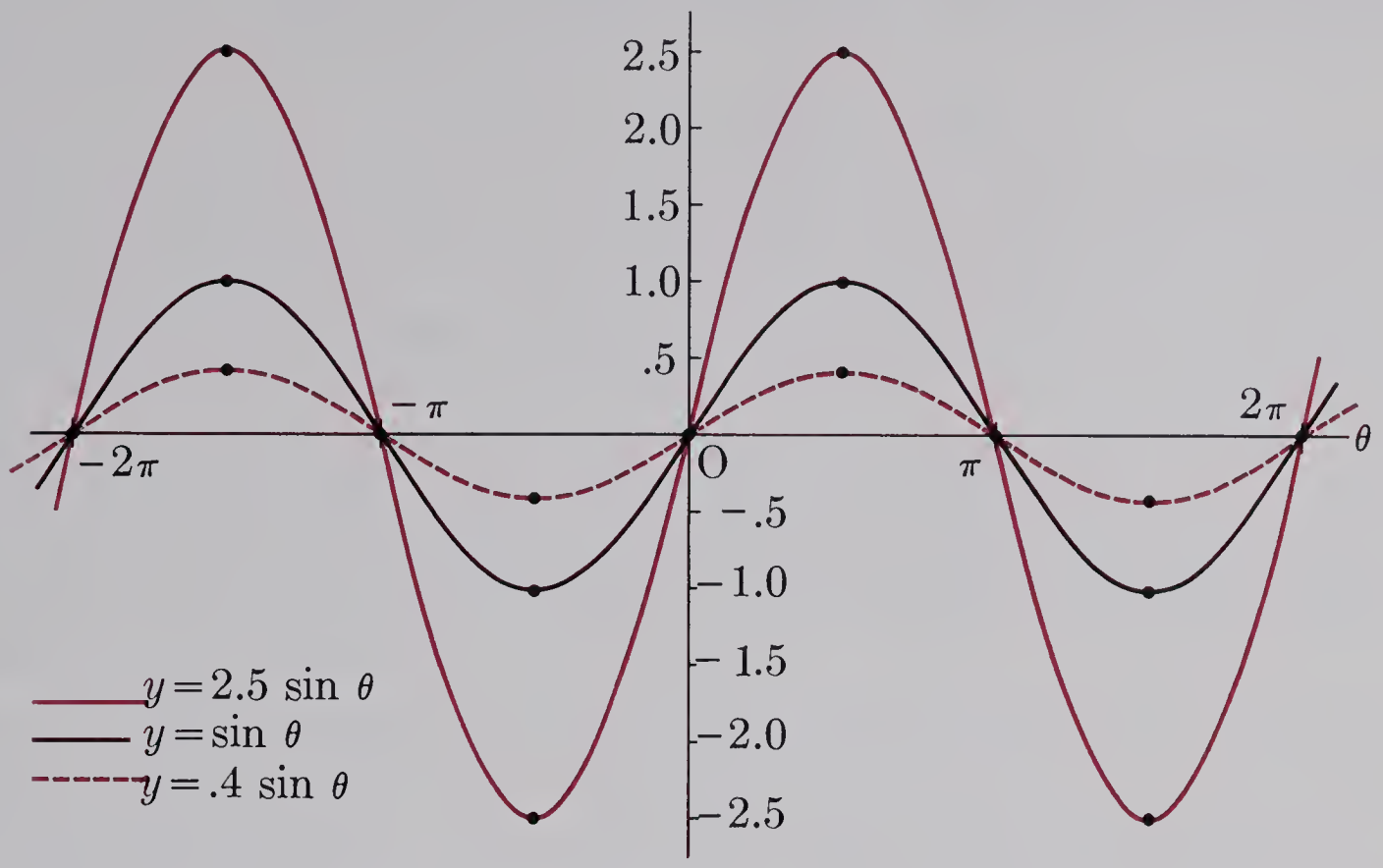


Figure 12-6

EXERCISE 12—7

1. Calculate a table of values for $(\theta, 2 \sin \theta)$ at 30° intervals for θ between 0° and 180°, and draw the graph between −360° and 360°.

2. Sketch the graphs of $y = 0.5 \sin \theta$, $y = \sin \theta$, and $y = 3 \sin \theta$, between $\theta = -2\pi$ and $\theta = 2\pi$, relative to the same set of axes. Compare the amplitudes.

3. State the amplitude of the following:

   (a) $3 \sin \theta$ (b) $\frac{1}{5} \sin \theta$ (c) $4.3 \sin \theta$

   (d) $.002 \sin \theta$ (e) $2\pi \sin \theta$ (f) $3 \cdot 10^6 \sin \theta$

4. Calculate a table of values for $(\theta, 1.5 \cos \theta)$ at 30° intervals from 0° to 90°. Draw the graph of $y = 1.5 \cos \theta$ between $\theta = 0°$ and 720°.

5. Sketch the graphs of $y = a \cos \theta$ with $a = .8$, 1, and 2.0 between $\theta = -2\pi$ and $\theta = 2\pi$. Compare the amplitudes.

6. State the amplitude of the following:

   (a) $7 \cos \theta$ (b) $.03 \cos \theta$ (c) $5 \cdot 10^3 \cos \theta$

   (d) $6 \cdot 10^{-2} \cos \theta$ (e) $\sqrt{2} \cos \theta$ (f) $\frac{2}{5} \cos \theta$

12 8 The Periodicity of sin kθ and cos kθ

Another interesting case of great importance in science occurs when the argument is $k\theta$, where k is a positive real number.

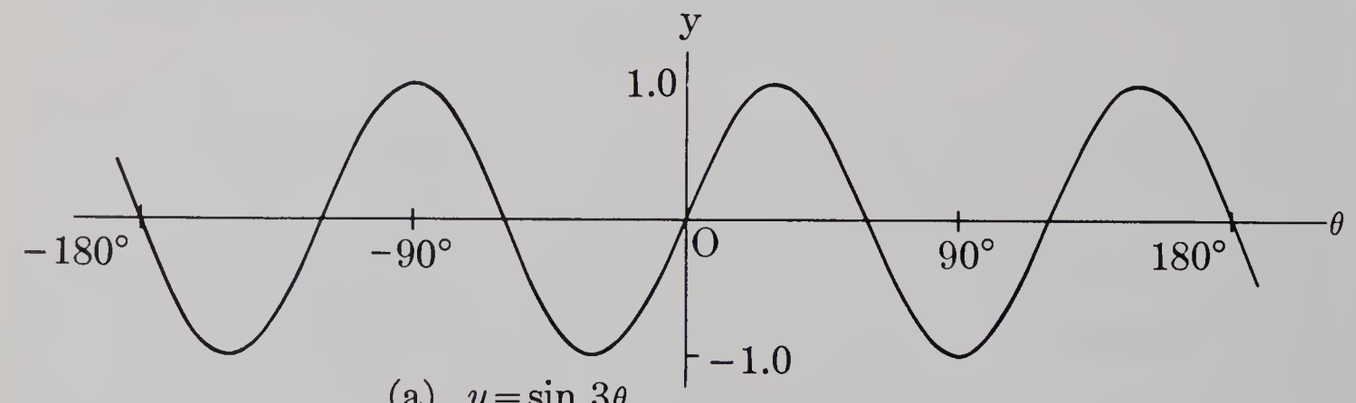
The function defined by $y = \sin 3\theta$ is used as an example.

Table of Values for $y = \sin 3\theta$

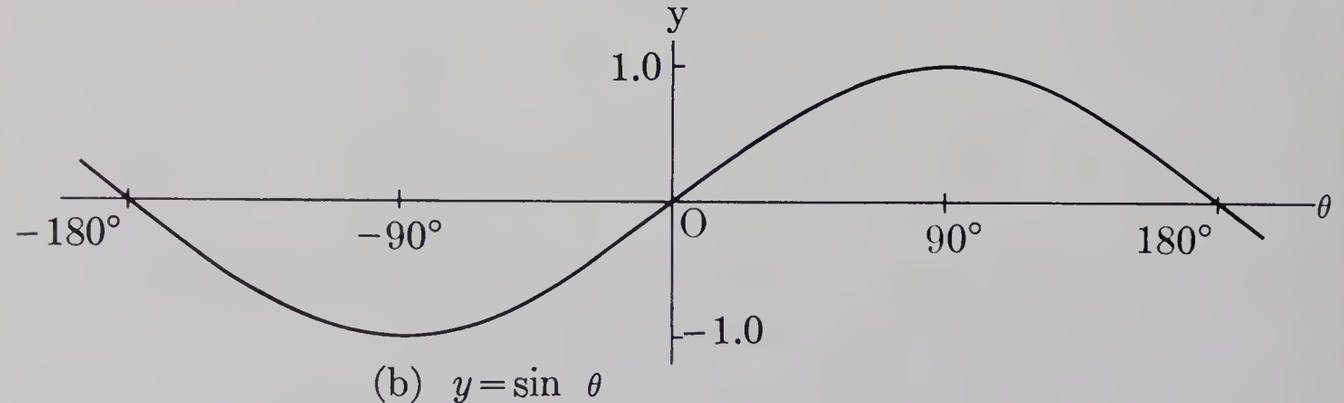
θ	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°
3θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
$\sin 3\theta$	0	.2588	.5000	.7071	.8660	.9659	1	.9659	.8660	.7071	.5000	.2588	0

θ	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	120°
3θ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
$\sin 3\theta$	−.2588	−.5000	−.7071	−.8660	−.9659	−1	−.9659	−.8660	−.7071	−.5000	−.2588	0

θ	125°	130°	135°	140°	145°	150°	155°	160°	165°	170°	175°	180°
3θ	375°	390°	405°	420°	435°	450°	465°	480°	495°	510°	525°	540°
$\sin 3\theta$	.2588	.5000	.7071	.8660	.9659	1	.9659	.8660	.7071	.5000	.2588	0



(a) $y = \sin 3\theta$



(b) $y = \sin \theta$

Figure 12-7

The graph for $k=1$, that is $y=\sin\theta$, is sketched on the same scale for comparison. The amplitude is not changed; it is *one* in both cases. However the maxima and minima are closer together when $k=3$ than when $k=1$, and the same applies to the zeros. In fact the whole graph has been *squeezed* in the direction parallel to the θ-axis. Therefore, the periods of the two functions must de different.

DEFINITION: The *period, in* θ, of the sine function, $\sin k\theta$, is the difference in the value of θ between two adjacent zeros of $y=\sin k\theta$ at which y changes sign from negative to positive as θ increases.

If we examine Figure 12-7 (a) and (b), we see that for $y=\sin 3\theta$, the period is $\frac{2\pi}{3}$, and for $y=\sin\theta$, the period is 2π.

Note that if θ alone is the argument of the sine function, we usually shorten the phrase 'the period in θ' to merely 'the period'.

Thus the period in θ is only one third as great for $y=\sin 3\theta$ as for $y=\sin\theta$. Alternatively, we could say that $y=\sin 3\theta$ repeats 3 times as rapidly when θ increases as does $y=\sin\theta$.

When you sketch the graphs for other values of k in the exercises, you will see in every case the period in θ is $\frac{2\pi}{k}$.

Alternate Definition: The period, in θ, of the sine function $\sin k\theta$ is $\frac{2\pi}{k}$.

Sketch graphs for the cases $k=1$, 3, and .5 are superimposed for comparison in Figure 12-8.

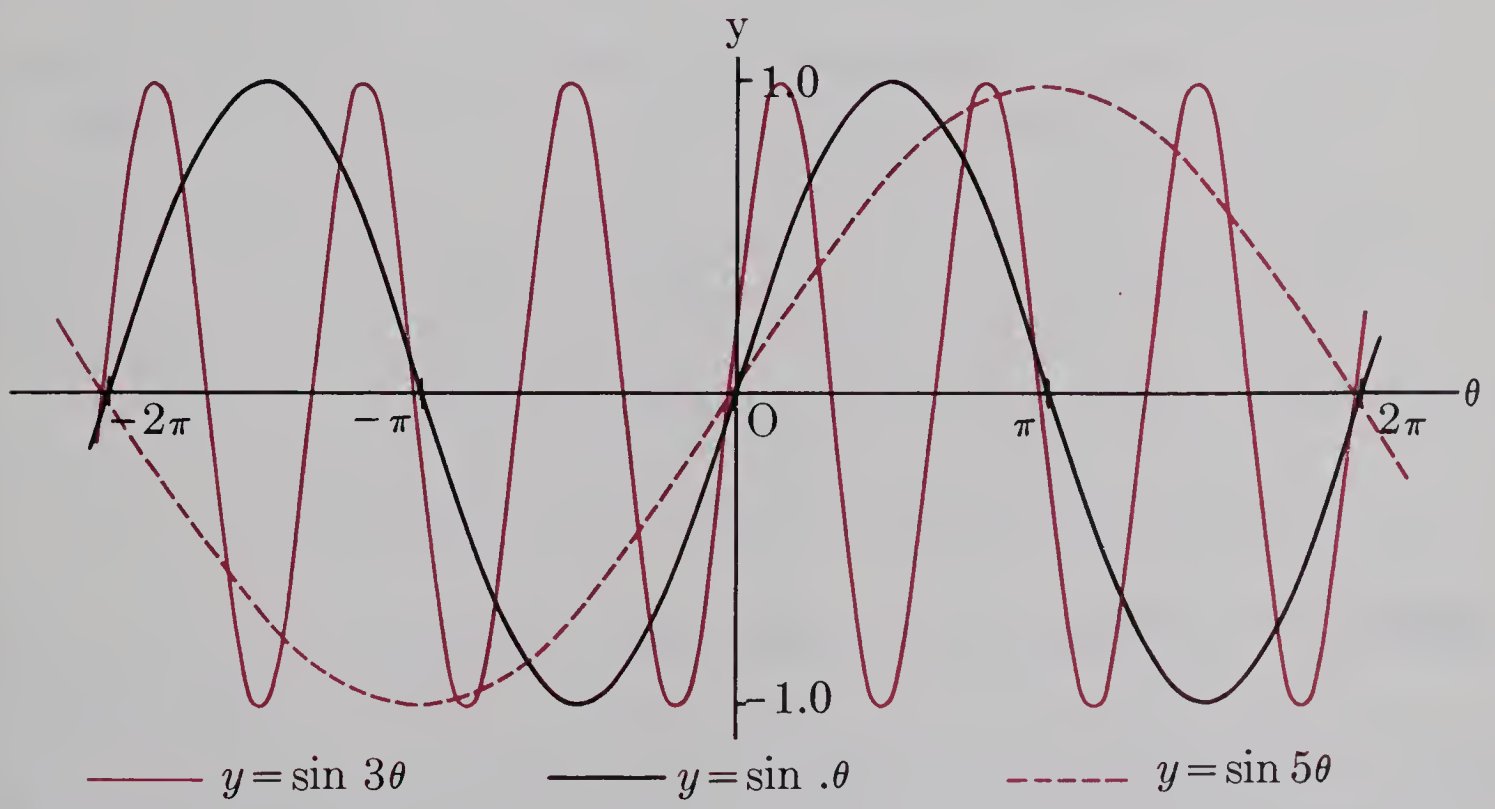


Figure 12-8

EXERCISE 12—8

(In all graphs show at least two full periods of the function.)

1. Sketch the graphs of the functions determined by
   (a) $y=\sin 2\theta$ (b) $y=\sin 4\theta$ (c) $y=\sin \frac{\theta}{2}$ (d) $y=\sin \frac{\theta}{4}$.
   State the periods in θ of these functions.

2. Sketch the graphs of the functions determined by
   (a) $y=\sin 3\theta$ (b) $y=\sin 4\theta$
   relative to the same set of axes. When does the combined pattern formed by the two functions begin to repeat for the first time for positive θ?

3. Sketch the graphs of the functions determined by
   (a) $y=\cos \theta$ (b) $y=\cos 2.5\theta$ (c) $y=\cos 5\theta$ (d) $y=\cos .4\theta$.
   State the periods in θ of these functions.

4. Repeat (2) for the curves given by $y=\cos 2\theta$ and $y=\cos 5\theta$.

5. Examine the graphs of the pairs of functions defined as follows:
   (a) $y=\sin 2\theta$, $y=\sin 4\theta$ (b) $y=\sin \frac{\theta}{2}$, $y=\sin 2\theta$
   (c) $y=\cos \frac{\theta}{3}$, $y=\cos \frac{\theta}{2}$ (d) $y=\cos 2\theta$, $y=\cos \frac{\theta}{3}$
   What do you notice about the period of the pattern of the pairs and the individual periods?

6. (a) Show that the difference in the values of θ between two adjacent zeros of $y=\sin k\theta$ at which y changes sign from positive to negative is equal to the period in θ.
   (b) Show that the period in θ if $y=\sin k\theta$ is equal to the difference between the values of θ at two adjacent maxima.
   (c) Show that the period in θ of $y=\sin k\theta$ is equal to the difference between the values of θ at two adjacent minima.

7. Prove that $\sin k\theta=\sin k(\theta+n\frac{2\pi}{k})$ and $\cos k\theta=\cos k(\theta+n\frac{2\pi}{k})$.

12 9 The Functions $a \sin k\theta$ and $a \cos k\theta$

We have studied the functions defined by $y=a \sin \theta$ and $y=\sin k\theta$. Now we consider the function defined by $y=a \sin k\theta$.

Consider the specific case of the function defined by $y=2.5 \sin 3\theta$. The graph of $\{(x, y)|y=2.5 \sin 3\theta, \theta\in \mathrm{Re}\}$ is shown in Figure 12-9. For comparison we show the graphs of the functions defined by $y=2.5 \sin \theta$ and $y=\sin 3\theta$, which we studied in the previous sections.

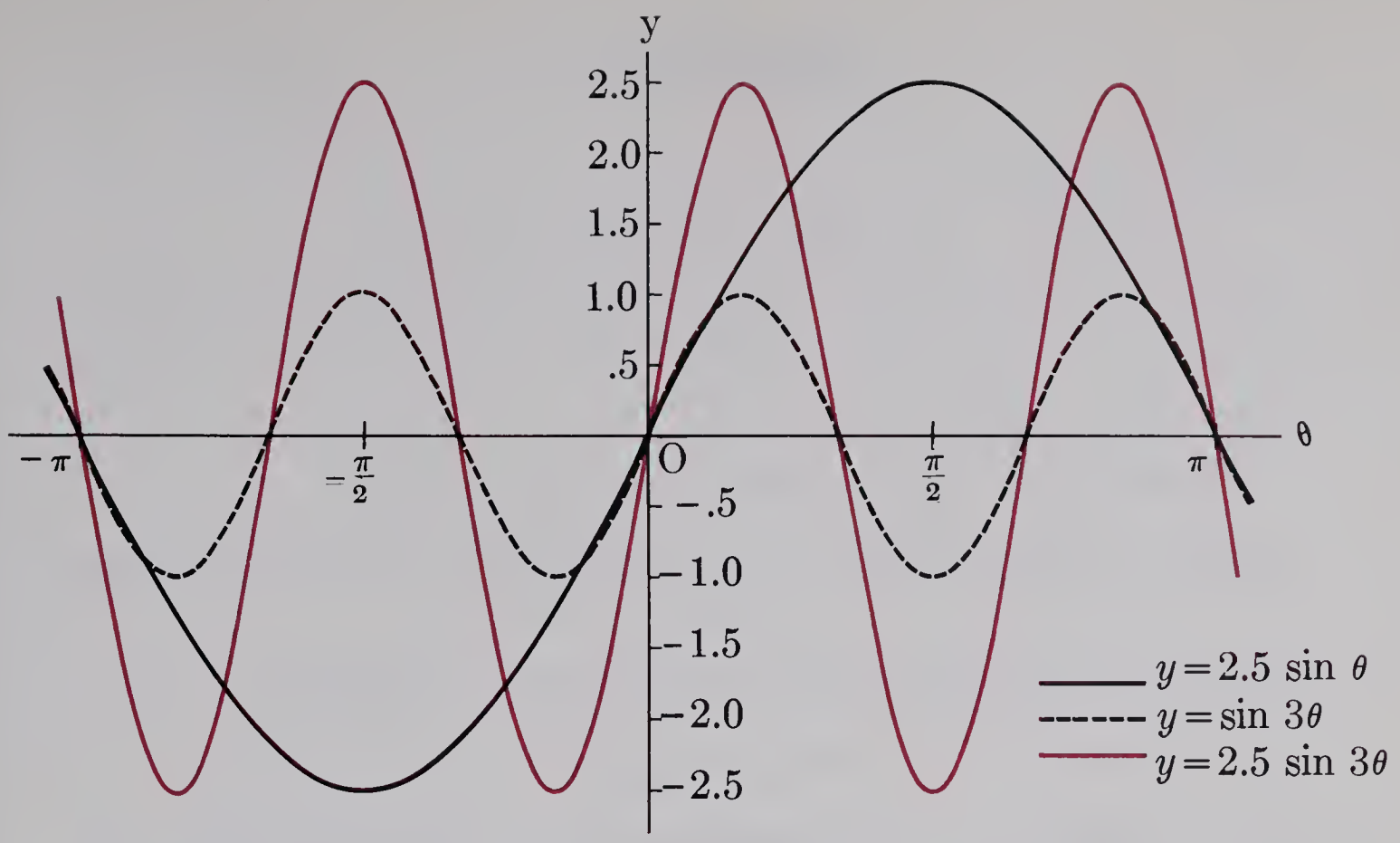


Figure 12-9

Several facts can be noted from either tables of values or the graphs:

(i) $2.5 \sin 3\theta$ and $2.5 \sin \theta$ have the same amplitude.

(ii) The values of θ for which $2.5 \sin 3\theta$ and $\sin 3\theta$ have maxima or minima are the same.

(iii) The values of θ for which $2.5 \sin 3\theta$ and $\sin 3\theta$ are zero are the same; that is, the zeros of the functions coincide.

(iv) The periods of $2.5 \sin 3\theta$ and $\sin 3\theta$ are equal.

Thus we see that the change in periodicity in θ between $2.5 \sin 3\theta$ and $2.5 \sin \theta$ does not affect the amplitude, and that the change in amplitude between $2.5 \sin 3\theta$ and $\sin 3\theta$ does not affect the periodicity in θ. The amplitude and period in θ are two *independent* properties.

The function defined by $y = 2.5 \sin 3\theta$ has been used as an illustration, but the conclusions hold for the general function defined by $y = a \sin k\,\theta$.

If $y = a \sin k\theta$, then $\frac{y}{a} = \sin k\theta$, if $a \neq 0$, and so when $y = 0$, $\sin k\theta = 0$.

This equation determines the values of θ for which the function is zero; they are $\{ \ldots, -\frac{\pi}{k}, 0, \frac{\pi}{k}, \ldots \}$.

The same equation, $\sin k\theta = 0$, determines the zeros of the function $y = \sin k\theta$, and so the function $\sin k\theta$ has the same zeros as $a \sin k\theta$. Hence the period in θ of the two functions is the same, $\frac{2\pi}{k}$, and it is independent of the amplitude.

In summary, therefore, $y = a \sin k\theta$ defines a function of θ for which a is *the amplitude* and $\frac{2\pi}{k}$ is the period in θ.

EXERCISE 12—9

1. Sketch the graphs of

$y=\sin 2\theta$, $y=3\sin 2\theta$, and $y=3\sin\theta$

relative to the same axes. State the amplitude and period of each.

2. Sketch the graphs of

$y=\cos 4\theta$, $y=2\cos\theta$, and $y=2\cos 4\theta$

relative to the same axes. State the amplitude and period of each.

3. State the amplitude and period of each of the following:

(a) $3.5\sin 2\theta$ (b) $\sqrt{2}\sin 3\theta$ (c) $\frac{1}{4}\cos 7\theta$

(d) $\frac{\sqrt{3}}{2}\cos\frac{\theta}{4}$ (e) $10\sin\pi\theta$ (f) $.2\sin\frac{3\theta}{4}$

(g) $\sqrt{5}\cos 5\pi\theta$ (h) $4\sin\frac{\pi\theta}{5}$ (i) $5\cos\sqrt{3}\theta$

(j) $\frac{3}{2}\sin\frac{\pi\theta}{\sqrt{5}}$ (k) $3.2\cos\frac{\theta}{12}$ (l) $\frac{1}{\sqrt{3}}\sin\frac{\theta}{\sqrt{3}}$

(m) $4\cos\pi\theta$ (n) $2\pi\sin 4\theta$ (o) $.\dot{1}\dot{6}\sin\sqrt{3}\theta$

(p) $5\times 10^5\cos\theta$ (q) $.1\sin 377.4\theta$ (r) $10^{-6}\sin 2\pi\theta$

12 10 The Phase of $a\sin(\theta+d)$

If we draw the graph of $y=2\sin(\theta+\frac{\pi}{3})$, we can compare it with the graph of $y=2\sin\theta$ which we studied in Section 12-7. Figure 12-10 shows the two graphs drawn to the same axes for comparison.

Table of values of $y=2\sin(\theta+\frac{\pi}{3})$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\theta+\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{7\pi}{3}$
$\sin\left(\theta+\frac{\pi}{3}\right)$	.8660	1.0	.8660	.5000	0	−.5000	−.8660	−1.0	−.8660	−.5000	0	.5000	.8660
$2\sin\left(\theta+\frac{\pi}{3}\right)$	1.732	2.0	1.732	1.0	0	−1.0	−1.732	−2.0	−1.732	−1.0	0	1.0	1.732

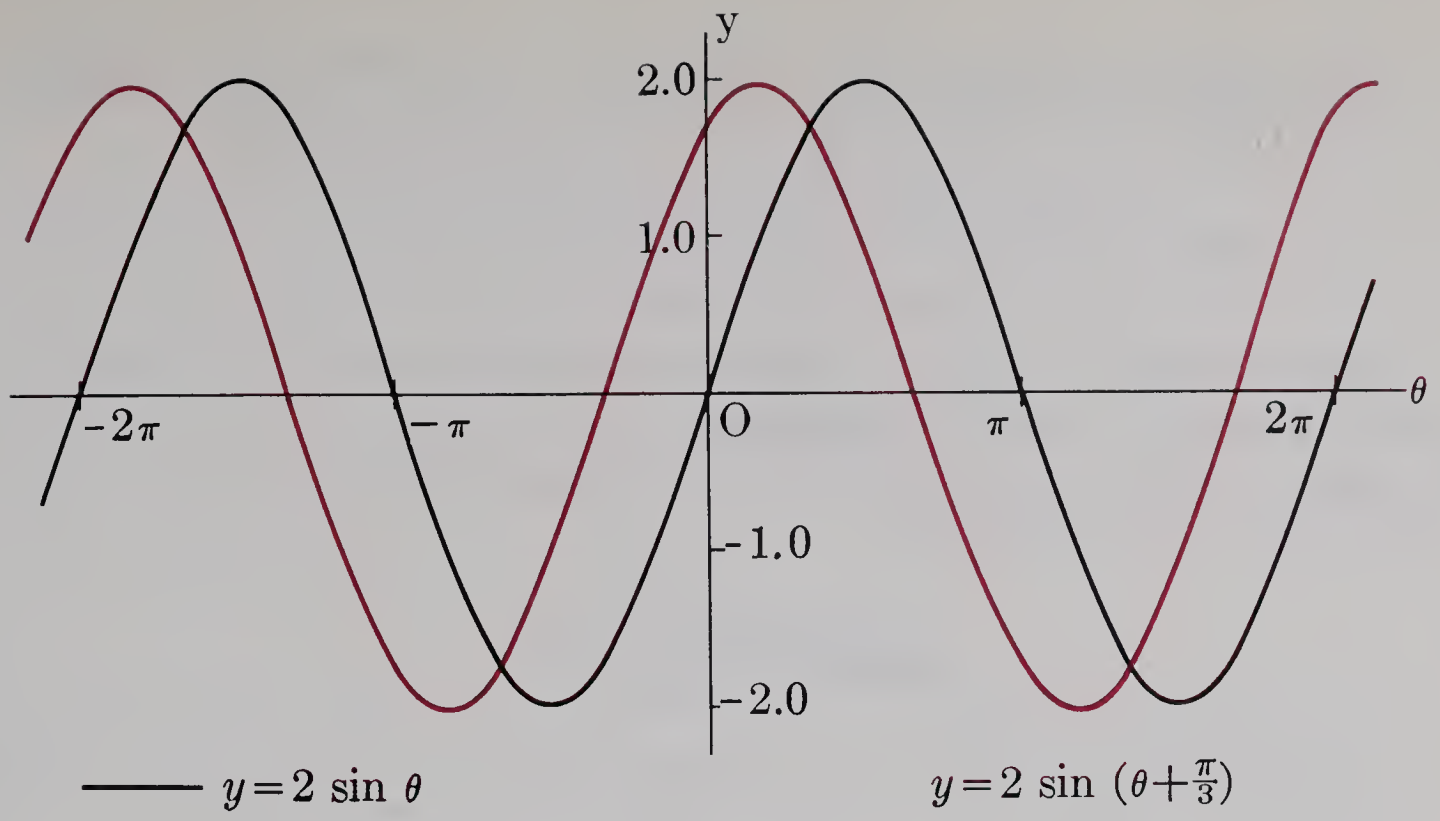


Figure 12-10

We notice immediately that the relationship between these two curves resembles that between $y = \sin\theta$ and $y = \cos\theta$, which we discussed in Section 12-6. The two curves have the same shape and size but are moved or shifted relative to each other in the direction of the θ-axis.

DEFINITIONS

If $y = a \sin(\theta + d)$, then the constant angle d is called the phase angle of the function with respect to $y = a \sin\theta$.

The relative shift of the curve $y = a \sin(\theta + d)$ with respect to $y = a \sin\theta$ is called a phase shift.

Generally the range of d is restricted so that $-\pi \leq d \leq \pi$.

(If $y = a \cos(\theta + d)$, then d is the phase angle of the function with respect to $y = a \cos\theta$.)

If we examine the two graphs shown we see that
the zeros of $y = a \sin\theta$ are $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$, and
the zeros of $y = a \sin(\theta + \frac{\pi}{3})$ are $\{\ldots, -\frac{7\pi}{3}, -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} \ldots\}$.
Thus the phase angle, $\frac{\pi}{3}$, determines the displacement of the zeros of $y = a \sin(\theta + \frac{\pi}{3})$ with respect to those of $y = a \sin\theta$. This displacement is most easily calculated by finding the value of θ for which the argument is zero; this zero is at $\theta + \frac{\pi}{3} = 0$, that is, $\theta = -\frac{\pi}{3}$, and the phase shift is $\frac{\pi}{3}$ to the left.

In the general case of $y = a \sin(\theta + d)$, the phase shift is given by $\theta = -d$, which is the condition for the argument of the sine function to be zero.

If d is positive (in our example $d = \frac{\pi}{3}$), the function $y = a \sin(\theta + d)$ is said to *lead* the function $y = a \sin\theta$; the corresponding zeros of $y = a \sin(\theta + d)$ occur *before or to the left* of the zeros of $y = a \sin\theta$.

If d is negative then $y=a\sin(\theta+d)$ is said to *lag* the function $y=a\sin\theta$. Note that in this terminology

$$y=\cos\theta=\sin\left(\theta+\tfrac{\pi}{2}\right) \text{ leads } y=\sin\theta \text{ by } \tfrac{\pi}{2},$$

and

$$y=\sin\theta=\cos\left(\theta-\tfrac{\pi}{2}\right) \text{ lags } y=\cos\theta \text{ by } \tfrac{\pi}{2}.$$

Note also that a lead by π has the same effect as a lag by π; $\sin\theta$ and $\sin(\theta\pm\pi)$ are said to be in antiphase.

The phase angle is a property of the trigonometric functions which is independent of the amplitude.

EXERCISE 12—10

1. Draw the graph of *one* of the functions defined as follows and compare it with the graph of $y=a\sin\theta$, where a is the same amplitude in each case as that of the given function. Sketch the other graphs.

   (a) $y=3\sin\left(\theta+\frac{\pi}{6}\right)$ (b) $y=\frac{1}{2}\sin\left(\theta-\frac{3\pi}{4}\right)$

   (c) $y=\sin(\theta+25°)$ (d) $y=2\sin(\theta-120°)$

   State the amplitude and phase angle with respect to $y=\sin\theta$ in each case.

2. Sketch the graphs of the functions defined as follows and compare them with the graph of $y=a\cos\theta$, where a is the same amplitude in each case as that of the given function:

   (a) $y=1.2\cos\left(\theta+\frac{\pi}{4}\right)$ (b) $y=.8\cos\left(\theta-\frac{2\pi}{3}\right)$

   (c) $y=2\cos(\theta-75°)$ (d) $y=\cos(\theta+150°)$

   State the amplitude and phase angle with respect to $y=\cos\theta$ in each case.

3. Give the phase shift relative to $\sin\theta$ for each of the following:

   (a) $\sqrt{2}\sin\left(\theta+\frac{\pi}{3}\right)$ (b) $2\sin\left(\theta-\frac{3\pi}{4}\right)$ (c) $4\sin(\theta-144°)$

   (d) $.3\sin\left(\theta-\frac{\pi}{\sqrt{2}}\right)$ (e) $2.5\sin\left(\theta+\sqrt{2}\right)$ (f) $\frac{1}{3}\sin\left(\theta+\frac{\pi}{4}\right)$

4. At what values of θ in the interval $-4\pi<\theta<4\pi$ or $-720°<\theta<720°$, are the following equal to zero?

   (a) $2\sin\left(\theta+\frac{2\pi}{3}\right)$ (b) $\cos\left(\theta-\frac{\pi}{4}\right)$ (c) $3\cos\left(\theta+\frac{\pi}{6}\right)$

   (d) $.5\sin(\theta-63°)$ (e) $1.2\cos(\theta+40°)$ (f) $\sqrt{2}\sin(\theta-60°)$

5. For what values of θ in the domain $0<\theta<6\pi$ or $0<\theta<1080°$ do the following attain their maximum values?

   (a) $\sin(\theta+45°)$ (b) $2\cos\left(\theta-\frac{\pi}{3}\right)$ (c) $\frac{1}{2}\cos\left(\theta+\frac{3\pi}{4}\right)$

   (d) $\frac{2}{5}\sin\left(\theta+\frac{5\pi}{6}\right)$ (e) $\cos(\theta-125°)$ (f) $2\sin\left(\theta-\frac{2\pi}{3}\right)$

   For what values of θ do the minima occur?

12 11 The Functions $a \sin(k\theta + d)$ and $a \cos(k\theta + d)$ (Supplementary)

We can now study the more general function defined by

$$y = a \sin (k\theta + d),$$

where a is a positive real number,

k is a positive real number, and

d is the phase angle, a real number such that $-\pi \leq d \leq \pi$.

The graph of a particular case,

$$y = 2.5 \sin (3\theta + \tfrac{\pi}{4}),$$

is given in Figure 12-11 with the graph of $y = 2.5 \sin 3\theta$ for comparison.

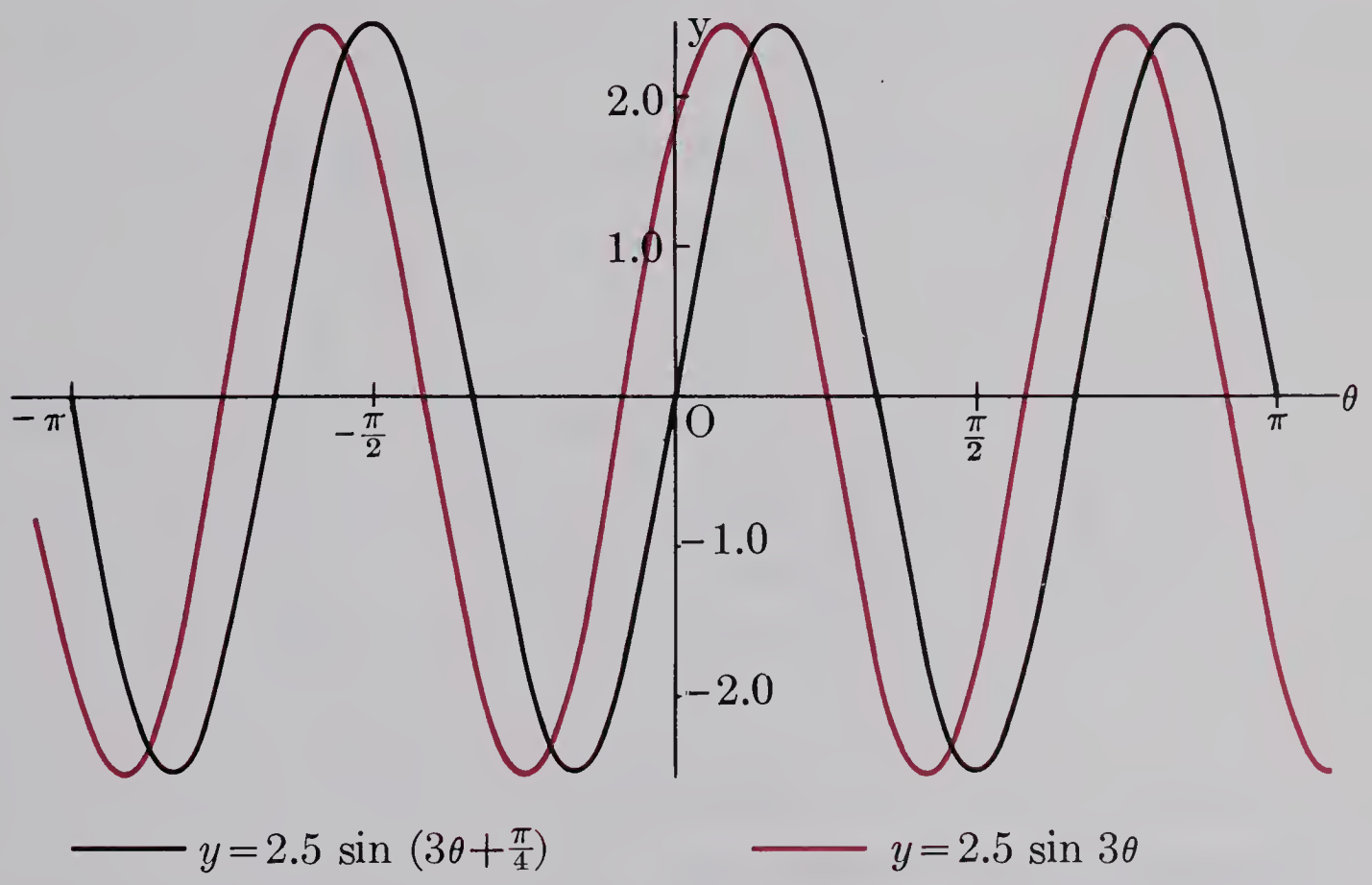


Figure 12-11

Both functions have the same amplitude and the same period in θ, but they are different in phase.

The phase angle of $y = 2.5 \sin (3\theta + \frac{\pi}{4})$ is $\frac{\pi}{4}$. Note, however, that this is not now the shift of the graph to the left from the graph of $y = 2.5 \sin 3\theta$; the left shift is given by $3\theta + \frac{\pi}{4} = 0$, that is, $\theta = -\frac{\pi}{12}$.

The graphs of three other similar functions determined by

(a) $y = 1.5 \sin (3\theta + \frac{\pi}{4})$ (b) $y = 2.5 \sin (2\theta + \frac{\pi}{4})$ (c) $y = 2.5 \sin (3\theta + \frac{2\pi}{3})$

are given in Figure 12-12 for comparison.

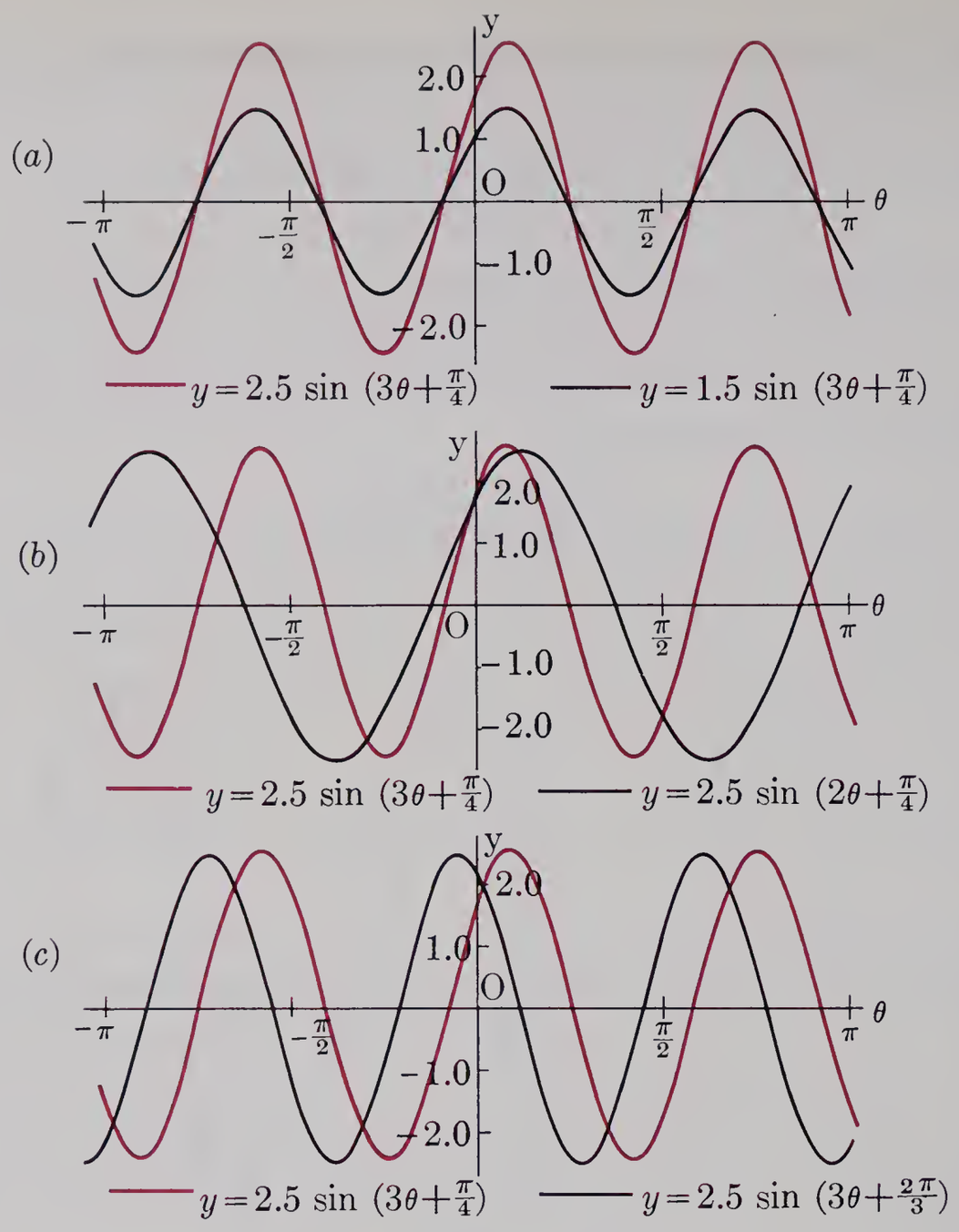


Figure 12-12

From these graphs it follows that

(i) a change in the amplitude leaves the period in θ and phase angle unchanged;

(ii) a change in the period in θ leaves the amplitude and phase angle unchanged;

(iii) a change in the phase angle leaves the amplitude and period in θ unchanged.

The amplitude, period in θ, and phase angle are thus independent properties of the functions; however, the phase shift in θ is dependent on both the phase angle and the period in θ, but it is independent of the amplitude.

EXERCISE 12—11 (Supplementary)

1. Sketch the graphs of the functions defined by

   (a) $y=.5 \sin \theta$ (b) $y=.5 \sin (\theta+\frac{\pi}{4})$

   (c) $y=\sin (\theta+\frac{\pi}{4})$ (d) $y=.5 \sin 3\theta$

   (e) $y=\sin 3 (\theta+\frac{\pi}{4})$ (f) $y=.5 \sin 3(\theta+\frac{\pi}{4})$

   State the amplitude, periodicity, and phase angle and shift of each relative to $\sin \theta$.

2. Sketch the graphs of the functions defined by

   (a) $y=\cos \theta$ (b) $y=2 \cos (\theta-\frac{2\pi}{3})$

   (c) $y=2 \cos \theta$ (d) $y=2 \cos (\theta-\frac{2\pi}{3})$

   (e) $y=\cos 4\theta$ (f) $y=2 \cos 4(\theta-\frac{2\pi}{3})$.

   State the amplitude, periodicity, and phase angle and shift of each relative to $\cos \theta$.

3. What is the phase angle and shift of $y=-2 \sin \theta$ relative to $y=\sin \theta$?

4. What is the phase angle and shift of $y=\sin (3\theta+\pi)$ relative to $y=\sin \theta$?

12 12 Chapter Summary

The sine and cosine functions of θ exist in the domain of the set of real numbers, Re, and $-1 \leq \sin \theta \leq 1$, $-1 \leq \cos \theta \leq 1$, respectively.

$\sin (-\theta)=-\sin \theta$, $\cos (-\theta)=\cos \theta$, $\tan (-\theta)=-\tan \theta$.

The functions $\sin \theta$ and $\cos \theta$ are periodic with period 2π.

The zeros of $\sin \theta$ are $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}=\{n\pi, n \in \mathrm{I}\}$.

The zeros of $\cos \theta$ are $\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}=\{(n+\frac{1}{2})\pi, n \in \mathrm{I}\}$.

$\sin \theta=1$ when $\theta=\{\ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\}=\{(2n+\frac{1}{2})\pi, n \in \mathrm{I}\}$.

$\sin \theta=-1$ when $\theta=\{\ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}=\{(2n-\frac{1}{2})\pi, n \in \mathrm{I}\}$.

$\cos \theta=1$ when $\theta=\{\ldots, -2\pi, 0, 2\pi, \ldots\}=\{2n\pi, n \in \mathrm{I}\}$.

$\cos \theta=-1$ when $\theta=\{\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots\}=\{(2n+1)\pi, n \in \mathrm{I}\}$.

For the function determined by $y=a \sin (\theta+d)$,
a is the amplitude, 2π is the period in θ, and d is the phase angle.

For the function determined by $y=a \sin k\theta$,
a is the amplitude, and $\frac{2\pi}{k}$ is the period in θ.

REVIEW EXERCISE 12—12

1. Without using the tables find the value of

(a) $\sin 240°$ (b) $\sin(-45°)$ (c) $\sin(-840°)$

(d) $\cos 315°$ (e) $\cos(-135°)$ (f) $\cos 750°$

(g) $\sin -\frac{\pi}{6}$ (h) $\sin \frac{15\pi}{4}$ (i) $\sin -\frac{3\pi}{2}$

(j) $\cos \frac{7\pi}{2}$ (k) $\cos -\frac{5\pi}{6}$ (l) $\cos -\frac{13\pi}{4}$.

2. State the amplitude of the following:

(a) $15 \sin x$ (b) $\frac{1}{3} \sin 3x$ (c) $4.5 \sin 2\theta$

(d) $.25 \sin(\theta+\frac{\pi}{3})$ (e) $4 \sin(3\theta+\frac{\pi}{2})$ (f) $1.25 \sin 5(\theta+\frac{\pi}{4})$

Graph the functions in (b) and (d).

3. State the amplitude and phase angle relative to $\sin\theta$ of the following:

(a) $\sin(\theta+\frac{\pi}{3})$ (b) $3 \sin(\theta-\frac{\pi}{2})$ (c) $2.5 \sin(\theta+50°)$

(d) $.375 \sin(\theta-35°)$ (e) $7.5 \sin(\theta+\frac{3\pi}{4})$ (f) $4 \sin(\theta-135°)$

4. State the amplitude and period of the following:

(a) $\sin 3\theta$ (b) $2 \sin(.5\theta)$ (c) $3 \sin 2\theta$

(d) $.25 \sin(.75\theta)$ (e) $\sin(\frac{1}{12}\theta)$ (f) $7.5 \sin(6.25\theta)$

Graph the functions (b) and (c).

5. State the amplitude and phase angle relative to $\cos\theta$ of the following:

(a) $2 \cos(\theta-\frac{\pi}{3})$ (b) $1.25 \cos(\theta+30°)$ (c) $3 \cos(\theta+\frac{5\pi}{6})$

(d) $\cos(\theta-120°)$ (e) $3 \sin\theta$ (f) $1.5 \sin(\theta-\frac{\pi}{4})$

6. State the amplitude, period, and phase angle and shift relative to $\sin\theta$ of

(a) $2.5 \sin 3(\theta+\frac{\pi}{4})$ (b) $\frac{1}{5} \sin(2\theta-\frac{\pi}{3})$ (c) $3 \cos 2\theta$

(d) $3.5 \sin\frac{1}{2}(\theta-25°)$ (e) $4 \sin(3\theta+45°)$ (f) $2 \cos 3(\theta-60°)$.

7. Show that (a) $\sin\theta=0$ for the set $\{\theta|\theta=n\pi,\ n\in I\}$,

(b) $\sin\theta=1$ for the set $\{\theta|\theta=(2n+\frac{1}{2})\pi,\ n\in I\}$,

(c) $\sin\theta=-1$ for the set $\{\theta|\theta=(2n-\frac{1}{2})\pi,\ n\in I\}$.

8. Show that (a) $\cos\theta=1$ for $\{\theta|\theta=2n\pi,\ n\in I\}$,

(b) $\cos\theta=-1$ for $\{\theta|\theta=(2n+1)\pi,\ n\in I\}$,

(c) $\cos\theta=0$ for $\{\theta|\theta=(n+\frac{1}{2})\pi,\ n\in I\}$.

Chapter 13

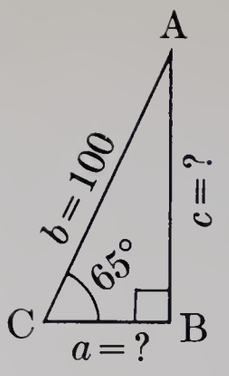
The Solution of Triangles

13 1 The Solution of the Right-Angled Triangle

To solve a triangle means to find all the parts of the triangle. In dealing with such problems, a diagram representing all the given facts should accompany the solution. When the triangle is named ABC, it is convenient to denote the lengths of the sides opposite $\angle A$, $\angle B$, $\angle C$, by a, b, and c, respectively; we shall adopt this convention.

In all our numerical work we shall assume that the angles are measured only to the nearest degree, that is, for acute angles to two significant figures at most. The tables of sines, cosines, and tangents on page 392 are constructed for angles measured in an integral number of degrees, and the values of the functions are given to four significant figures. However we cannot assume that any computed results will be more accurate than the given angle measurements. Therefore in all cases we shall not expect our results to be more accurate than two significant figures, and in some cases the results will not be as accurate as that; however we will always state our results with two significant figures. It should be remembered that, while some values of the trigonometric function are rational numbers, for example $\sin 30° = .5000$, most of the values are irrational numbers and so the tables only give a four-figure approximation, for example $\sin 65° \simeq .9063$.

Example 1



Given: A triangle ABC, having AC = 100, $\angle ACB = 65°$, $\angle ABC = 90°$.

Required: (i) length of AB
(ii) length of BC
(iii) $\angle BAC$.

Solution: (i) To find the length of AB:

$$\frac{AB}{100} = \sin 65°$$

$$\therefore AB = 100 \sin 65°$$

$$\simeq 100 \times .9063.$$

$$\therefore AB \simeq 91 \text{ units.}$$

(ii) To find the length of BC:

$$\frac{BC}{100} = \cos 65°$$

$$\therefore BC = 100 \cos 65°.$$

$$\therefore BC \simeq 100 \times .4226.$$

$$\therefore BC \simeq 42 \text{ units.}$$

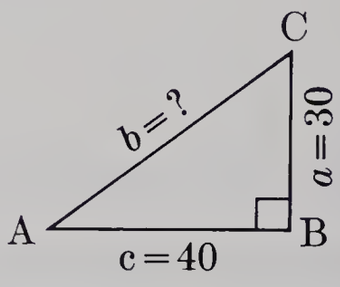
(iii) To find $\angle BAC$:

$$\angle BAC + \angle ABC + \angle ACB = 180°$$

But $\angle ACB + \angle ABC = 65° + 90°$.

$$\therefore \angle BAC = 180° - 155° = 25°.$$

Example 2



Given: $\triangle ABC$ with AB = 40, BC = 30, $\angle ABC = 90°$.

Required: To find (i) length of AC
(ii) $\angle BAC$
(iii) $\angle ACB$.

Solution: (i) To find the length of AC:

$$AC^2 = AB^2 + BC^2 = 40^2 + 30^2 = 1600 + 900 = 2500$$

$$\therefore AC = 50$$

(ii) To find $\angle BAC$:

$$\tan A = \frac{BC}{AB}$$

$$\therefore \tan A = \frac{30}{40}$$

$$= 0.75.$$

$$\therefore \angle BAC \simeq 37°.$$

(iii) To find $\angle ACB$:

$$\tan C = \frac{AB}{BC}$$

$$\therefore \tan C = \frac{40}{30}$$

$$\simeq 1.3333.$$

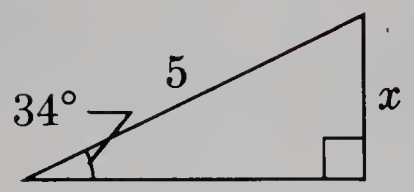
$$\therefore \angle ACB \simeq 53°.$$

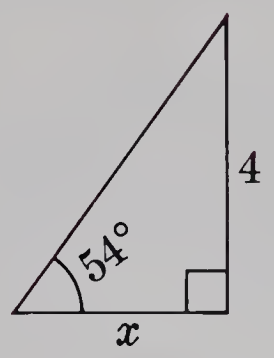
You are advised to derive your results directly from the given data. We could obtain $\angle BAC$ from the fact that $\angle BAC = 90° - \angle ACB = 37°$. However, if our first computation of $\angle ACB$ were incorrect, then the value of $\angle BAC$ would also be incorrect. Using the methods of solution suggested, we can use $\angle BAC + \angle ACB = 90°$ as a check.

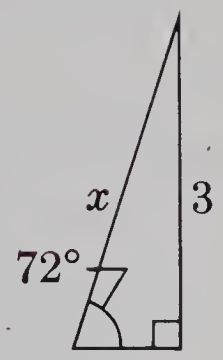
EXERCISE 13—1

All answers should be given to two significant figures.

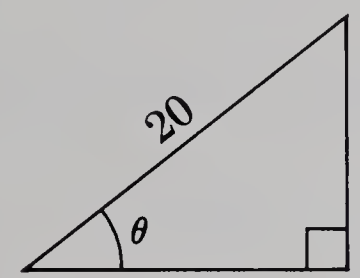
In each of the following find the measure of the unknown side:



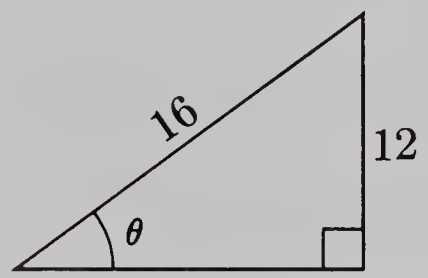




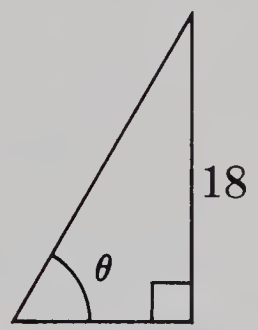
Find the measure of angle θ (in degrees) in each of the following right-angled triangles:



16



9



7. ABC is a right-angled triangle having $\angle B = 90°$, $a = 200$, $b = 250$. Find the remaining parts of the triangle. $\angle A = 53°$ $\angle C = 37°$ $c = 150$
8. In $\triangle ABC$, $\angle C = 90°$, $a = 36$, $b = 80$. Find $\angle A$, $\angle B$, and c. $\angle A = 24°$ $\angle B = 66$ $c = 88$
9. In $\triangle ABC$, $\angle A = 90°$, $a = 8$ inches, $\angle B = 55°$. Find the lengths of the remaining sides and the third angle.
10. In $\triangle ABC$, $\angle B = 90°$, $a = 10$ inches, $\angle C = 67°$. Find b, c, and $\angle A$.
11. The base of an isosceles triangle is 30 inches and the altitude corresponding to this base is 24 inches. Solve the triangle.
12. The sides of an equilateral triangle are 3 inches long. Find the length of a median.
13. In $\triangle ABC$, AB = 4.5 cm., AC = 6.5 cm., and the altitude from A to BC is 3.5 cm. Find the length of BC and all the angles of $\triangle ABC$ (2 cases).
14. In $\triangle ABC$, AB = 13 inches, BC = 17 inches, and $\angle ABC = 67°$. AD is the altitude from A to BC. Find the lengths of AD and AC, $\angle ACB$ and $\angle BAC$.

13.2 The Law of Sines

In the last section we reviewed the solution of right-angled triangles. We now wish to find a method of solving any triangle if we are given three of its parts.

Three cases arise in which the triangle is uniquely determined by given data:

(1) one side and the two angles,
(2) two sides and the included angle,
(3) three sides, provided that the sum of the lengths of any two sides is greater than the length of the remaining side.

A fourth case arises when two sides and the nonincluded angle are given, but in this case the triangle is not always uniquely determined.

To solve the first case listed above, we use the law of sines. This is a set of proportions between the values of the sine functions of the angles and the lengths of the sides of a triangle.

Law of Sines

The sides of a triangle are proportional to the sines of the angles opposite these sides.

We present two methods of proof.

First Method

Given: An acute-angled $\triangle ABC$ whose sides are a, b, and c units in length.

Required: To prove $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

Proof: Consider $\triangle ABC$ and choose its vertices so that A is $(0, p)$, B is $(-q, 0)$, C is $(r, 0)$.

Then $AO \perp BC$ at O.

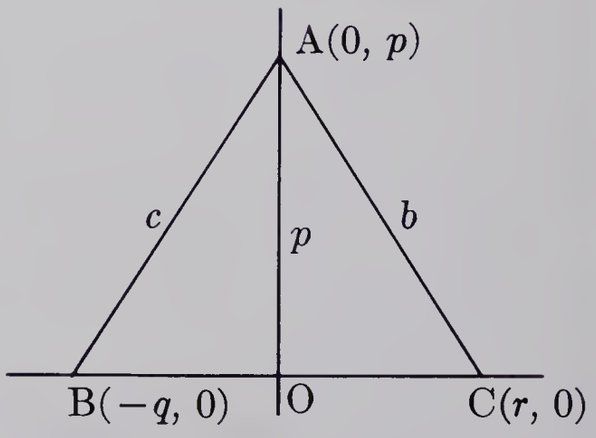
$AO = p = b \sin C$

and $AO = p = c \sin B$

$\therefore\ b \sin C = c \sin B$

and $\sin B \neq 0$, $\sin C \neq 0$.

$$\frac{b}{\sin B}=\frac{c}{\sin C}.$$



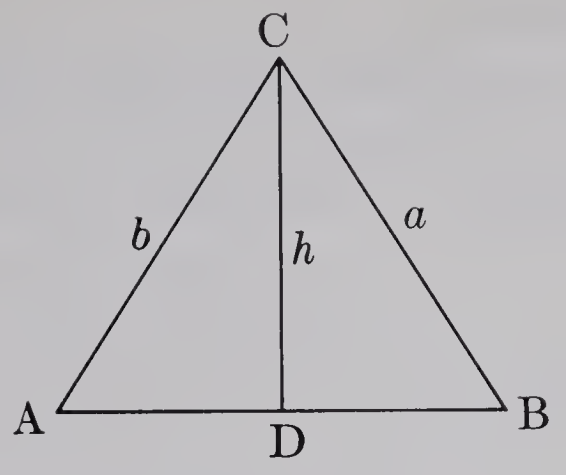
Similarly we can show that

$$\frac{a}{\sin A}=\frac{b}{\sin B}.$$

$$\therefore\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{and also} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

Second Method

Given: Let ABC be an acute-angled triangle whose sides are a, b, and c units in length.



Required: To prove

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Proof: Draw a line segment $CD \perp AB$ and let its length be represented by h units.

In right-angled triangle BDC,

$$\sin B=\frac{CD}{BC}=\frac{h}{a}$$

$$\therefore\ a \sin B=h.$$

In right-angled triangle ADC,

$$\sin A=\frac{CD}{AC}=\frac{h}{b}$$

$$\therefore\ b \sin A=h.$$

$$\therefore\ a \sin B=b \sin A.$$

Now $\sin A \neq 0$ and $\sin B \neq 0$

$$\therefore\quad \frac{a}{\sin A}=\frac{b}{\sin B}.$$

Similarly, by drawing a perpendicular from A to the opposite side it may be proved that

$$\frac{b}{\sin B}=\frac{c}{\sin C}.$$

$$\therefore\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{and also} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

To use the law of sines to solve a triangle in which we are given two angles and a side, we find the third angle from $\angle A+\angle B+\angle C=180°$. Then rearranging the above results so that an unknown side is in the numerator of the first ratio, we solve for this side.

Example 1

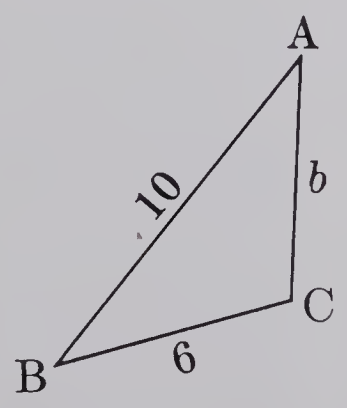
Given $\triangle ABC$ having $a=6$, $c=10$, determine the ratio of sin A to sin C.

Solution: $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$

By the property of alternation of a proportion

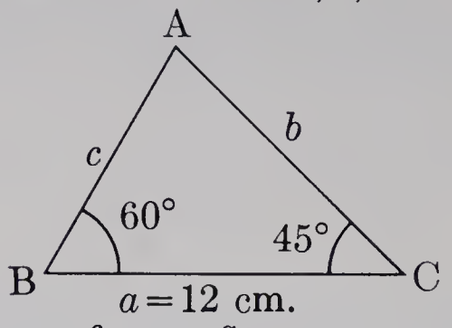
$$\frac{\sin A}{\sin C}=\frac{a}{c}=\frac{6}{10}=\frac{3}{5}$$

$\therefore$ sin A : sin C = 3 : 5.



Example 2

In $\triangle ABC$, $\angle B = 60°$, $\angle C = 45°$, $a = 12$ cm. Find b, c, and $\angle A$.



Solution: $\angle A + \angle B + \angle C = 180°$

$\angle A + 60° + 45° = 180°$

$\angle A = 180° - 105° = 75°$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$b = \frac{a \sin B}{\sin A}$$

$$= \frac{12 \sin 60°}{\sin 75°}$$

$$\simeq \frac{12 \times 0.8660}{0.9659}$$

$\therefore b \simeq 11$ cm.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

$$= \frac{12 \sin 45°}{\sin 75°}$$

$$\simeq \frac{12 \times 0.7071}{0.9659}$$

$\therefore c \simeq 8.8$ cm.

In $\triangle ABC$, $\angle A = 75°$, $b = 11$ cm., $c = 8.8$ cm.

EXERCISE 13—2

1. Examine the proofs given in this section and then prove the law of sines for a triangle with one obtuse angle. Does it make any difference to the method?
2. In $\triangle ABC$ find the values of a and b, given $\angle A = 30°$, $\angle C = 80°$, $c = 100$ cm.
3. In $\triangle ABC$ find the values of a and c, given $\angle B = 60°$, $\angle A = 40°$, $b = 1000$ cm.
4. Given $\angle A = 45°$, $\angle B = 60°$, and $a = 100$, find c in $\triangle ABC$.
5. In $\triangle ABC$ find the ratio of b to c, given $\angle B = 60°$, $\angle C = 45°$.
6. Find a, c, and $\angle A$ in $\triangle ABC$, given $b = 3$, $\angle C = 65°$, and $\angle B = 45°$.
7. In each of the following, find the measures of the remaining parts of the triangle:
   (a) $\triangle ABC$ with $\angle A = 40°$, $\angle C = 75°$, $b = 10$ cm.
   (b) $\triangle PQR$ with $\angle P = 84°$, $q = 7.5$ inches, $\angle R = 56°$
   (c) $\triangle XYZ$ with $\angle Y = 28°$, $x = 3.6$ inches, $\angle Z = 77°$
   (d) $\triangle RST$ with $\angle R = 46°$, $r = .57$ cms., $\angle T = 68°$
   (e) $\triangle UVW$ with $\angle W = 37°$, $u = 28$ cms., $\angle V = 110°$

13 3 The Law of Cosines

To solve the other two cases when the triangle is not always uniquely determined, we need a formula involving three sides of a triangle and one angle; the result required is known as the Law of Cosines.

Law of Cosines

In a triangle, the square on any side is equal to the sum of the squares on the other two sides minus twice the product of those two sides multiplied by the cosine of the contained angle.

In symbols: In any $\triangle ABC$, $a^2=b^2+c^2-2bc\cos A$.

Again we present two proofs.

First Method

Given: Let $\triangle ABC$ be an acute-angled triangle whose sides are a, b, and c units in length (Figure 13-1).

Required: To prove

$$a^2=b^2+c^2-2bc\cos A,$$
$$b^2=c^2+a^2-2ca\cos B,$$
and $$c^2=a^2+b^2-2ab\cos C.$$

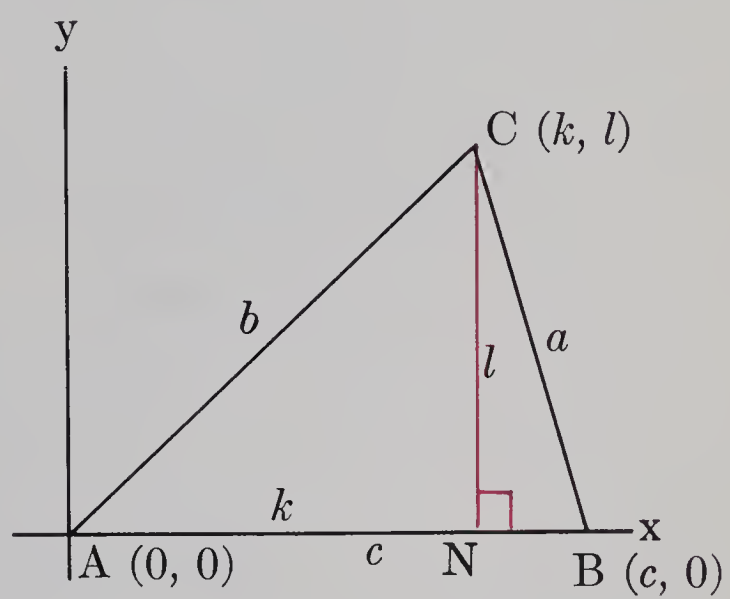


Figure 13-1

Proof: Consider any triangle ABC and choose its vertices so that A is (0, 0), B is $(c, 0)$, C is (k, l), and $BC=a$, $CA=b$, $AB=c$.

$$a^2=BN^2+CN^2 \qquad \text{(Pythagoras)}$$
$$=(c-k)^2+l^2$$
$$=c^2-2ck+k^2+l^2$$
$$=c^2+k^2+l^2-2ck$$
also $b^2=k^2+l^2$ (Pythagoras)
and $k=b\cos A$. (defn.)
$\therefore\ a^2=c^2+b^2-2cb\cos A$ or $a^2=b^2+c^2-2bc\cos A$.

Similarly we can prove

$$b^2=c^2+a^2-2ca\cos B \text{ and}$$
$$c^2=a^2+b^2-2ab\cos C.$$

Second Method

Given: An acute-angled $\triangle ABC$ whose sides are a, b, c units in length (Figure 13-2).

Required: To prove

$$a^2=b^2+c^2-2bc\cos A,$$
$$b^2=c^2+a^2-2ac\cos B,$$
and $$c^2=a^2+b^2-2ab\cos C.$$

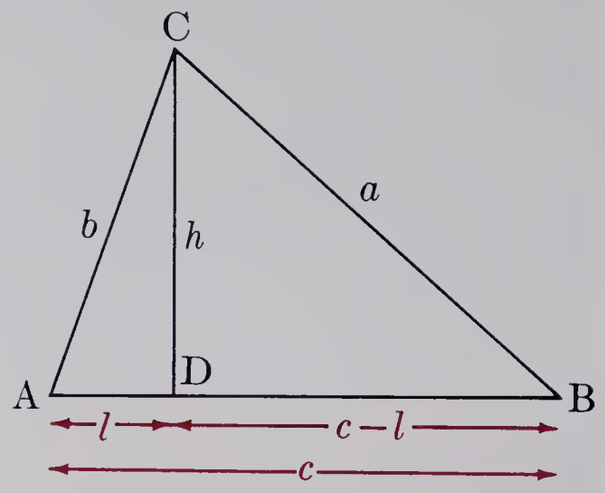


Figure 13-2

Proof: Draw line segment $CD \perp AB$ and let its length be h units.
Let the length of the line segment AD be l units.
$\therefore$ Length of $BD = c-l$.
We use the Theorem of Pythagoras.

In right-angled triangle BDC,

$$h^2+(c-l)^2=a^2$$
$$\therefore\ h^2=a^2-(c-l)^2.$$

In right-angled triangle ADC,

$$h^2+l^2=b^2$$
$$h^2=b^2-l^2$$

$$\therefore\ a^2-(c-l)^2=b^2-l^2.$$
$$a^2-c^2+2cl-l^2=b^2-l^2$$
$$a^2=b^2+c^2-2cl$$

But in $\triangle ADC$, $l=b\cos A$.

$$\therefore\ a^2=b^2+c^2-2bc\cos A.$$

Similarly, we can prove

$$b^2=a^2+c^2-2ac\cos B$$
and $$c^2=a^2+b^2-2ab\cos C.$$

From the above set of formulae we may readily deduce the following forms by algebraic rearrangement of the equations:

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \qquad \cos B=\frac{c^2+a^2-b^2}{2ca} \qquad \cos C=\frac{a^2+b^2-c^2}{2ab}$$

The proof for the case of an obtuse-angled triangle is required as an exercise.

Note: If $\triangle ABC$ is such that $\angle ABC = 90° = 1$ rt. $\angle$, then $\cos B = \cos 90° = 0$ and so $b^2=a^2+c^2$. Thus the cosine law formulae are seen to be generalizations of the Theorem of Pythagoras.

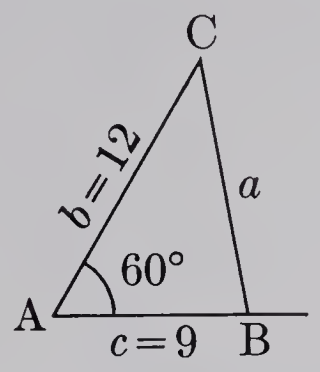
We will first consider an example in which two sides and the included angle are given.

Example 1

Find a, $\angle B$, and $\angle C$ in $\triangle ABC$, which has $\angle A = 60°$, $b = 12$, $c = 9$.

Solution: By the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos 60°$$
$$a^2 = 12^2 + 9^2 - 2(12 \times 9) \times \tfrac{1}{2}$$
$$= 144 + 81 - 108$$
$$= 225 - 108$$
$$a^2 = 117$$
$$\therefore\ a = \sqrt{117} \simeq 10.8.$$



$\therefore$ the length of BC is 11. (to two significant figures)

The angles B and C can be found by again using the cosine law or by using the sine law; the latter is simpler. (See *Notes* (a), (b).)

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin B = \frac{12 \times \sin 60°}{10.8} \qquad \sin C = \frac{9 \times \sin 60°}{10.8}$$

$$\simeq \frac{.8660}{.9} \qquad \simeq \frac{.8660}{1.2}$$

$$\simeq .9622. \qquad \simeq .7217.$$

$$\angle B = 74° \text{ or } 106° \qquad \angle C = 46° \text{ or } 134°$$

$C = 134°$ is impossible. (See *Note* (c).)

We test the other two possibilities as follows:

$$\angle A + \angle B + \angle C = 60° + 74° + 46° = 180°$$
$$\angle A + \angle B + \angle C = 60° + 106° + 46° \neq 180°$$

$\therefore$ In $\triangle ABC$, $a = 11$, $\angle B = 74°$, $\angle C = 46°$.

Notes:

(a) The general rule is to use only the data to calculate a required value, if this is possible. The reason for this rule is to avoid the accumulation of errors due to round-off (or chop-off); see the note for Example 2 on the next page.

(b) If there is a choice of formulae which may be used, then the simplest is used; this usually occurs when a previously calculated value has to be used in the next calculation.

(c) When considering the test to decide which pair of values of the two angles is correct, the large obtuse angle is always impossible, and so need not be tested formally, if it is greater than the supplement of the given angle.

Now we consider an example in which all three sides are given.

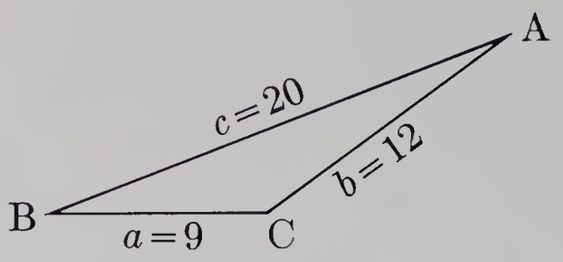
Example 2

Given $\triangle ABC$ with sides $a=9$, $b=12$, $c=20$. Find all the angles.

Solution: $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

$= \dfrac{20^2+9^2-12^2}{2\times 20\times 9}$

$= \dfrac{481-144}{360}$



$\cos B \simeq 0.9361$

$\therefore \angle B \simeq 21°$.

The use of the sine law to calculate $\angle A$ and $\angle C$ would require the calculated value of $\angle B$. Therefore in accordance with the general rule (*Note* (a) previous page), we use the cosine law (see also *Note* below).

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$= \dfrac{12^2+20^2-9^2}{2\times 12\times 20}$

$= \dfrac{544-81}{480}$

$= \dfrac{463}{480}$

$\simeq .9646$

$\therefore \angle A \simeq 15°$.

$\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$= \dfrac{9^2+12^2-20^2}{2\times 9\times 12}$

$= \dfrac{225-400}{216}$

$= \dfrac{-175}{216}$

$\simeq -.8102$

$\therefore \angle C \simeq 180° - 36° = 144°$.

Remember $\cos (180° - A) = -\cos A$.

Check: $\angle A + \angle B + \angle C = 15° + 21° + 144° = 180°$

In $\triangle ABC$, $\angle A = 15°$, $\angle C = 144°$, $\angle B = 21°$.

Note: In this example we use the cosine law three times though this is longer than using the calculated value of $\angle B$ and the sine law to find $\angle A$ and $\angle C$. However, in general, the values obtained in the latter way for $\angle A$ and $\angle C$ are not as accurate as those obtained above. In fact, in the example given, the use of the sine law gives $\angle A = 16°$ and $\angle C = 143°$, which are not correct to the nearest degree.

What happens to the value of cos A if $a > b+c$? Why is it impossible in this case for any triangle to be formed?

EXERCISE 13—3

All answers should be given to two significant figures.

1. Given $\triangle ABC$ having the following parts (draw a sketch to represent the triangle in each case):
   (i) $\angle C = 30°, b = 20, a = 22$, find c (ii) $\angle B = 45°, a = 30, c = 50$, find b
   (iii) $\angle A = 60°, b = 20, c = 30$, find a (iv) $\angle B = 90°, a = 12, c = 5$, find b

2. Solve ABC given that
   (a) $c = 50, b = 50, \angle A = 63°$ (b) $a = 15, c = 11, \angle B = 40°$
   (c) $b = 10, a = 4, \angle C = 32°$ (d) $b = 120, c = 100, \angle A = 56°$
   (e) $c = 8, a = 5, \angle B = 43°$ (f) $b = 6, a = 4, \angle C = 37°$.

3. (i) If $a = 3, b = 4, c = 6$, find the cosine of the largest angle.
   (ii) If $a = 4, b = 5, c = 6$, find the cosine of the smallest angle.
   (iii) If $a = \sqrt{5}, b = 2\sqrt{2}, c = 4$, find the cosine of the smallest angle.

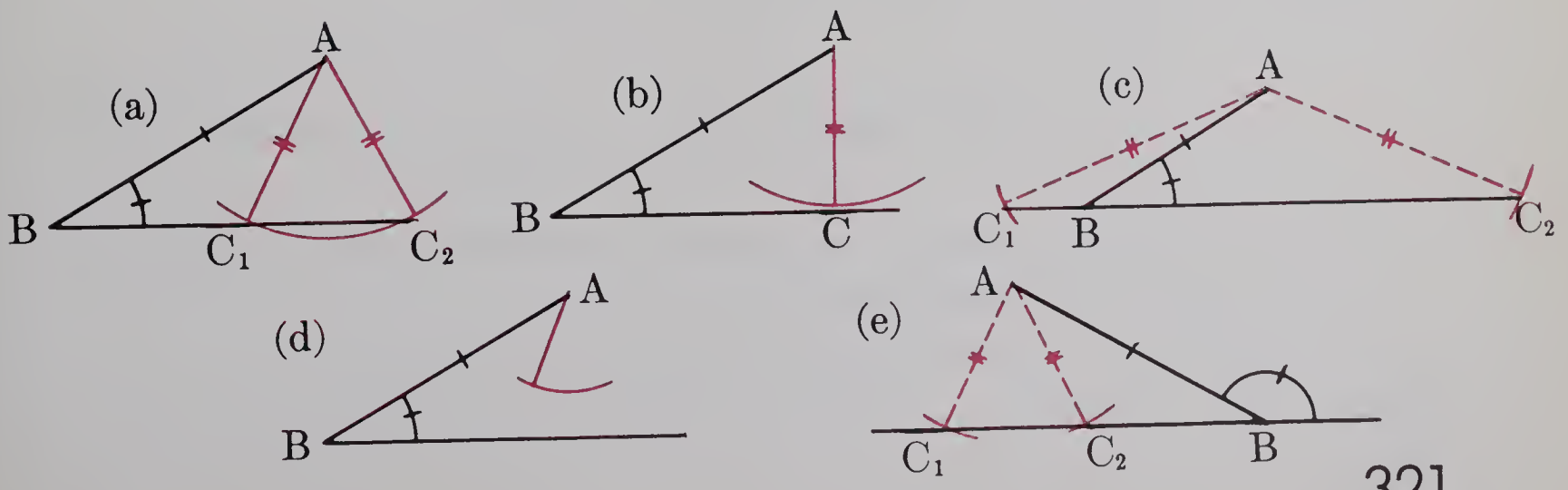
4. Solve $\triangle ABC$ (if possible) given
   (i) $a = 20, b = 30, c = 40$ (ii) $a = 15, b = 20, c = 25$
   (iii) $a = 10, b = 15, c = 20$ (iv) $a = 7, b = 6, c = 15$.

5. Find the remaining parts of the triangles in (1).

6. Prove the law of cosines, $a^2 = b^2 + c^2 - 2bc \cos A$, when $\angle A$ is obtuse.

13 4 The Ambiguous Case

When we are given two sides and a nonincluded angle from which to construct a triangle, we know that we can usually construct two triangles (diagram a). In the case when one of the other angles is a right angle (diagram b) and in the case when the side opposite the given angle is greater than the adjacent side (diagram c), we can construct a unique triangle. In other cases we cannot construct any triangle (diagrams d and e).



To solve the problem by trigonometry, we use the sine law. The ambiguous case (a) arises when we are given a, c, and $\angle A$ from which we can calculate $\sin C$ and $0 < \sin C < 1$. $\angle C$ can be either acute or obtuse because in both cases $\sin C$ is positive. If $\sin C = 1$, then $\angle C = 90°$ and we have the unique right-angled triangle (b). If $\sin C > 1$, we have the case (d) when no triangle can be constructed because $\sin C$ must be less than 1 for all real angles C. In cases (c) and (e) we may use a diagram to show the true situation; we can find values for C in these cases but they are not always valid values.

Example 1

Given $\triangle ABC$, with $a=2$, $b=3$, $\angle A=37°$, find c, B, and C.

Solution:

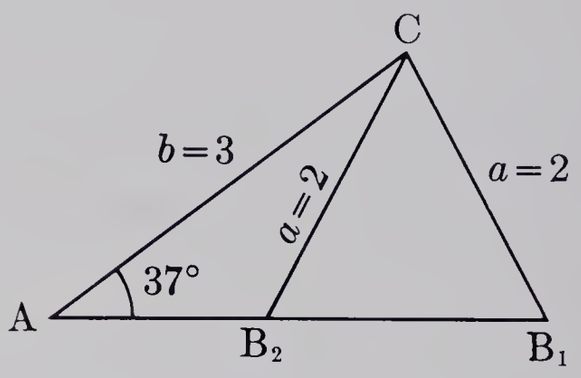
$$\frac{\sin B}{b}=\frac{\sin A}{a}$$

$$\therefore \sin B=\frac{3\times\sin 37°}{2}$$

$$\simeq\frac{3\times.6018}{2}$$

$$\simeq.9027.$$

$$\therefore \angle B\simeq 65° \text{ or } 115°.$$



We write $AB_1=c_1$, $AB_2=c_2$, and
$\angle AB_1C=\angle B_1$, $\angle AB_2C=\angle B_2$, $\angle ACB_1=\angle C_1$, and $\angle ACB_2=\angle C_2$.

If $\angle B_1=65°$,
$\angle C_1=180°-(37°+65°)$
$=180°-102°=78°$.

$$\frac{c_1}{\sin C_1}=\frac{a}{\sin A}$$

$$\therefore c_1\simeq\frac{2\times.9781}{.6018}\simeq 3.3.$$

If $\angle B_2=115°$,
$\angle C_2=180°-(37°+115°)$
$=180°-152°=28°$.

$$\frac{c_2}{\sin C_2}=\frac{a}{\sin A}$$

$$\therefore c_2\simeq\frac{2\times.4695}{.6018}\simeq 1.6.$$

Check: Use $c^2=a^2+b^2-2ab\cos C$.

$a^2+b^2-2ab\cos C_1$
$\simeq 2^2+3^2-2\times2\times3\times.2079$
$\therefore c_1^2\simeq 4+9-2.4948\simeq 10.51.$
$\therefore c_1\simeq 3.2.$

$a^2+b^2-2ab\cos C_2$
$\simeq 2^2+3^2-2\times2\times3\times.8829$
$\therefore c_2^2\simeq 4+9-10.5948\simeq 2.405.$
$\therefore c_2\simeq 1.5.$

Note that the check is not precise in this case but it is within the accuracy that may be expected.

In $\triangle ABC$,

either $c_1=3.2$, $\angle C_1=78°$, $\angle B_1=65°$, *or* $c_2=1.5$, $\angle C_2=28°$, $\angle B_2=115°$.

Example 2

Given $\triangle ABC$, $b=2$, $c=3$, $\angle B=60°$, find a, A, and C.

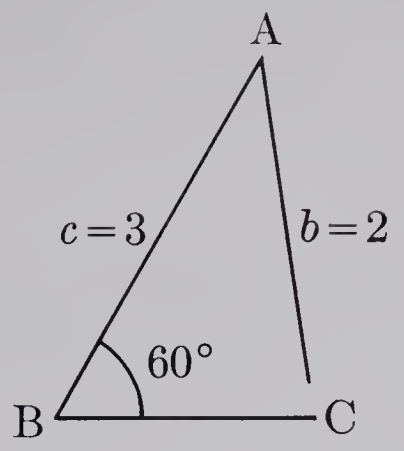
Solution: $\dfrac{\sin C}{c}=\dfrac{\sin B}{b}$

$$\sin C \simeq \frac{3\times .8660}{2}$$

$$\simeq 1.2990$$

$\therefore$ no real $\angle C$ exists.

$\therefore$ no real $\triangle ABC$ exists.



EXERCISE 13—4

All answers should be given to two significant figures.

Find the remaining parts of the following triangles if possible:

1. $a=8$, $b=9$, $A=61°$
2. $a=6$, $b=7$, $A=53°$
3. $a=18$, $b=35$, $A=23°$
4. $a=40$, $b=64$, $A=41°$
5. $a=30$, $b=24$, $A=75°$
6. $a=30$, $b=18$, $A=123°$
7. $b=5$, $c=6$, $B=97°$
8. $c=50$, $a=80$, $C=36°$
9. $b=30$, $c=75$, $C=17°$
10. $b=95$, $c=96$, $B=114°$
11. $b=76$, $c=58$, $C=49°$
12. $a=20$, $c=25$, $C=58°$

13 5 Applications of Trigonometry

The solution of acute- and obtuse-angled triangles can be applied to many problems. Remember that there are four cases:

(1) one side and two angles:
use the angle sum to find the other angle;
use the sine law to find the other two sides;
use the cosine law with the calculated angle for a check.

(2) three sides:
use the cosine law for all three angles;
use the angle sum as a check.

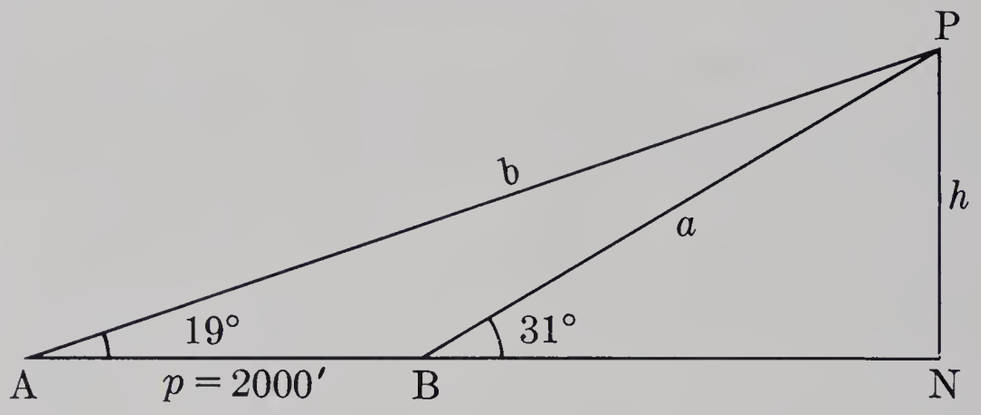
(3) two sides and the included angle:
use the cosine law for the other side;
use the sine law for the other angles;
use the angle sum as a check.

(4) two sides and a nonincluded angle:
use the sine law for a second angle:
use the angle sum for the third angle;
use the sine law for the third side;
use the cosine law with the third angle as a check.

Example 1

The angle of elevation of the top of a mountain peak P, as observed from a point A on a level plain below, is 19°. The angle of elevation from a second point B, 2000 feet closer to the base of the mountain and in the same vertical plane as A and P, is 31°. Find the height of the peak.

Solution:



In the diagram which illustrates the problem, PN, the perpendicular from P to the horizontal ABN, represents the height of the mountain.

Let the height of the mountain be h feet.

Then $h = a \sin 31° = b \sin 19°$.

In $\triangle ABP$, $p = 2000$, $\angle PAB = 19°$.

$$\angle APB = \angle NBP - \angle BAP = 31° - 19° = 12°$$

and $$\frac{a}{\sin \angle PAB} = \frac{p}{\sin \angle APB} = \frac{b}{\sin \angle PBA}.$$

$$\therefore a = \frac{2000 \sin 19°}{\sin 12°} \simeq \frac{2000 \times .3256}{.2079} \simeq 3{,}132.$$

$$\therefore h \simeq 3{,}132 \times .5150 \simeq 1{,}600.$$

or

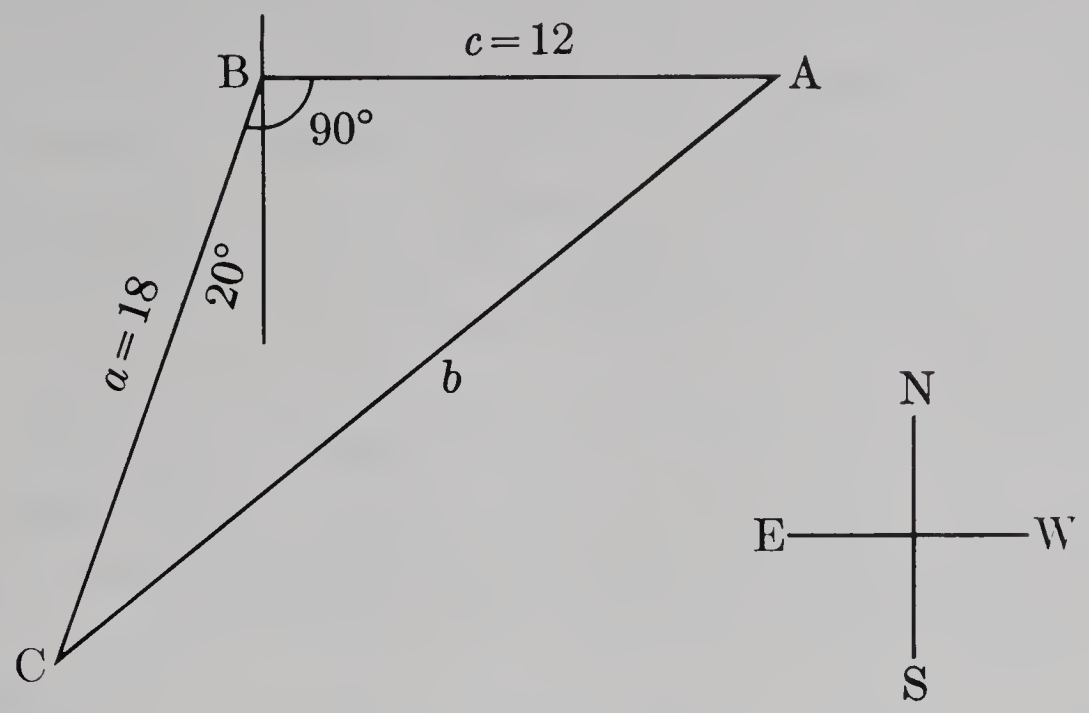
$$b = \frac{2000 \sin 31°}{\sin 12°} \simeq \frac{2000 \times .5150}{.2079} \simeq 4{,}954.$$

$$h \simeq 4{,}954 \times .3256 \simeq 1{,}600.$$

The height of the mountain is 1,600 feet.

Example 2

Athens is 12 miles due east of Brantford, and Carthage is 18 miles S 20°W of Brantford. How far is Carthage from Athens and in what direction?



Solution: Let the distance from Carthage to Athens $= b$ miles.
The direction from Athens to Carthage is determined by $\angle BAC$.

In $\triangle ABC$

$$c = 12,\ a = 18,\ \angle B = 90° + 20° = 110°$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$\simeq 18^2 + 12^2 - 2 \times 18 \times 12(-.3420)$$
$$\simeq 324 + 144 + 147.8$$
$$= 615.8$$
$$b \simeq 25$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin A = \frac{a \sin B}{b}$$
$$= \frac{18 \sin 110°}{25}$$
$$\simeq \frac{18 \times .9397}{25}$$
$$\simeq .6766$$
$$A \simeq 39°$$

Direction AC is 39° S of W, or equivalently S 51°W.
Carthage is 25 miles S 51°W of Athens.

EXERCISE 13—5

All answers should be given to two significant figures.

1. The angle of elevation of the top of a mountain peak was observed to be 22° from a point A and 29° from a point B, 1600 feet nearer the base of the mountain. If the points A, B, and the mountain peak are all in the same vertical plane, what is the height of the mountain top?
2. A and B, two observation stations, are situated 30 miles apart on a straight north-south coastline. The bearings of an enemy destroyer are S 35°W from A and N 65°W from B. How far is the destroyer from A and B?
3. Two lighthouses, X and Y, are located 20 miles apart, X being due north of Y. An observer at X spots a ship in distress and determines its bearing to be S 65°W; from Y the bearing of the same ship is N 42°W. Find the distances of the ship from X and Y.
4. The distance AB across a lake is to be found. A line segment AC, 500 yards in length, is marked off and angles BAC and ACB are measured and found to be 79° and 67° respectively. Find the distance AB.
5. A ship sails due west from a harbour for 40 nautical miles. It then sails in the direction S 63°W for 22 nautical miles. How far is it then from the harbour?
6. An endless piece of rope 52 feet long is stretched taut to form a triangle with two sides 14 feet and 20 feet in length. Find the angle of the triangle between these two sides.
7. A tree has its base at a certain point X between two observers stationed at two points, A and B, in the same plane as X. They find the measures of angles XAB and XBA to be 70° and 55° respectively. If the distance from A to B is 1500 feet, how far is X from A and B?
8. From the top of a cliff 360 feet high, the angle of depression of the top of a tower on level horizontal ground below is 54°. From the foot of the cliff the angle of elevation of the top of the tower is 38°. Find the height of the tower.
9. The distances between three cities A, B, and C are as follows: distance from A to B is 140 miles, distance from B to C is 180 miles, and distance from C to A is 220 miles. If A lies due north of B, and C is in an easterly direction from A, find the bearing of C from B.
10. A large triangular field has sides of 300 yards, 550 yards, and 700 yards respectively. Find the measure of the smallest angle contained by the sides.

11. Three towns A, B, and C are situated at the vertices of a triangle and joined by straight roads. The internal angle at B is 67°, and the distance from A to B is 26 miles and from C to B is 12 miles. Find the distance from A to C.

12. A hillside slopes at 18° to the horizontal and the line of greatest upward slope is due north. At local noon a tree which grows vertically casts a shadow 75 feet long. The angle of elevation of the sun at noon is 65°. How tall is the tree? (Remember that the angle of elevation is measured from the horizontal.)

13. A golf shot is 18° off the direct line from the tee to the hole. The shot is found to be 240 yards and the ball lies 80 yards from the hole after the shot. What is the distance from the tee to the hole if it is more than 240 yards?

14. The diagonals of a parallelogram are 6 inches and 9 inches long and they intersect at an angle of 65°. Find the lengths of the sides of the parallelogram.

13 6 Chapter Summary

The Law of Sines: The sides of a triangle are proportional to the sines of the angles opposite these sides.

In $\triangle ABC$: $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$, $\qquad \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

The Law of Cosines: In a triangle, the square on any side is equal to the sum of the squares on the other two sides minus twice the product of those two sides multiplied by the cosine of the contained angle.

In $\triangle ABC$: $a^2=b^2+c^2-2bc\cos A$, $b^2=c^2+a^2-2ca\cos B$, and $c^2=a^2+b^2-2ab\cos C$.

Solution of Triangles

(1) Given one side and two angles.
Use angle sum, sine law twice, and cosine law for check.

(2) Given three sides.
Use cosine law three times, angle sum for check.

(3) Given two sides and the included angle.
Use cosine law, sine law twice, angle sum for check.

(4) Given two sides and a nonincluded angle (ambiguous case).
Use sine law, angle sum, sine law, and cosine law for check.
(There may be two, one, or no solutions in this case.)

REVIEW EXERCISE 13—6

Solve completely the triangle ABC (if possible), given

1.	$b=7$	$C=115°$	$B=25°$	2.	$a=36$	$b=18$	$A=133°$
3.	$a=32$	$b=12$	$C=118°$	4.	$a=12$	$b=22$	$c=18$
5.	$A=19°$	$B=32°$	$c=60$	6.	$a=7$	$b=18$	$B=123°$
7.	$B=72°$	$C=45°$	$a=10$	8.	$a=8$	$b=3$	$c=9$
9.	$B=26°$	$A=123°$	$c=14$	10.	$b=12$	$c=3$	$A=36°$
11.	$a=18$	$b=12$	$A=43°$	12.	$a=3\sqrt{5}$	$b=2\sqrt{7}$	$c=5\sqrt{2}$
13.	$a=2$	$b=4$	$c=7$	14.	$A=55°$	$b=5$	$c=3.5$

15. Toronto is 220 miles S 35°W from Ottawa. Montreal is 100 miles S 80°E from Ottawa. How far is Toronto from Montreal?

16. Two forest ranger stations 10 miles apart in an E-W line report a fire at S 25°W and S 36°E respectively. How far is the fire from each station?

17. A man observes the angles of elevation of a TV mast on the top of a building to be 65° to the top of the mast and 54° to the bottom of the mast. He knows the height of the mast is 50 ft. How far is he from the building and how high is the building?

18. Two men set out walking from the same point. One goes due east at 5 m.p.h., the other southwest at 6 m.p.h. At what rate are they separating?

19. An airplane heads due west with an airspeed of 375 m.p.h. A wind blows at 45 m.p.h. in a direction N 35°E. What is the plane's ground speed and direction of travel?

20. An airplane has an airspeed of 200 m.p.h. and a north wind at 45 m.p.h. slows its ground speed to 170 m.p.h. In what direction does the plane travel? Is there a unique result?

Chapter 14

Sequences and Series

14 1 Definition of a Sequence

You will recall that a function was defined as a relation in which, to each member of the domain, there corresponds a unique member of the range. In this chapter we shall study certain *ordered* sets determined by functions. These sets are called sequences.

DEFINITION: A function f whose domain is a subset of the set of integers defines an ordered set called a *sequence*.

Note that if the domain is a finite subset of the integers, the sequence is described as *finite*. If the domain is I^+ (or other infinite subset of the integers), then the sequence is *infinite*. In the following examples, the domain is I^+, unless otherwise specified.

Example 1

List the members of the ordered set of values $\{f(n)\}$ determined by

$$f(x) = 2.5 - x.$$

Solution: $f(1) = 2.5 - 1 = 1.5$
$f(2) = 2.5 - 2 = .5$
$f(3) = 2.5 - 3 = -.5$ and so on.

The ordered set is $\{1.5, .5, -.5, \ldots\}$

This set is called a *sequence* and 1.5, .5, $-.5$, . . . are the *terms* of this sequence. It is important to realize that although we enclose the terms in braces as above, there is an important difference from our previous use of this notation; in the case of a sequence, the set is *ordered*. The order of the terms in the sequence must correspond to the order inherent in the domain of the integers. It should also be noted that a sequence may have two or more equal terms since they are the values of a function over a subset of the integers. In working with sets which are not ordered, we list only the unequal elements.

In Example 1 the terms of the sequence were rational numbers. The following example offers a sequence whose terms are irrational.

Example 2

List the terms in the sequence defined by $g(x) = \log x$.

Solution: The ordered set of values determined by $g(x) = \log x$ is
$\{\log 1, \log 2, \log 3, \ldots, \log n, \ldots\}$.

Log n is the *nth term* or *general term* of this sequence.

Note: log 1, the first term, is often written as t_1.
log n, the nth term, appears as t_n.
log $(n+1)$, the $(n+1)$st term or t_{n+1}, is, of course, just as "general" as the nth term (or, for that matter, the $(n-1)$st term).

EXERCISE 14—1

1. For the sequence defined by $f = \{(x, y) | y = x+1; x \in \mathrm{I}\}$, state
   (i) $f(2)$ (ii) $f(20)$ (iii) $f(n)$
   (iv) $f(n+1)$ (v) $f(2k+3)$ (vi) $f(5r-1)$.

2. For the sequence whose nth term is $3n$, state
   (i) the first term (ii) the tenth term (iii) the $(n+1)$st term.
   (iv) t_4 (v) t_{100} (vi) t_k

3. Given the sequence $\{2, 4, 6, \ldots, 2n \ldots\}$, state
   (i) the 25th term (ii) the $(2n)$th term (iii) the $(2k+5)$th term.
   (iv) t_6 (v) t_k (vi) t_{2k+5}

4. For the sequence defined by $g = \{(x, y) | y = (-1)^x; x \in \mathrm{I}^+\}$, state
   (i) $g(1)$ (ii) $g(2)$ (iii) $g(5)$ (iv) $g(12)$
   (v) $g(n)$ (vi) $g(2n)$ (vii) $g(99)$ (viii) $g(2n+1)$
   What is the range of the function defined by g?
   What is the sequence defined by g?

5. For the sequence defined by $f = \{(x, y) | y = x^2; x \in \mathrm{I}^+\}$, find
   (i) $f(2)$ (ii) $f(50)$ (iii) $f(k)$
   (iv) $f(k+5)$ (v) $f(17)$ (vi) $f(k-1)$

6. List the first three terms of the sequence defined by
$$g = \left\{(x, y) | y = \frac{x+2}{x}; x \in \mathrm{I}^+\right\}.$$

7. List the first three terms of the sequence whose nth term is

(a) $\frac{n(n-1)}{2}$ (b) $5n^2-2n$ (c) $(-1)^n$

(d) 2^{n+3} (e) $(n+1)^2$ (f) $\cos n$

8. Use function notation as in (6) to define sequences with the following terms:

(a) $2, 4, 6, \ldots, 2n, \ldots$

(b) $5, 25, 125, \ldots, 5^n, \ldots$

(c) $\sqrt{3}, \sqrt{21}, 7\sqrt{3}, \ldots, \sqrt{3}(\sqrt{7})^{n-1}, \ldots$

9. Write the first four terms of the sequence whose nth term is given by

$$t_n=2n-1+(n-1)(n-2)(n-3).$$

10. In the sequence for which $t_n=\frac{n+2}{5n}$,

(a) find the value of n for which $t_n=0.4$;

(b) show that t_n cannot be more than $\frac{3}{5}$ for any value of n.

11. A certain radioactive substance has a half-life of approximately 50 hours. (One half of the original material is still present after 50 hours.) If the original amount were 1000 grams, how many grams would remain after 100 hours? 150 hours? $50n$ hours?

12. A ball is dropped from a height of 12 feet. If it bounces back $\frac{1}{3}$ of the original height, how high does it rebound on the first bounce? the second bounce? the nth bounce?

13. The number of bacteria in a laboratory culture doubles every hour. If the original number of bacteria were 100, what would be the number after 1 hour? after 10 hours? after n hours?

14. Another way to define a sequence is by a formula which allows us to list the terms after a given start. For example, we may have a function whose values are given by

$$f(1)=1,\ \ f(2)=1,\ \ f(n)=f(n-1)+f(n-2) \text{ for } n\geq 3.$$

This rule is called a *recursion formula* and defines a famous sequence whose terms are known as the *Fibonacci* numbers*. The first few of these numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

*Consult your school library for information on Fibonacci (Leonardo of Pisa) who met this sequence in a problem dealing with rabbits.

(a) List the four Fibonacci numbers which follow 55 in the above sequence.

(b) Find the next two terms in each of the sequences defined recursively by $t_n = t_{n-1} + t_{n-2}$ for $n \geq 3$:

(i) 3, 5, 8, 13, 21

(ii) $-1, -2, -3, -5$

(iii) $1, -1, 0, -1$.

15. List the first six terms in the sequences defined by the following recursive formulae:

(a) $f(n) = f(n-1) + 1;\ f(1) = 1$

(b) $f(n) = \frac{1}{2}\left[f(n-1) + \frac{2}{f(n-1)}\right];\ f(1) = 1$

(c) $f(n) = nf(n-1);\ f(1) = 2$

16. List the first four terms in the sequences defined recursively by:

(a) $f(n) = f(n-1) + Pi;\quad f(1) = P(1+i)$

(b) $f(n) = (1+i)f(n-1);\ f(1) = P(1+i)$

17. What is the recursion formula which defines the sequence

$$2, 6, 4, 5, 4\tfrac{1}{2}, 4\tfrac{3}{4}, 4\tfrac{5}{8}, \ldots ?$$

18. Suggest a recursion formula which defines the sequence

$$1, 2, 5, 29, 866, \ldots$$

14 2 Deducing a Rule for a Sequence

Given the first three terms

$$1, 3, 5,$$

what is the fourth term of the sequence? *Assuming* that the set of odd numbers is the set involved, we write $t_4 = 7$.
However the sequence for which

$$t_n = n^3 - 6n^2 + 13n - 7,$$

also has 1, 3, 5 as its first three terms but $t_4 = 13$. (See (9) in Exercise 14-1.)

Example

Find a *possible* nth term for the sequence

$$1, 4, 9, 16, \ldots$$

Solution: These terms may be written as

$$1^2, 2^2, 3^2, 4^2, \ldots$$

and then we may write $t_n = n^2$.

Alternatively, we may think of this sequence as defined by a function f whose domain is I^+ and whose value is given by $f(x) = x^2$.

It is often helpful to compare the terms of the given sequence with the sequence of positive integers to discover a rule expressing t_n in terms of n. In order to make this comparison we may write

$$\begin{array}{ccc} 1, & 3, & 5 \\ \text{as } 2(1)-1, & 2(2)-1, & 2(3)-1. \end{array}$$

This suggests $t_n = 2n-1$ as a *possible* or *logical* nth term. It must be understood that other expressions could be given for t_n. (We saw above that $t_n = n^3-6n^2+13n-7$ also denotes a sequence for which the first three terms are 1, 3, and 5.)

EXERCISE 14—2

By comparing the given terms with the sequence of positive integers, discover a possible expression for the general term (t_n) in each of the following sequences:

1. 2, 3, 4, 5
2. 3, 5, 7, 9
3. $4, 4^2, 4^3, 4^4$
4. 1, 8, 27, 64
5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$
6. 5, 25, 125, 625
7. $\sqrt{7}, 7, 7\sqrt{7}, 49$
8. (1)(2), (2)(3), (3)(4), (4)(5)
9. $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$
10. $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}$
11. $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}$
12. $2+\frac{1}{3}, 2+\frac{1}{9}, 2+\frac{1}{27}, 2+\frac{1}{81}$
13. $4\frac{1}{2}, 4\frac{1}{4}, 4\frac{1}{8}, 4\frac{1}{16}$
14. $8, 7\frac{1}{2}, 7\frac{1}{3}, 7\frac{1}{4}$
15. $1\frac{1}{2}, 2\frac{2}{3}, 3\frac{3}{4}, 4\frac{4}{5}$
16. $2^1+1, 2^2+1, 2^3+1, 2^4+1$
17. 4, 10, 28, 82
18. $2^1-1, 2^2-1, 2^3-1, 2^4-1$
19. 2, 8, 26, 80
20. 5, 13, 35, 97
21. 1, 5, 19, 65
22. $\frac{5}{6}, \frac{13}{36}, \frac{35}{216}, \frac{97}{1296}$
23. Suggest an additional possible nth term in (1) and (2).

14 3 The Arithmetic Sequence

Consider the sequence

$$\{5, 8, 11, 14, \ldots\}.$$

Note that the difference between any pair of consecutive terms is 3. Such a sequence which has a *common difference* between successive terms is known as an *arithmetic sequence.*

Example 1

Find t_n for the *arithmetic* sequence

$$5, 8, 11, 14, \ldots$$

Solution:

$$\begin{aligned} t_1 &= 5 = 5+0(3) \\ t_2 &= 8 = 5+1(3) \\ t_3 &= 11 = 5+2(3) \\ t_4 &= 14 = 5+3(3) \\ &\vdots \qquad\quad \vdots \\ \therefore \quad t_n &= 5+(n-1)3 \qquad \text{(by analogy)} \end{aligned}$$

If the first term is a and the common difference is d, then the sequence is

$$a, a+d, a+2d, \ldots$$

and $t_n = a+(n-1)d$

is the general term of the arithmetic sequence.

Note that $t_n = a+dn-d$

$$= dn+(a-d).$$

This is a linear expression in n and any linear expression in n may represent the general term of an arithmetic sequence.

Note: We may define the arithmetic sequence by a recursion formula as follows:

$$t_n = t_{n-1}+d;\ t_1 = a.$$

Example 2

(a) Find the first three terms of the arithmetic sequence for which

$$t_n = -11n+65.$$

(b) Find the 100th term of this sequence.

(c) Is there a term equal to -11?

Solution: (a) *Method 1:* If $t_n = -11n+65$,

then $t_1 = -11(1)+65 = 54$

$t_2 = -11(2)+65 = 43$

$t_3 = -11(3)+65 = 32.$

Method 2:

$$\begin{aligned} t_n &= -11n+65 \\ &= -11(n-1)+65-11 \\ &= 54+(n-1)(-11) \end{aligned}$$

$\therefore a = 54 = t_1$ and $d = -11$.

$\therefore t_2 = a+d = 54-11 = 43$

and $t_3 = a+2d = 54-22 = 32$.

(b)

$$\begin{aligned} t_{100} &= a+99d \\ &= 54+99(-11) \\ &= 54-1089 \\ &= -1035 \end{aligned}$$

(c)

If $t_n = -11$,

$-11n+65 = -11$, $n \in I^+$

$-11n = -76$.

There is no integral solution. Therefore there is no term of the sequence equal to -11.

Example 3

Find an arithmetic sequence whose fourth term is 0, and whose eleventh term is -42.

Solution: If the sequence is $a, a+d, a+2d, \ldots,$

then $a+3d = 0$

and $a+10d = -42$.

Subtracting, $-7d = 42$.

$\therefore d = -6$.

But $a+3(-6) = 0$.

$\therefore a = 18$.

The sequence is 18, 12, 6, 0, . . .

Summary

The general arithmetic sequence may be defined as

$$\{a, a+d, a+2d, \ldots, a+(n-1)d, \ldots\}$$

or defined by

$$\{(n, y) | y = a+(n-1)d;\ n \in I^+\}$$

or by

$$t_n = f(n) = a+(n-1)d.$$

EXERCISE 14—3

1. Do the terms in each of the following sets form an arithmetic sequence? If so, what is the common difference?
   (a) $a, a+2d, a+4d$ (b) $4, 0, -4$
   (c) $1, 0, -1$ (d) $1, -1, 1$
   (e) $\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$ (f) $\frac{5}{\sqrt{2}}, \frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

2. If an arithmetic sequence is such that $a=-2$ and $d=5$, find
   (a) the 10th term (b) the nth term
   (c) the $(n+1)$st term (d) the $2n$th term.

3. If the nth term of an arithmetic sequence is given by $f(n)=2n-1$, find
   (a) $f(1)$ (b) $f(50)$
   (c) $f(2n)$ (d) $f(2n+1)-f(2n)$

4. How many multiples of 4 are there from 8 to 84 inclusive?

5. How many multiples of 3 are there from 3 to 2187 inclusive?

6. Is -16.5 a term in the arithmetic sequence
   $$5.4, 3.1, \ldots ?$$

7. If the nth term of an arithmetic sequence is $g(n)=10-\frac{5}{2}n$, find
   (a) the 14th term,
   (b) the value of the first negative term.

8. If $a-5b$, $2a+b$, and $5a+2b$ are consecutive terms of an arithmetic sequence, show that $2a-5b=0$.

9. Find the value of the middle term of the arithmetic sequence
   $$2, 7, 12, \ldots, 502.$$

10. Find the value of the 10th term from the end in the sequence of (9).

11. In an arithmetic sequence, the 10th term is $2x-3y$ and the 20th term is $12x-3y$. Find the 30th term.

12. In a sequence, $t_n=4-\frac{1}{n}$. Is the sequence arithmetic? Find t_{100}.

13. Can there be more than one form for the general term t_n of an arithmetic sequence?

14. Write the first three terms and the nth term of the arithmetic sequence defined by the recursion formula $t_n = t_{n-1} + 8$; $t_1 = -3$.

15. Write the first four terms of the sequence defined by

$$t_n = 2t_{n-1} - t_{n-2};\ t_1 = x;\ t_2 = x + y.$$

14.4 The Geometric Sequence

Consider the sequence

$$\{3, 15, 75, 375, \ldots, 3(5)^{n-1}, \ldots\}.$$

This is an example of another important type of sequence called a *geometric sequence.* The ratio of any term to the preceeding term remains constant.

Note: $\dfrac{t_2}{t_1} = \dfrac{15}{3} = \dfrac{5}{1}$

$\dfrac{t_3}{t_2} = \dfrac{75}{15} = \dfrac{5}{1}$

. .

. .

$$\frac{t_n}{t_{n-1}} = \frac{3(5)^{n-1}}{3(5)^{n-2}} = 5.$$

5 is called the *common ratio* of the sequence.

In general, if the common ratio is r, $\dfrac{t_n}{t_{n-1}} = r$,

and we obtain the recursion formula $t_n = rt_{n-1}$.

Thus each term is obtained by multiplying the preceding term by r. If the first term is a, then the second term is ar and so on. The sequence may be written as

$$\{a, ar, ar^2, \ldots, ar^{n-1}, \ldots\}.$$

It will be useful to remember $t_n = ar^{n-1}$.

If $r = 1$, the sequence is
$\{a, a, a, a, \ldots\}$.
If $r = -1$, the sequence is
$\{a, -a, a, a, \ldots\}$.
If $r = 0$, the sequence becomes
$\{a, 0, 0, 0, \ldots\}$.

These 3 values of r will be excluded from our work on geometric sequences.

Example 1

In a geometric sequence, the fifth and eighth terms are 160 and -1280.

Find (a) the third term, (b) the nth term, (c) the $2n$th term.

Solution: (a) If the sequence is $a, ar, ar^2, \ldots,$

$$t_5 = 160. \quad \therefore ar^4 = 160.$$
$$t_8 = 1280. \quad \therefore ar^7 = -1280.$$
$$\therefore \frac{ar^7}{ar^4} = r^3 = -8.$$

This equation in r has one real root, -2.

$$\therefore a(-2)^4 = 160.$$
$$16a = 160$$
$$a = 10$$
$$\therefore t_3 = ar^2$$
$$= 10(-2)^2$$
$$= 40.$$

(b)
$$t_n = ar^{n-1}$$
$$= 10(-2)^{n-1}$$

(c)
$$t_{2n} = ar^{2n-1}$$
$$= 10(-2)^{2n-1}$$
$$= -10(2)^{2n-1} \qquad \text{(Why?)}$$

Example 2

If a, x, b are consecutive terms of a geometric sequence, express x in terms of a and b.

Solution: Since a, x, b form a geometric sequence, then

$$\frac{x}{a} = \frac{b}{x}$$

or $x^2 = ab$.

$\therefore x = \pm\sqrt{ab}$.

(x is sometimes referred to as the geometric mean of a and b or the mean proportional between a and b.)

Summary

The general geometric sequence may be defined as

$$\{a, ar, ar^2, \ldots, ar^{n-1}, \ldots\}$$

or defined by

$$\{(n, y) | y = ar^{n-1}; n \in \mathrm{I}^+\}$$

or by

$$t_n = f(n) = ar^{n-1}.$$

EXERCISE 14—4

1. Do the terms in each of the following sets form a geometric sequence? If so, what is the common ratio?

   (a) $y, y+2, y+4$ (b) $4y, 2y, y$ (c) $\frac{3}{20}, \frac{3}{40}, \frac{3}{80}$

   (d) $-1, 0, 1$ (e) $-1, 1, -1$ (f) $2\sqrt{5}, 10, 10\sqrt{5}$

2. A geometric sequence has $a=20$ and $r=\frac{1}{10}$. Find (a) t_3, (b) t_n.

3. If the nth term of a geometric sequence is $t_n=9(4)^{n-1}$, find expressions for (a) t_1, (b) t_3, (c) t_{2n}, (d) $t_{n+1} \div t_n$.

4. If the numbers in each of the following sets are consecutive terms of a geometric sequence, find x and y:

   (a) $14, 4, x, y$ (b) $x, y, 8, 10$

   (c) $x, 1, \frac{5}{3}, y$ (d) $2\sqrt{5}, 10\sqrt{2}, x, y$

5. For the geometric sequence $4\sqrt{5}, 20, 20\sqrt{5}, 100, \ldots$, find t_9.

6. If $x+a$, $x^2+2ax+a^2$, and $(x-a)(x^2-a^2)$ are consecutive terms in a geometric sequence, show that either a or x must be zero.

7. In a geometric sequence, the third and sixth terms are 13.5 and 364.5. Find an expression for the general term.

8. In a geometric sequence $-\frac{3}{4}, y, -\frac{64}{27}$ are consecutive terms. Find y.

9. If the original price of an article ($\$x$) is increased by 20% and the resulting price is again increased by 20%, show that

   (a) the three prices form a geometric sequence;

   (b) the total increase is 44% of the original price.

10. In a geometric sequence, $a=1000$ and $r=1.05$. Write unsimplified expressions for the first three terms and the general term.

11. If \$1000 is lent at 5% per annum, compounded annually, what is the *amount* of the loan at the end of 1, 2, 3, and n years?

12. Find the common ratio in a geometric sequence in which the sum of the first two terms is $\frac{1}{16}$ of the sum of the next two terms.

13. For the sequence $4, 3, \ldots$, find t_4 if the sequence is

    (a) arithmetic, (b) geometric.

14. The sum of three numbers in a geometric sequence is $\frac{65}{3}$ and the sum of three numbers in an arithmetic sequence is 15. Each set of three numbers has the same middle term. Find the three numbers in the geometric sequence.

14.5 Series and Sum of a Series

DEFINITION: The indicated sum of the terms of a sequence is called a *series*.

If we indicate the sum of the terms of the arithmetic sequence

$$2, 5, 8, 11, \ldots, 62$$

by writing

$$2+5+8+11 \ldots +62,$$

the result is referred to as an arithmetic series.

Similarly $3+9+27+81$

is a geometric series. In this case the sum of the series is 120.

We shall use the symbol S_n to represent the sum of n terms of a series. In this example of a geometric series,

$$S_1=3,\ S_2=12,\ S_3=39,\ S_4=120.$$

There is no general method which will produce a formula for the sum of n terms of a series. In fact, for many series, there is no general formula.

Sometimes the sum of a series may be predicted by *inductive* methods. We examine the sum of two, three, four terms, and so on, and then attempt to *predict* the sum of n terms.

Example 1

Find a formula for the sum of n terms of the series of odd natural numbers

$$1+3+5+7+ \ldots$$

Solution:

$$S_1=1=1^2$$
$$S_2=1+3=4=2^2$$
$$S_3=1+3+5=9=3^2$$
$$S_4=1+3+5+7=16=4^2$$

Thus it seems likely that $S_n=n^2$.

Check for $n=5$:

$$S_5=1+3+5+7+9=25=5^2$$

and for $n=6$:

$$S_6=1+3+5+7+9+11=36=6^2.$$

(Note that we have not proved that $S_n=n^2$. We have predicted a formula and have checked our prediction for $n=5$ and $n=6$. A formal proof is considered in Section 12 of this chapter.)

Example 2

Use inductive reasoning to obtain a formula for the sum of the series

$$1+2+3+4+\ldots n.$$

Solution: *Method 1*:

$$S_1 = 1 = \frac{(1)(2)}{1}$$

$$S_2 = 1+2 = 3 = \frac{(2)(3)}{2}$$

$$S_3 = 1+2+3 = 6 = \frac{(3)(4)}{2}$$

$$S_4 = 1+2+3+4 = 10 = \frac{(4)(5)}{2}$$

Hence $S_n = \frac{(n)(n+1)}{2}$ would seem to be the required formula.

Check for $n=6$:

$$S_6 = 1+2+3+4+5+6 = 21 = \frac{(6)(7)}{2} = \frac{6(6+1)}{2}$$

Method 2:

$$S_2 = 1+2 = 3 = 2\left(\frac{1+2}{2}\right)$$

$$S_3 = 1+2+3 = 6 = 3\left(\frac{1+3}{2}\right)$$

$$S_4 = 1+2+3+4 = 10 = 4\left(\frac{1+4}{2}\right)$$

In each case the sum is the average of the first and last terms, multiplied by the number of terms in the sum. For S_n this "average term" is $\frac{1+n}{2}$ and the number of terms is n.

Thus $S_n = n\left(\frac{1+n}{2}\right) = n\left(\frac{n+1}{2}\right)$.

Example 3

Use the result in Example 2 to predict the sum of the first 60 terms of the arithmetic series $8+11+14+\ldots+(3n+5)+\ldots$

Solution:

$$t_{60} = 3(60)+5 = 185$$

$$\begin{aligned} S_{60} &= 8+11+14+\ldots+185 \\ &= [3(1)+5]+[3(2)+5]+[3(3)+5]+\ldots+[3(60)+5] \\ &= 3(1)+3(2)+3(3)+\ldots+3(60)+60(5) \\ &= 3(1+2+3+\ldots+60)+60(5) \\ &= \frac{3(60)(61)}{2}+300 \qquad \text{(using Example 2)} \\ &= 5{,}490+300 = 5{,}790 \end{aligned}$$

EXERCISE 14—5

Use inductive methods of reasoning to obtain the sum of the first 20 terms and also the sum of the first n terms for the following series:

1. $1+1+1+\ldots$

2. $1-1+1-1+\ldots+(-1)^{n-1}+\ldots$

3. $11+13+15+\ldots+(2n+9)+\ldots$

4. $2+4+6+8+\ldots+2n+\ldots$

5. $\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\frac{1}{(3)(4)}+\ldots+\frac{1}{(n)(n+1)}+\ldots$

6. $\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots$

7. $\frac{1}{(1)(6)}+\frac{1}{(6)(11)}+\frac{1}{(11)(16)}+\ldots+\frac{1}{(5n-4)(5n+1)}+\ldots$

8. $\frac{1}{(1)(7)}+\frac{1}{(7)(13)}+\frac{1}{(13)(19)}+\ldots+\frac{1}{(6n-5)(6n+1)}+\ldots$

9. For the series in (8), find n, if $S_n=\frac{16}{97}$.

In (10−13) assume that $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$ and use the method of Example 3, Section 14-5.
Find a formula for the sum of each of the following series:

10. $11+13+15+\ldots+(2n+9)$

11. $2+4+6+\ldots+2n$

12. $12+8+4+\ldots+(16-4n)$

13. $(a-4)+(a-8)+(a-12)+\ldots+(a-4n)$

14. Show that a formula for the sum of the series

    $a+(a+d)+(a+2d)+\ldots+[a+(n-1)d]$ is $\frac{n}{2}[2a+(n-1)d]$.

15. By comparing S_1, S_2, S_3, etc., for each of the series

    $$1+2+3+\ldots+n \text{ and } 1^3+2^3+3^3+\ldots+n^3,$$

    predict the sum of n terms of the latter series. Check for $n=6$.

16. If the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$, find k if $S_k = 136$.

17. Use the result in (15) to find p if the sum of the cubes of the first p natural numbers is 225.

14 6 The Sum of the General Arithmetic Series

In the previous section we have said that there is no general method which will produce a formula for the sum of n terms of a series. There is, however, a method which allows us to sum any arithmetic series. We offer first a particular example and then the general case.

Example 1

Find the sum of 10 terms of the arithmetic series

$$9+13+17+ \ldots$$

Solution:

$$\begin{aligned} t_{10} &= a+9d \\ &= 9+9(4) \\ &= 45 \end{aligned}$$

$$\therefore \quad S_{10} = 9+13+17+ \ldots +41+45$$

$$\text{or} \quad S_{10} = 45+41+37+ \ldots +13+9.$$

Adding corresponding terms of these series yields

$$S_{10}+S_{10} = 54+54+54+ \ldots +54+54.$$

There are 10 terms on the right side, each equal to 54.

Thus $2S_{10} = 10(54)$

$$\text{or} \quad S_{10} = \tfrac{10}{2}(54)$$

$$= 270.$$

Example 2

Find the sum of n terms of the series

$$a+(a+d)+(a+2d)+ \ldots +[a+(n-1)d].$$

Solution: If we represent t_n by l (the "last" of the n terms), then $t_{n-1} = l-d$.

Then $S_n = a+(a+d)+(a+2d)+ \ldots +(l-d)+l$

$$\text{or} \quad S_n = l+(l-d)+(l-2d)+ \ldots +(a+d)+a.$$

Adding corresponding terms of these series (as in Example 1) yields

$$S_n+S_n = (a+l)+(a+l)+(a+l)+ \ldots +(a+l)+(a+l).$$

$$\therefore \quad 2S_n = n(a+l);$$

$$\therefore \quad S_n = \tfrac{1}{2}n(a+l).$$

Replacing l by $a+(n-1)d$, we have $S_n=\frac{n}{2}[a+a+(n-1)d]$

$$=\frac{n}{2}[2a+(n-1)d].$$

Note that S_n may be written as

$$S_n=an+\frac{n}{2}(n-1)d \qquad \text{or} \qquad S_n=\frac{d}{2}n^2+\left(a-\frac{d}{2}\right)n.$$

The expression for the sum of n terms of an arithmetic series will always be a quadratic in n with a zero absolute term.

We may use either of the forms derived above as a formula for the sum of the series,

$$a+(a+d)+(a+2d)+\ \ldots\ +[a+(n-1)d].$$

$$\boxed{S_n=\frac{n}{2}(a+l)} \quad \text{or} \quad \boxed{S_n=\frac{n}{2}[2a+(n-1)d]}$$

Example 3

Find the sum of (i) 20 terms (ii) n terms
of the arithmetic series $4+9+14+\ \ldots$

Solution: (i) $a=4,\ d=5$

$$\therefore\ S_{20}=\tfrac{20}{2}[2(4)+(20-1)(5)]$$
$$=10[8+95]$$
$$=1030.$$

It is also possible to write

$$S_n=\frac{d}{2}n^2+\left(a-\frac{d}{2}\right)n.$$

$$\therefore\ S_{20}=\tfrac{5}{2}(20)^2+(4-\tfrac{5}{2})20$$
$$=1000+30$$
$$=1030.$$

(ii) $$S_n=\frac{n}{2}[2(4)+(n-1)5]$$

$$=\frac{n(5n+3)}{2}$$

Example 4

How many terms of the arithmetic series $30+27+24+\ldots$ must be taken to yield a sum of 165?

Solution: If $S_n=165$ and $a=30$, $d=-3$,

then $\frac{n}{2}[2(30)+(n-1)(-3)]=165.$

$$n(60-3n+3)=330$$
$$-3n^2+63n=330$$
$$-3n^2+63n-330=0$$
$$n^2-21n+110=0$$
$$(n-10)(n-11)=0$$
$$\therefore n=10 \text{ or } n=11.$$

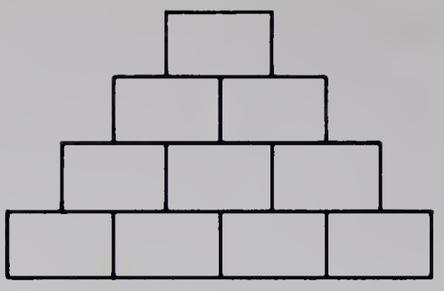
Therefore either 10 or 11 terms yield a sum of 165.
(What does this suggest for t_{11}? Check.)

EXERCISE 14—6

For each of the following arithmetic series, find the sum in parentheses:

1. $6+10+14+\ldots$ (S_{20})
2. $14+9+4+\ldots$ (S_{10})
3. $4.5+4.6+4.7+\ldots$ (S_{100})
4. $(a+x)+(a+2x)+(a+3x)+\ldots$ (S_{11})
5. An arithmetic series has $a=10$, $d=10$. Find t_{100} and S_{100}.
6. An arithmetic series has $a=x$, $d=-b$. Find t_{20} and S_{20}.
7. In the arithmetic series $4+5+6+\ldots$, find n if $S_n=99$.
8. In the arithmetic series $9+7+5+\ldots$, find n if $S_n=-24$.
9. Find an arithmetic series such that the sum of the first twenty terms is 820 and the sum of the next twenty terms is 3240.
10. An arithmetic series has 9 terms and the middle term is 21. Find S_9.
11. Show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.
12. A runner trains by running $\frac{1}{4}$ mile one day, $\frac{1}{2}$ mile the next day, $\frac{3}{4}$ mile the third day, and so on. Find (a) the distance he runs on the 11th day, (b) the total distance he runs in 15 days.
13. An arithmetic series is such that $t_{13}=-50$ and $S_{20}=-750$. Find the series.

14. A gable end is built of bricks with 30 bricks in the bottom row, decreasing one brick per row to one in the top row as shown. Find the total number of bricks required.



15. Find the sum of $2n$ terms of the series
$1-2+3-4+5 \ldots +(2n-1)-(2n)+\ldots$
Use two different methods if possible.

16. A man earns \$5000 in his first year and receives an increase of \$300 per year each year thereafter. If his maximum salary is \$8500, what amount of money does he receive over a period of 20 years?

17. Find the first four terms of the arithmetic series for which the sum of n terms is $5n^2+3n$.

18. In an arithmetic series $S_n=3n^2+4n$. Find S_{20}.

19. In an arithmetic series $S_n=2n^2-7n$. Find S_{12}.

20. In an arithmetic series $t_1=a$ and $t_{11}=b$; find S_{21} in terms of a and b.

21. Find an expression for the *product* of n terms of the geometric sequence $a, ar, ar^2, \ldots ar^{n-1}$.

14 7 The Sum of the General Geometric Series

We now consider the sum of the series associated with the general geometric sequence, first in a particular example.

Example 1

Find the sum of the geometric series

$$9+6+4+\ldots+9\left(\tfrac{2}{3}\right)^{n-1}.$$

Solution: Since $a=9$ and $r=\frac{2}{3}$,

$$S_n=9+6+4+\ldots+9\left(\tfrac{2}{3}\right)^{n-2}+9\left(\tfrac{2}{3}\right)^{n-1}$$

and rS_n, that is, $\frac{2}{3}S_n=\quad 6+4+\ldots+9\left(\tfrac{2}{3}\right)^{n-2}+9\left(\tfrac{2}{3}\right)^{n-1}+9\left(\tfrac{2}{3}\right)^{n}$.

Subtracting corresponding terms in the above array, we obtain

$$S_n-\tfrac{2}{3}S_n=9+0+0+\ldots+0+0-9\left(\tfrac{2}{3}\right)^{n}.$$

$$S_n\left(1-\tfrac{2}{3}\right)=9\left[1-\left(\tfrac{2}{3}\right)^{n}\right]$$

$$S_n=\frac{9\left[1-\left(\tfrac{2}{3}\right)^{n}\right]}{1-\tfrac{2}{3}}$$

$$S_n=27\left[1-\left(\tfrac{2}{3}\right)^{n}\right]$$

Example 2

Find the sum of the geometric series

$$a+ar+ar^2+\ldots+ar^{n-1}.$$

Solution:

$$S_n=a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}$$
$$rS_n=\quad ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}+ar^n$$

Subtracting corresponding terms in the above array, we obtain

$$S_n-rS_n=a+0+0+\ldots+0+0-ar^n$$
$$\therefore S_n(1-r)=a(1-r^n)$$

$$\text{or } S_n=\frac{a(1-r^n)}{1-r}. \qquad (r\neq 1)$$

Multiplying both numerator and denominator of the fraction on the right-hand side by -1, we obtain

$$S_n=\frac{a(r^n-1)}{r-1}.$$

While the two expressions for S_n are equal, the latter form is more convenient from a mechanical point of view when $r>1$, while the first form is ordinarily used when $r<1$. Remember that with a geometric sequence or series, the values $r=0,\ 1,\ -1$ are excluded by agreement.

We may use either of these formulas for the sum of the series

$$a+ar+ar^2+\ldots+ar^{n-1}$$

$$\boxed{S_n=\frac{a(1-r^n)}{1-r} \quad (r<1)} \qquad \text{or} \qquad \boxed{S_n=\frac{a(r^n-1)}{r-1} \quad (r>1)}$$

Example 3

Find S_n and S_7 for the geometric series

$$18-9+\tfrac{9}{2}\ldots$$

Solution:

$$a=18,\ r=-\tfrac{1}{2}$$
$$\therefore S_n=\frac{18[1-(-\tfrac{1}{2})^n]}{1-(-\tfrac{1}{2})}$$
$$=12[1-(-\tfrac{1}{2})^n].$$
$$\therefore S_7=12[1-(-\tfrac{1}{2})^7]$$
$$=12[1-(-\tfrac{1}{128})]$$
$$=\frac{12(129)}{128}$$
$$=\frac{387}{32}.$$

Example 4

The sum of the first four terms of a geometric series is to the sum of the first eight as 1:257. Find the common ratio.

Solution: If the series is $a+ar+ar^2+\ldots$, then

$$\frac{S_4}{S_8}=\frac{\dfrac{a(r^4-1)}{r-1}}{\dfrac{a(r^8-1)}{r-1}}=\frac{1}{257}$$

$$\frac{r^4-1}{r^8-1}=\frac{1}{257}$$

$$\frac{1}{r^4+1}=\frac{1}{257}$$

$$r^4+1=257$$

$$r^4=256$$

$$r=\pm 4$$

Is it possible to find the first term from the given information?

EXERCISE 14—7

Find the sum of the following geometric series:

1. $3+6+12+\ldots$ to 6 terms.
2. $3-6+12-\ldots$ to 7 terms.
3. $.03+.003+.0003+\ldots$ to n terms.
4. $-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$ to k terms.
5. $\sqrt{3}-3+3\sqrt{3}-\ldots$ to 9 terms.
6. $1+(x-y)+(x-y)^2+\ldots$ to $2n$ terms.
7. Given that $(\frac{3}{4})^7=.133$ (approx.), find (to the nearest tenth) S_7 for the geometric series $24+18+\ldots$
8. Given that $(\frac{5}{6})^8=.23$ (approx.), find (to one decimal place) the value of S_8 for the geometric series $30-25+\ldots$
9. Find the common ratio of a geometric series if the sum of the first five terms is to the sum of the first ten terms as 1:33.
10. Use the table of $(1+i)^n$ to find S_{10} for the geometric series
$$(1.04)+(1.04)^2+(1.04)^3+\ldots$$

11. Find S_{20} for the geometric series

$$1+(1.02)^2+(1.02)^4+\ldots$$

12. Find S_{12} for the geometric series

$$1+\frac{1}{1.05}+\frac{1}{(1.05)^2}+\ldots$$

13. One cent is placed on the first square of a chessboard which has 64 squares. Two cents are placed on the second square, four on the third, and so on, until each square is covered. Given that $2^{64}=1.84\times10^{19}$ (approx.), find the total value of the money on the board.

14. Three numbers form a geometric series whose sum is 28. The cube of the second number is 512. Find the three numbers.

15. Three numbers whose sum is 26 form a geometric series. The product of the numbers is 216. Find the series.

16. Show that S_n for the series

$$3+\tfrac{3}{2}+\tfrac{3}{4}+\ldots+3(\tfrac{1}{2})^{n-1}$$

is always less than 6 for any n.

17. For the series in (16), how much less than 6 is S_8?

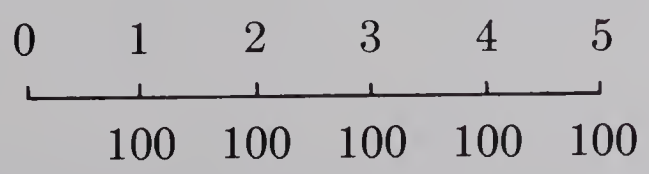
18. Find the sum of n terms of the series $2+8+26+80+\ldots$

19. Three numbers form a geometric series whose sum is 65. If the first, second, and third numbers are multiplied by 6, 7, and 4 respectively, the results form an arithmetic series. What are the original three numbers?

14.8 Annuities

DEFINITION: If a fixed sum of money is payable at any regular intervals, the sequence of payments is called an *annuity*. The time interval between any two payments is called the *period* of the annuity and is taken as 1 year unless otherwise stated.

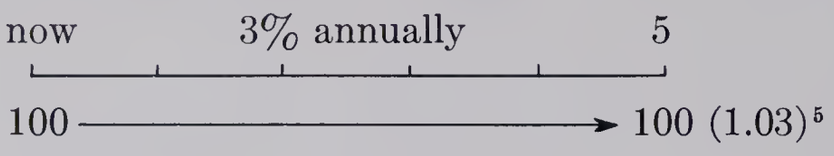
Suppose a man deposits \$100 in a bank account at the end of each year for 5 years. The sequence of payments constitutes an annuity and may be represented on a diagram as shown below.



In this case, we may say that the annuity *begins now and runs for 5 years.*

14 9 Amount of an Annuity

In Chapter 5 we learned that \$100 invested at 3% compounded annually accumulates to $\$100\ (1.03)^5$ at the end of 5 years. This is illustrated in the following diagram:



Generally, where the principal P is invested at $100i$ % per annum for n years, the accumulated value A is given by

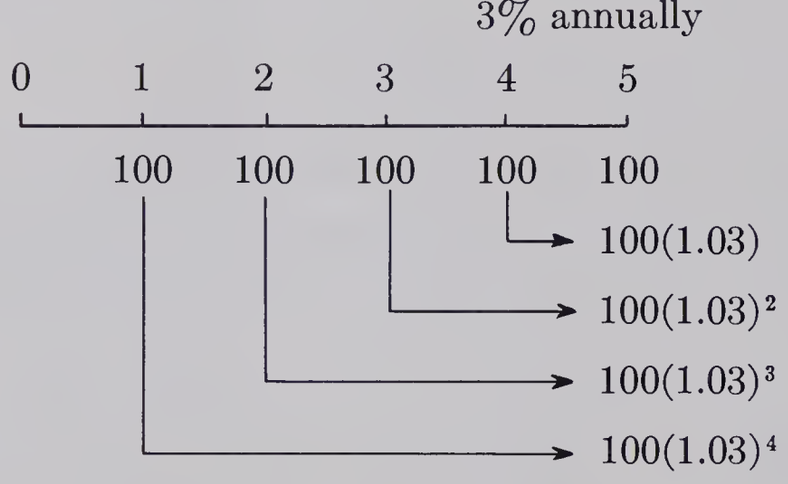
$$A = P(1+i)^n.$$

In the case of an annuity, the amounts of the individual payments may be summed by an application of the formula for the sum of a geometric series. Consider the following example.

Example 1

If a man deposits \$100 at the end of each year for 5 years in a bank which pays interest at 3% compounded annually, find the total amount on deposit immediately after the fifth deposit has been made.

Solution:



The indicated sum of the five accumulated values forms a geometric series.

The amount of the annuity five years hence is

$100+100(1.03)+100(1.03)^2+100(1.03)^3+100(1.03)^4$.

The sum is obtained from the formula

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ with } a=100,\ r=1.03,\ n=5.$$

$$\text{Thus } S_5 = \frac{100[(1.03)^5-1]}{1.03-1} \simeq \frac{100(1.15927-1)}{.03} \qquad \text{(tables)}$$

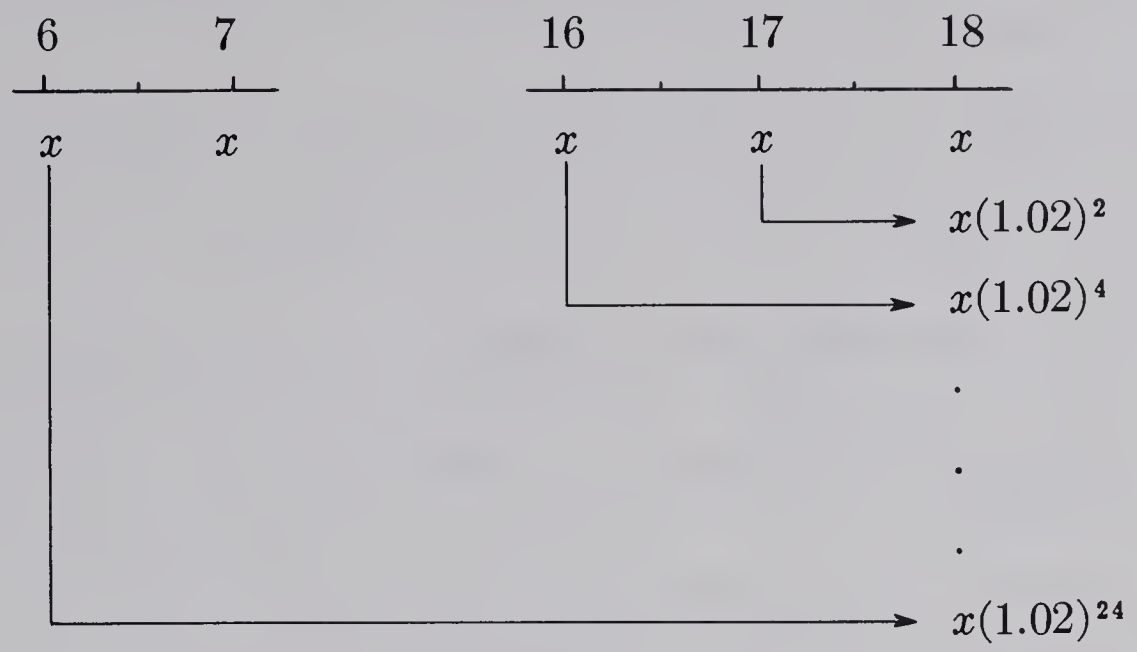
$$= \frac{15.927}{.03} = 530.9.$$

Therefore the amount of the annuity is \$530.90.

Example 2

A father makes annual deposits in a trust fund which pays interest at 4% per annum compounded semi-annually. If the first deposit is made on his son's sixth birthday, what should be the amount of the annual deposit so that the fund shall total \$5000 when the deposit is made on the boy's eighteenth birthday?

Solution:



The indicated sum of the accumulated values produces the geometric series

$$x+x(1.02)^2+x(1.02)^4+\ \ldots\ +x(1.02)^{24}.$$

Now $a=x, \quad r=(1.02)^2, \quad n=13.$

$$\therefore S_{13}=\frac{x[(1.02)^{26}-1]}{(1.02)^2-1}$$

But $S_{13}=5000.$

$$\therefore 5000=\frac{x[(1.02)^{26}-1]}{(1.02)^2-1}.$$

$$5000\simeq\frac{x(1.67342-1)}{1.0404-1} \qquad \text{(tables)}$$

$$\therefore x(.67342)=5000\ (.0404).$$

$$\therefore x=\frac{202}{.67342}\simeq 299.961.$$

$\therefore$ The annual deposit is \$299.96 (nearest cent).

Note: 1. Since the interest tables are correct only to 5 figures, all our answers must be rounded off to 5 figures.

2. Always assume that payments are made at the end of each period, unless stated otherwise.

EXERCISE 14—8

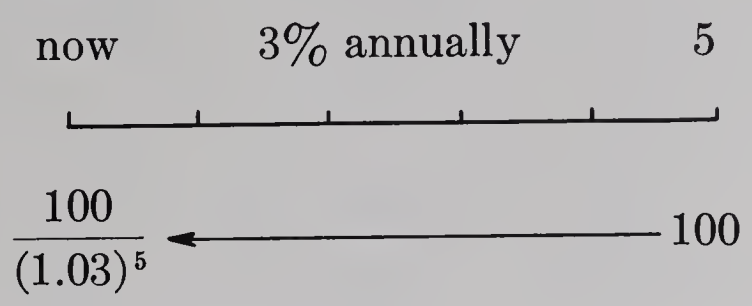
1. A man receives an annuity of \$300 each year for 10 years. If he deposits the payments in an account which pays interest at $4\frac{1}{2}\%$ per annum, find the total value of the deposits at the end of the 10-year period.
2. An annuity of \$500 begins now and runs for 8 years. If the payments are invested at 4% compounded quarterly, find the total invested at the end of the 8-year period.
3. A person makes 20 annual deposits of \$100 each in a trust fund which bears interest at 4% compounded semi-annually. If the first deposit is made on April 1st, 1966, find the accumulated value of the deposits on April 1st, 1986.
4. You save \$70 each half year and deposit it in a fund which pays 5% compounded semi-annually. If the total number of deposits is 16, what is the accumulated amount on deposit *two years after* the date of the last deposit?
5. A young man deposits a certain sum of money each half year in a fund which bears interest at 5% compounded semi-annually. What sum must be deposited so that, immediately after the 6th deposit has been made, \$350 will be available to buy a motor scooter?
6. What sum of money invested at the beginning of each year for 12 years in a fund which pays 6% per annum compounded annually, will accumulate to \$12,000 three years after the date of the last deposit?
7. Find the amount deposited annually, in an account paying 5% compounded semi-annually, if the first payment was made on January 1, 1950 and the total value immediately after the deposit on January 1, 1965 was \$20,000.
8. An investor deposits \$$x$ at the end of each half year in a fund which pays interest at 6% compounded semi-annually. Find x if the accumulated amount on deposit immediately after the 19th deposit has been made is \$5000.
9. Annually on July 1 from 1960 to 1970 inclusive, a man places \$1000 in an account which pays interest of 4% per annum compounded semi-annually. Find the accumulated value of the account on July 1, 1973.
10. A man invests \$1000 per year placing his money at the end of each year in a fund paying an annual rate of 4%. Find the number of years needed for the total amount on deposit to accumulate to \$20,000. (Use logarithms and obtain an answer correct to one decimal place.)

14 10 Present Value of an Annuity

In Chapter 5 we found that the value *now* of \$100 five years hence is

$$100 \times \frac{1}{(1.03)^5},$$

if money is worth 3% per annum.
This may be shown on a diagram:



$\frac{100}{(1.03)^5}$ (or $100(1.03)^{-5}$) is called the *present value* of the \$100 five years hence.

Generally, if the rate is $100i$ % per annum, the present value (P.V.) of A, n interest periods hence, is given by

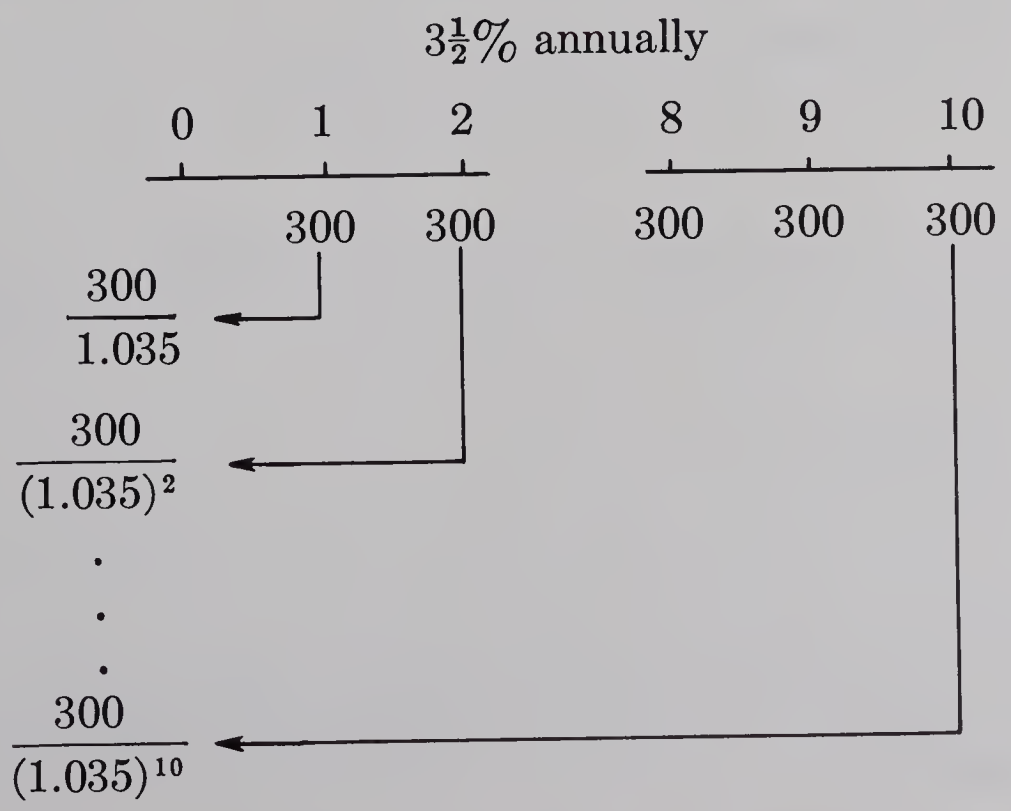
$$\text{P.V.} = \frac{A}{(1+i)^n} \quad \text{or} \quad A(1+i)^{-n}.$$

Many problems dealing with annuities are concerned with the present value of a sequence of payments. Consider the following examples.

Example 1

Find the present value of an annuity of \$300 which starts now and runs for 10 years, if money is worth $3\frac{1}{2}$% per annum.

Solution:



The present value of the annuity may be represented by the geometric series

$$\frac{300}{1.035}+\frac{300}{(1.035)^2}+\cdots+\frac{300}{(1.035)^{10}}.$$

It will be convenient to rewrite this as

$$\frac{300}{(1.035)^{10}}+\frac{300}{(1.035)^9}+\cdots\frac{300}{1.035},$$

so that $a=\dfrac{300}{(1.035)^{10}}$, $r=1.035$, $n=10$.

$$\therefore \text{P.V.}=\frac{300}{(1.035)^{10}}\times\frac{[(1.035)^{10}-1]}{(1.035)-1}$$

$$=\frac{300}{(1.035)-1}\times\frac{(1.035)^{10}-1}{(1.035)^{10}}$$

$$=\frac{300}{.035}\times\left(1-\frac{1}{(1.035)^{10}}\right)$$

$$\simeq\frac{300}{.035}\times(1-0.70892) \qquad \text{(tables)}$$

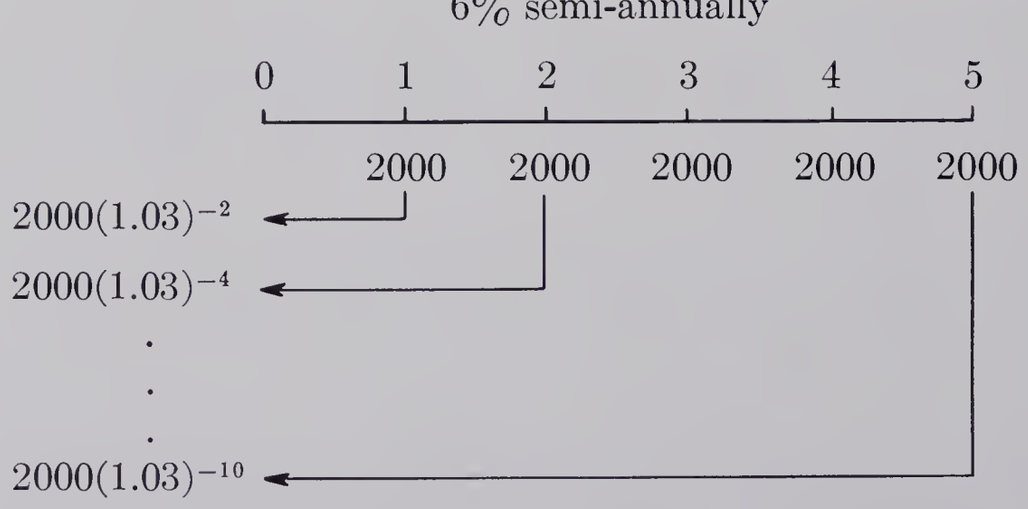
$$=\frac{300\ (0.29108)}{.035}=\frac{87324}{35}\simeq 2494.97.$$

The present value of the annuity is \$2495.00 (five significant figures).

Example 2

The price of a house is \$6000 down and \$2000 per year at the end of each year for five years. What is the equivalent cash price if money is worth 6% per annum compounded semi-annually?

Solution:



(The use of the negative exponents permits a slightly more compact solution.)

The present value of the five payments is

$$2000(1.03)^{-10}+2000(1.03)^{-8}+\ \ldots\ +2000(1.03)^{-2}.$$

In this case $a=2000(1.03)^{-10}$

$r=(1.03)^2$

$n=5.$

$$\therefore \text{P.V.}=2000(1.03)^{-10}\times\frac{[(1.03)^{10}-1]}{(1.03)^2-1}$$

$$=\frac{2000}{(1.03)^2-1}\times(1.03)^{-10}[(1.03)^{10}-1]$$

$$=\frac{2000}{(1.03)^2-1}\times[1-(1.03)^{-10}]$$

$$\simeq\frac{2000}{.09273}\times(1-.74409) \qquad \text{(tables)}$$

$$=\frac{2000}{.09273}\times(.25591)$$

$$=\frac{511.82}{.09273}$$

$$\simeq 5519.465$$

$=5519.50$ (five significant figures).

The equivalent cash price of the house is

$\$(6000+5519.50)$

$=\$11519.50$

EXERCISE 14—9

1. Find the present value of an annuity of \$300 which begins now and runs for ten years if the rate of interest is 4% per annum compounded semi-annually.

2. A man is to pay back a loan by paying \$1000 at the end of each year for six years. What amount in cash would discharge the debt now, if money is worth 5% per annum compounded semi-annually?

3. The purchase price of a summer cottage is \$2000 down and \$1000 at the end of each year for four years. Find the equivalent cash price of the cottage if money earns 5% per annum.

4. Find the value on June 1, 1967 of an annuity of \$50 payable on each June 1 from 1968 to 1977 inclusive. Money earns 6% per annum compounded semi-annually.

5. Find the value on May 1, 1965 of an annuity of 15 annual payments, the first payment on May 1, 1965. The rate of interest is 4% per annum.

6. If money earns 5% per annum, what is the annual payment of a ten-payment annuity, the first payment one year hence, if \$1000 is invested now?

7. \$5000 is invested at 4% per annum compounded semi-annually. Calculate the amount of the annual payment of an 8-payment annuity if the first payment is one year hence.

8. You have \$10,000 deposited at 5% per annum. Find the largest annual withdrawal possible from this fund, if the first of ten withdrawals is made immediately.

14 11 Deferred Annuities

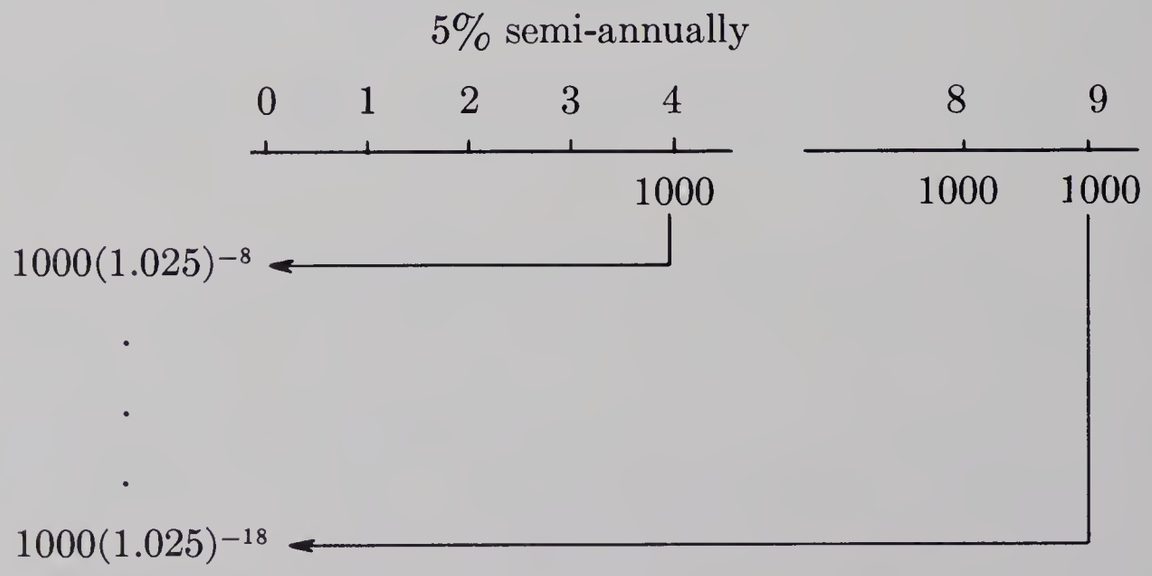
When we say that an annuity begins now, the first payment is understood to be one year hence.

If an annuity begins in five years, the first payment will be at the end of 6 years. Such an annuity is said to be *deferred* for five years.

Example 1

Find the present value of an annuity of \$1000 which is deferred for 3 years and runs for 6 years if the prevailing interest rate is 5% per annum compounded semi-annually.

Solution:



The present values of the 6 payments form the series
$1000(1.025)^{-18}+1000(1.025)^{-16}+ \ldots 1000(1.025)^{-8}$.
Then $a=(1.025)^{-18}$, $r=(1.025)^2$, $n=6$.

$$\therefore \text{P.V.} = 1000(1.025)^{-18} \times \frac{[(1.025)^{12}-1]}{(1.025)^2-1}$$

$$= \frac{1000}{(1.025)^2-1} \times [(1.025)^{-6}-(1.025)^{-18}]$$

$$\simeq \frac{1000}{.05063}(0.86230-.64117) \qquad \text{(tables)}$$

$$= \frac{221.13}{.05063}$$

$$\simeq 4367.57$$

Therefore the present value of the annuity is \$4367.60 (5 figures).

EXERCISE 14—10

1. Find the present value of an annuity of \$500 which is deferred for 5 years and runs for 4 years, if money earns 4% per annum.

2. Find the value now of a 10-payment annuity if the first payment is made 8 years hence and money is worth 5% per annum, compounded semi-annually.

3. A father wishes to give his daughter an annuity of \$1000 with the first of 4 payments on her 18th birthday. If money earns 6% per annum, what is the value of the annuity on her 10th birthday?

4. An annuity of \$500 is payable every July 1st from 1969 to 1978 inclusive. If the rate of interest is 4% per annum, compounded semi-annually, what is the value of the annuity on July 1, 1966?

5. The first of 12 payments of an annuity of \$1000 is made on March 1st, 1972. If money earns 5% per annum, what amount must be invested on March 1st, 1967 to purchase this annuity?

6. What is the annual payment of an annuity of 10 payments which is deferred for 6 years and purchased now for \$8000? Money is worth 4% per annum compounded semi-annually.

7. On May 1, 1965, \$10,000 is deposited at 5% per annum compounded semi-annually. Find the annual payment if this money buys a 10-payment annuity with the first payment on May 1, 1969.

8. Find the present value of an annuity of \$3000 which is deferred for 5 years and runs for 6 years, if money is worth 6% per annum compounded semi-annually.

14.12 (Supplementary) Proof by Mathematical Induction

You will recall that earlier in this chapter we used special methods to develop formulae for the sum of the general arithmetic and geometric series. We *predicted* formulae for the sum of various series but did not develop these formulae deductively.

For example, we could predict that the sum S_n of n terms of

$$3+7+11+15+\ldots+(4n-1)+\ldots \text{ is } 2n^2+n.$$

Now consider the following formal proof by the principle of *mathematical induction.*

First test for $n=1$. $S_1=2(1)^2+1$

$=3$, which is correct since $t_1=3$.

Now suppose that the above formula for S_n holds for some value of n, say $n=k$.

$$\therefore S_k=2k^2+k. \quad \text{(by assumption)} \quad . \quad . \quad . \quad (1)$$

Does it follow that the formula must also hold for the next value of n, that is, $n=k+1$?

If we add the $(k+1)$st term of the series, that is,

$$t_{k+1}=4(k+1)-1=4k+3$$

to each member of (1), then

$$\begin{aligned} S_k+(4k+3) &= 2k^2+k+4k+3 \\ &= 2k^2+4k+2+k+1 \\ &= 2(k^2+2k+1)+(k+1) \\ &= 2(k+1)^2+(k+1). \end{aligned}$$

But $S_k+(4k+3)=S_{k+1}$

$$\therefore S_{k+1}=2(k+1)^2+(k+1),$$

which is exactly what our formula would have predicted. We have *proved* that *if* $S_n=2n^2+n$ for $n=k$, *then* it is also true for $n=k+1$.

Since the formula for S_n holds for $n=1$, it must also hold for $n=1+1$ or 2. (Why?)

Since it holds for $n=2$, it must also hold for $n=2+1$ or 3, and so on. Hence $S_n=2n^2+n$ is true for all positive integral values of n.

Proof by mathematical induction may be summarized in the following three steps:

1. Test the truth of the proposition for the smallest value of n $(n=1)$.
2. Assume that the proposition is true for a particular value of n, say $n=k$, and show that the proposition must also be true for the next higher value of n, that is, $n=k+1$.
3. Since the proposition is true for $n=1$, it must hold for $n=2$. Since it is true for $n=2$, it must also be true for $n=3$, and so on.

Note: This deductive method of proof is called proof by mathematical induction. Do not be misled by its historical name into thinking that we are reaching only an inductive prediction when we use this method. Proof by mathematical induction (or by the axiom of mathematical induction) is a rigorous deductive method leading to a logical conclusion. The name mathematical induction arises from step 3 above, in which we proceed through the *mathematical* sequence 1, 2, 3, . . . after making an *inductive* prediction in the beginning.

Example 1

Prove that $\frac{1}{(1)(4)}+\frac{1}{(4)(7)}+\frac{1}{(7)(10)}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$.

Solution: *Step 1:* Test $n=1$.

$S_1=\frac{1}{3(1)+1}=\frac{1}{4}$ which is true (since $S_1=t_1=\frac{1}{4}$).

Step 2: Assume that the formula holds for $n=k$;

that is, assume $S_k=\frac{k}{3k+1}$. (1)

Add $t_{k+1}=\frac{1}{(3k+1)(3k+4)}$, to each side of (1).

$$\therefore S_k+\frac{1}{(3k+1)(3k+4)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

But $S_k+\frac{1}{(3k+1)(3k+4)}=S_k+t_{k+1}=S_{k+1}$.

$$\therefore S_{k+1}=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{k+1}{3k+4} \qquad (\text{since } k\neq -\tfrac{1}{3})$$

(why?)

But $\frac{k+1}{3k+4}=\frac{k+1}{3(k+1)+1}$, which is exactly the value predicted by our formula. Hence if S_n holds for $n=k$, it also holds for $n=k+1$.

Step 3: Since the formula for S_n is true for $n=1$, it is also true for $n=2$, and so on for all $n\in I^+$.

EXERCISE 14—11

Offer proofs by mathematical induction for the following:

1. Prove that $1+2+3+\ldots+n=\frac{n(n+1)}{2}$.
2. Suggest a formula for the sum of the series $1+3+5+\ldots+(2n-1)$ and prove by mathematical induction that the formula holds for all values of n.
3. Prove that $2+4+6+\ldots+2n=n(n+1)$.
4. Prove that $a+(a+d)+(a+2d)+\ldots+a(n-1)d=\frac{n}{2}[2a+(n-1)d]$.
5. Prove that $4+9+14+\ldots+5n-1=\frac{n(5n+3)}{2}$.
6. Prove that $5+8+11+\ldots+(3n+2)=\frac{n(3n+7)}{2}$.
7. Prove that $1+4+7+\ldots+(3n-2)=\frac{3n^2-n}{2}$.
8. Prove that $3+9+27+\ldots+3^n=\frac{3}{2}(3^n-1)$.
9. Prove that $a+ar+ar^2+\ldots+ar^{n-1}=\frac{a(r^n-1)}{r-1}$.
10. Prove that $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$.
11. Prove that $\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.
12. Prove that $\frac{1}{(2)(5)}+\frac{1}{(5)(8)}+\frac{1}{(8)(11)}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{\frac{1}{2}n}{3n+2}$.
13. Prove that $1^3+2^3+3^3+4^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$.
14. Prove that
$$(1)(2)(3)+(2)(3)(4)+(3)(4)(5)+\ldots+(n)(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}.$$
15. Prove that $(3^1)(3^2)(3^3)\ldots(3^n)=(\sqrt{3})^{n(n+1)}$.
16. Prove that $n(n+1)$ is divisible by 2 for all integers n.
17. Prove that $n(n+1)(n+3)$ is divisible by 6 for all integers n.
18. Prove that $5^{2n}-1$ is a multiple of 3 for all integers n.

14 13 Chapter Summary

Sequence defined by a function. Finite and infinite sequences.

Terms of a sequence, the general term of a sequence, recursion formulae for the general term.

Arithmetic sequence, common difference, general term.

Geometric sequence, common ratio, general term.

Series and sum of a series. Inductive methods to predict sums of certain series.

Sum of the general arithmetic series.

Sum of the general geometric series.

Annuities, amount, present value, deferred annuities.

Mathematical induction (supplementary).

REVIEW EXERCISE 14—12

1. For the sequence defined by $f(x)=x^2$, write
   (1) $f(5)$ (2) $f(n)$ (3) $f(n+1)$ (4) $f(2k+3)$.

2. In the sequence whose nth term is given by $t_n=\frac{n^2+3n}{2}$,
   (a) find the value of n for which $t_n=27$;
   (b) show that $t_{n+1}-t_n=n+2$;
   (c) show that $t_{n+1}-2t_n+t_{n-1}=1$.

3. For the arithmetic sequence 14, 11, 8, . . . , find (a) t_{50} and t_n, (b) the number of the first term less than -200.

4. Show that if $-p-3r$, $p+2r$, $2p+5r$ is an arithmetic sequence, then $p+2r=0$.

5. Find S_{20} and S_n for the arithmetic series $10+14+18+\ldots$

6. How many terms of the series in (5) must be taken to yield a sum of 384?

7. For the series $20+17+14+\ldots+(20-3n)$, develop a formula for S_n, using the same method employed in the general case.

8. Find t_7 and t_n for the geometric sequence 7, 21, 63, . . .

9. In a geometric sequence, the 4th and 7th terms are $\frac{27}{2}$ and $\frac{729}{16}$ respectively. Find an expression for the nth term.

10. For the geometric series $\frac{3}{2}-\frac{9}{4}+\frac{27}{8}+\ldots$, find S_7 (as an improper fraction).

11. Find the common ratio of a geometric series if the ratio of the sum of the first six terms to the sum of the first twelve is 1:65.

12. Find a formula for the sum of n terms of the series

$$3+6+12+\ldots+3(2)^{n-1},$$

using the method employed to find the general formula for the sum of n terms of a geometric series.

13. If x^2-a^2, $x+a$, and $\dfrac{x-a}{x+a}$ are in geometric sequence, show that either a or x must be zero.

14. \$100 is deposited at the end of each year for 10 years in an account which pays interest at 5% per annum. Find the accumulated value of the deposits 3 years after the date of the last deposit.

15. What amount invested at the end of each year for 7 years will accumulate to \$10,000 immediately after the last deposit if money earns 6% per annum compounded semi-annually?

16. Find the present value of an annuity of \$1000 which begins now and runs for 8 years, if money is worth 4% compounded quarterly.

17. If money is worth 5% per annum, find the annual payment of an annuity which can be bought now for \$5000, the first of 6 payments to be 1 year hence.

18. An annuity of \$500 is deferred for 6 years and runs for 5 years. Find the present value of this annuity if the rate of interest is 4% per annum compounded semi-annually.

19. On each March 1st from 1968 to 1978 inclusive, \$1000 is deposited in a fund which pays interest at 6% per annum compounded semi-annually. Find the accumulated value of these deposits on March 1, 1982.

20. (Supplementary) Prove by mathematical induction that the sum of n terms of the series
$(2)(6)+(4)(9)+(6)(12)+\ldots+(2n)(3n+3)$ is $2n(n+1)(n+2)$.

21. (Supplementary) Prove by mathematical induction that the sum of n terms of the series $1+3+6+10+\ldots$ is $\dfrac{n(n+1)(n+2)}{6}$.

22. (Supplementary) Prove that n^3-n is divisible by 6 for all integers n.

23. (Supplementary) Prove $9^{3n}-1$ is a multiple of 7 for all integers n.

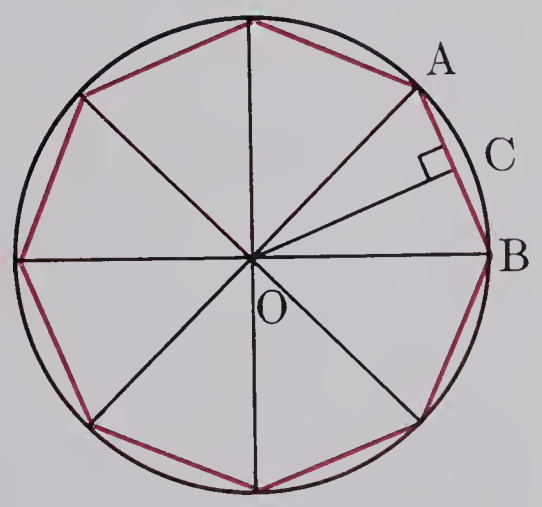
Appendix A

Circumference and Area of a Circle

A 1 Circumference

In Chapter 10 we stated that the circumference of a circle was the length of the circle. We also stated that there were certain logical objections to this definition. So far, we have only defined length of a line segment. It is, therefore, logically incorrect to define circumference as the length of the circle when we can only talk about the length of a line segment. Intuitively, we could imagine the circle cut at one point and "stretched out" to form a line segment.

In the diagram, a regular octagon has been inscribed in a circle by constructing two perpendicular diameters and bisecting each of the right angles so fomred. By the s.a.s. congruency theorem, the eight isosceles triangles are congruent and, hence, the sides of the octagon are equal in length and the angles are also equal.



By constructing $OC \perp AB$, we can easily prove that $\triangle AOC \equiv \triangle BOC$ and that $\angle AOC = 22\frac{1}{2}°$.

$$\therefore \frac{AC}{AO} = \sin 22\tfrac{1}{2}°.$$

$$\therefore AC = r \sin 22\tfrac{1}{2}° \text{ (where } r \text{ is the radius of the circle).}$$

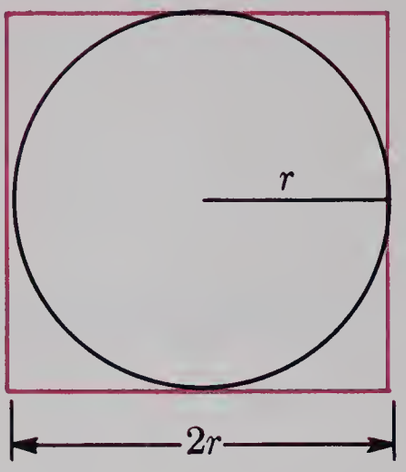
$$\therefore AB = 2r \sin 22\tfrac{1}{2}°.$$

$$\therefore \text{Perimeter of octagon} = 16\ r \sin 22\tfrac{1}{2}°.$$

The perimeter of any inscribed regular polygon, can be calculated in like manner. For a regular polygon of n sides, inscribed in a circle radius r, the perimeter (P_n) will be given by the formula

$$P_n = 2nr \sin \frac{180°}{n}.$$

When $n=3$, $P_3 \simeq 5.20r$
When $n=4$, $P_4 \simeq 5.66r$
When $n=6$, $P_6 \simeq 6.00r$
When $n=8$, $P_8 \simeq 6.12r$
When $n=16$, $P_{16} \simeq 6.24r$
When $n=32$, $P_{32} \simeq 6.27r$



As n increases P_n increases but, since each polygon is inscribed in a circle, P_n is finite. Indeed $P_n < 8r$, the perimeter of the square circumscribing the circle as shown in the diagram. Since P_n increases as n increases and since $P_n < 8r$ for all n, then P_n must approach arbitrarily close to some real number α as n increases without bound. We call this real number α the limit of the sequence or ordered set of numbers $\{P_n\}$ and we define this limit α to be the circumference of the circle.

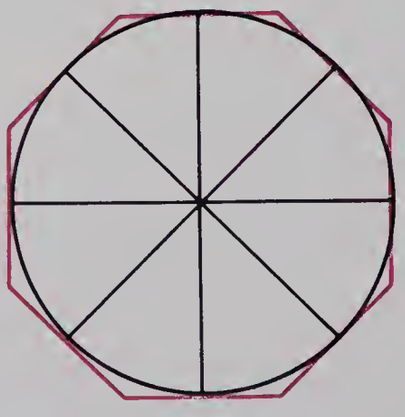
DEFINITION: Let P_n be the perimeter of a regular n-gon inscribed in a circle. The *circumference* C of the *circle* is the *limit* of P_n as the number of sides n of the n-gon increases without bound.

We may express this symbolically as $P_n \to C$. This is read, "As n increases without bound, P_n approaches arbitrarily close to C."

In the same way we can consider the circumscribed polygons and find that the perimeter P'_n of a circumscribed polygon is given by

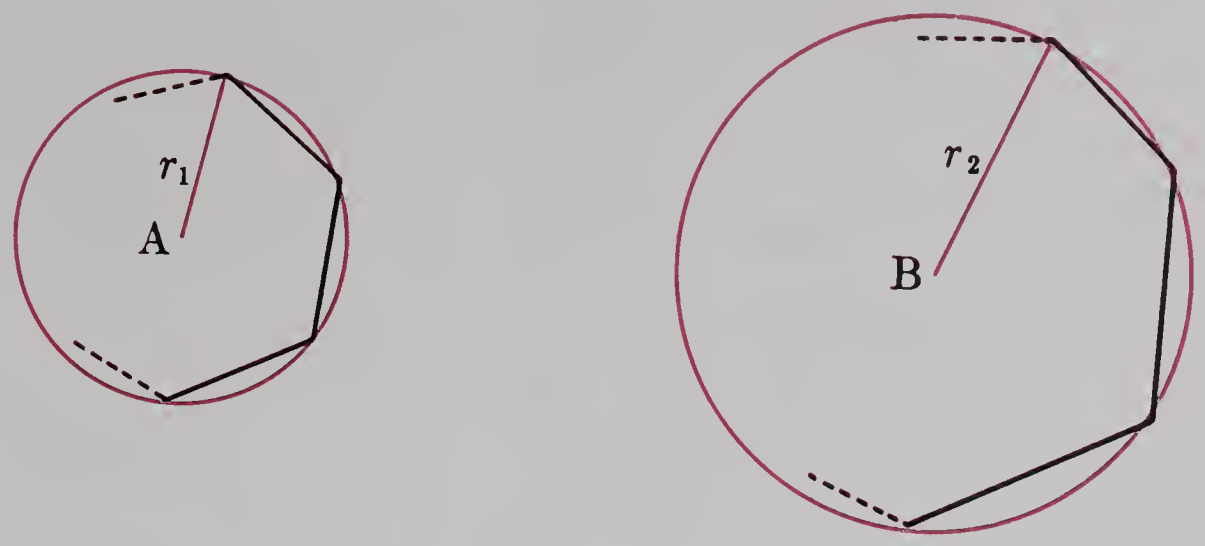
$$P'_n = 2nr \tan \frac{180°}{n}.$$

When $n=3$, $P'_3 \simeq 10.39r$.
When $n=4$, $P'_4 \simeq 8.00r$.
When $n=6$, $P'_6 \simeq 6.93r$.
When $n=8$, $P'_8 \simeq 6.63r$.
When $n=16$, $P'_{16} \simeq 6.37r$.
When $n=32$, $P'_{32} \simeq 6.30r$.



As n increases, P'_n decreases but is always greater than the circumference, and as n increases without bound, P'_n approaches arbitrarily close to C.

THEOREM: The ratio of the measure of the circumference of a circle to the measure of its diameter is constant.



Given: Two circles, centres A and B, radii r_1 and r_2 units respectively, and circumferences C_1 and C_2 units respectively.

Required: To prove $\dfrac{C_1}{2r_1}=\dfrac{C_2}{2r_2}$.

Proof: Inscribe in each circle a regular polygon of n sides.

Let the perimeters of the polygons be P_n and Q_n units respectively.

$$\therefore P_n = 2r_1\, n \sin\frac{180^\circ}{n} \quad \text{(Perimeter of inscribed polygon)}$$

$$\text{and } Q_n = 2r_2\, n \sin\frac{180^\circ}{n}.$$

$$\therefore \frac{P_n}{Q_n}=\frac{r_1}{r_2} \text{ for all } n.$$

$$\therefore \frac{P_n}{r_1}=\frac{Q_n}{r_2}. \qquad \text{(Alternation)}$$

As n increases, $P_n \to C_1$ and $Q_n \to C_2$.

$$\therefore \text{in the limit } \frac{C_1}{r_1}=\frac{C_2}{r_2}.$$

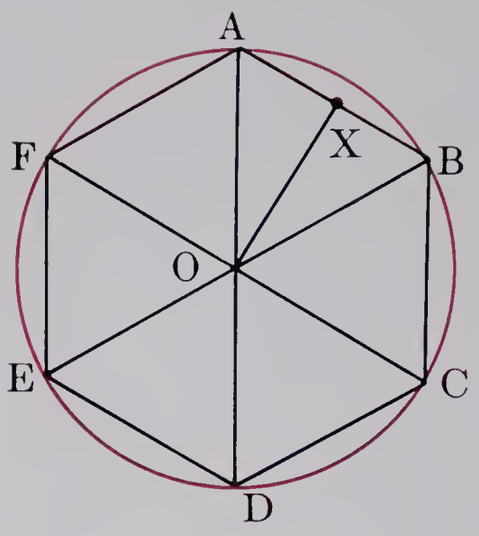
$$\therefore \frac{C_1}{2r_1}=\frac{C_2}{2r_2}.$$

The constant $\dfrac{C}{2r}$ for any circle is defined to be π.

Since $\dfrac{C}{2r}=\pi$, it follows that $C=2\pi r$.

A 2 Area of a Circle

In the diagram, ABCDEF is a regular hexagon inscribed in a circle. The area of $\triangle AOB$ is $\frac{1}{2}$ AB·OX and, since the six triangles are congruent, the area of the hexagon is $6 \times \frac{1}{2}$ AB·OX. The perimeter of the hexagon is 6AB. Therefore the measure of the area of ABCDEF $= \frac{1}{2}$ P·OX where P is the measure of the perimeter.

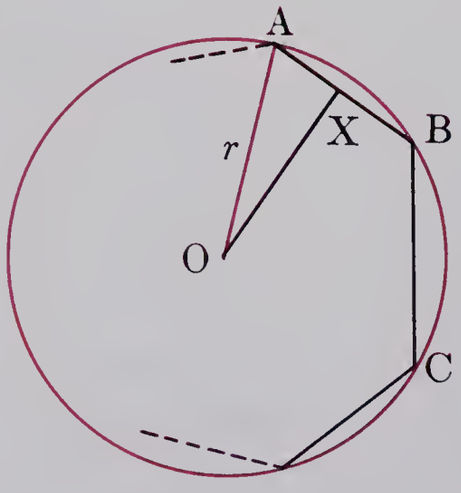


In the same way, the area of an inscribed *regular* n-gon will be $A_n = \frac{1}{2}$ P·OX where OX is the perpendicular from the centre of the circle to one side of the n-gon and P_n is the perimeter.

As the number of sides increases, the length of OX will also increase, as will the perimeter and area. The length of OX will approach the length of the radius. As P_n approaches C, OX approaches r. We may, therefore, state that the limit of OX is r. We can make OX as close to r as we please by taking n sufficiently large. This enables us to prove the formula for the measure of the area of a circle if we define the area of a circle as follows.

DEFINITION: Let A_n be the area of a regular n-gon inscribed in a circle. The *area* of the circle is the *limit* of A_n as the number of sides of the n-gon increases without bound.

THEOREM: The measure of the area of a circle is πr^2 where r is the measure of the radius.



Given: A circle, centre O, and radius r units.

Required: To prove $A = \pi r^2$ where A is the measure of the area of the circle.

Proof: Inscribe a regular n-gon ABC . . . in the circle.
Let the perimeter of the n-gon be P_n units.
Construct $OX \perp AB$.

$$\therefore A_n = \tfrac{1}{2}\, P \cdot OX.$$

As n increases, $P_n \to C$, $OX \to r$, and $A_n \to A$.

$$\begin{aligned} \therefore A &= \tfrac{1}{2}\, Cr \\ &= \tfrac{1}{2}(2\pi r)(r) \qquad (C = 2\pi r) \\ &= \pi r^2. \end{aligned}$$

Appendix B

Trigonometric Identities and Equations

B 1 Identities in Trigonometric Functions

In Chapter 12 we quoted various fundamental identities involving the trigonometric functions; they included

(a) the reciprocal identities,

(b) the Pythagorean identities.

These identities can be used to prove other identities. Recall that an equation in x, $x \in S$, is an identity if its solution set is the set S itself. For functions this may be restated as follows:

DEFINITION: An equality $f(x) = g(x)$ involving two functions f and g with the same domain is an identity if $f(x) = g(x)$ for *each and every* value of x in the domain.

There are three ways in which an identity may be verified:

(i) by changing the left member directly into the form of the right member,

(ii) by changing the right member directly into the form of the left member,

(iii) by changing both the left and right members into the same form and *changing each separately and independently of the other.*

Example 1

Prove $\sec A(1+\cos A)=1+\sec A$.

Solution: L.H.S. $=\sec A(1+\cos A)$

$=\sec A+\sec A\cdot\cos A$

$=\sec A+1$ (reciprocal identity)

$\therefore \sec A(1+\cos A)=1+\sec A$.

(Here we have written out every step in the solution; as usual the steps may be contracted and two or more of them performed mentally as soon as the processes become more familiar.)

Example 2

Prove $\dfrac{\csc^2\theta+\sec^2\theta}{\csc\theta\sec\theta}=\tan\theta+\cot\theta$.

Solution: L.H.S. $=\dfrac{\csc^2\theta+\sec^2\theta}{\csc\theta\sec\theta}$

$=\dfrac{\csc^2\theta}{\csc\theta\sec\theta}+\dfrac{\sec^2\theta}{\csc\theta\sec\theta}$

$=\dfrac{\csc\theta}{\sec\theta}+\dfrac{\sec\theta}{\csc\theta}$

$=\dfrac{1}{\sin\theta}\cdot\dfrac{\cos\theta}{1}+\dfrac{1}{\cos\theta}\cdot\dfrac{\sin\theta}{1}$ (definition)

$=\cot\theta+\tan\theta$ (definition)

$\therefore \dfrac{\csc^2\theta+\sec^2\theta}{\csc\theta\sec\theta}=\tan\theta+\cot\theta$

EXERCISE B—1

Prove the following identities:

1. $\dfrac{\sec^2\theta-1}{\tan\theta}=\tan\theta$
2. $\dfrac{2\tan x}{1+\tan^2 x}=2\sin x\cos x$
3. $\tan\theta+\cot\theta=\sec\theta\csc\theta$
4. $\sin^4 A-\cos^4 A=\sin^2 A-\cos^2 A$
5. $\cos^2\theta\cot^2\theta=\cot^2\theta-\cos^2\theta$
6. $\sin P(1+\cot P)=\cos P(1+\tan P)$
7. $\dfrac{1-\tan^2 B}{1+\tan^2 B}=\cos^2 B-\sin^2 B$
8. $\dfrac{\sec^2 X-2}{2-\csc^2 X}=\tan^2 X$
9. $\dfrac{\tan x}{\tan x+\cot x}=\sin^2 x$
10. $\dfrac{1+\sin^2\theta}{1+\cos^2\theta}=\dfrac{1+2\tan^2\theta}{2+\tan^2\theta}$
11. $\sec^2 y-\sin^2 y=\cos^2 y+\tan^2 y$
12. $\tan\phi+\dfrac{1}{\tan\phi}=\sec\phi\csc\phi$

13. $1+\sec A=\dfrac{\tan^2 A}{\sec A-1}$

14. $\dfrac{\sin B+\tan B}{1+\sec B}=\sin B$

15. $\csc\theta-\sin\theta=\dfrac{\cot\theta}{\sec\theta}$

16. $\dfrac{1+\cot^2 x}{1+\tan^2 x}=\cot^2 x$

17. $(\cos A+\sin A)^2-(\cos A-\sin A)^2=4\sin A\cos A$

18. $\dfrac{1}{1-\cos A}+\dfrac{1}{1+\cos A}=2\csc A\cot A$

19. $\dfrac{1}{1+\sin A}+\dfrac{1}{1-\sin A}=2\sec^2 A$

20. $\dfrac{\cos\theta}{1+\tan\theta}-\dfrac{\sin\theta}{1+\cot\theta}=\cos\theta-\sin\theta$

B 2 Further Identities Involving Trigonometric Functions

In the previous section we considered identities of the form

$$f(x)=g(x) \text{ for all } x\in S.$$

We may also have trigonometric identities where the arguments of the various functions are A, $\frac{\pi}{2}-A$, $\pi-A$, $\frac{\pi}{2}+A$, etc. To prove these identities, we use the complementary and supplementary angle identities.

We may also have identities involving trigonometric functions of two or more angles.

Example 1

Prove $\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$.

Solution: L.H.S. $=\cos^2 A-\sin^2 B$
$=1-\sin^2 A-(1-\cos^2 B)$
$=\cos^2 B-\sin^2 A=$ R.H.S.

Example 2

Prove $\cos A\sin(\frac{\pi}{2}-A)-\cos(\frac{\pi}{2}+A)\sin A=1$.

Solution: L.H.S. $=\cos A\sin(\frac{\pi}{2}-A)-\cos(\frac{\pi}{2}+A)\sin A$
$=\cos A\cos A+\cos(\pi-(\frac{\pi}{2}+A))\sin A$
$=\cos^2 A+\cos(\frac{\pi}{2}-A)\sin A$
$=\cos^2 A+\sin A\sin A$
$=\cos^2 A+\sin^2 A$
$=1=$ R.H.S.

Example 3

Prove $\dfrac{\cos^2A-\cos^2B}{\cos^2A\ \cos^2B}=\tan^2B-\tan^2A$

Solution: R.H.S. $=\tan^2B-\tan^2A$

$$=\frac{\sin^2B}{\cos^2B}-\frac{\sin^2A}{\cos^2A}$$

$$=\frac{1-\cos^2B}{\cos^2B}-\frac{1-\cos^2A}{\cos^2A}$$

$$=\frac{\cos^2A-\cos^2A\ \cos^2B-(\cos^2B-\cos^2B\ \cos^2A)}{\cos^2A\ \cos^2B}$$

$$=\frac{\cos^2A-\cos^2B}{\cos^2A\ \cos^2B}$$

$$\therefore\frac{\cos^2A-\cos^2B}{\cos^2A\ \cos^2B}=\tan^2B-\tan^2A.$$

Alternatively,

$$\text{L.H.S.}=\frac{\cos^2A-\cos^2B}{\cos^2A\ \cos^2B}$$

$$=\frac{1}{\cos^2B}-\frac{1}{\cos^2A}$$

$$=\sec^2B-\sec^2A$$

$$=1+\tan^2B-(1+\tan^2A)$$

$$=\tan^2B-\tan^2A$$

$$\therefore\frac{\cos^2A-\cos^2B}{\cos^2A\ \cos^2B}=\tan^2B-\tan^2A.$$

EXERCISE B—2

Prove the following identities:

1. $\sin^2A-\sin^2B=\cos^2B-\cos^2A$
2. $1-\sin(\frac{\pi}{2}+A)\sin(\frac{\pi}{2}-A)=\sin^2(\pi-A)$
3. $\dfrac{\sin(\frac{\pi}{2}+A)}{\cos(\pi+A)}=-1$
4. $\dfrac{\sin(\pi-A)}{\sin(\frac{\pi}{2}+A)}=\tan A$
5. $\csc^2A-\tan^2(\frac{\pi}{2}+A)=1$
6. $\tan A+\tan(90°-A)=\sec A\csc A$

7. $\tan^2 A - \tan^2 B = \dfrac{(\cos B + \cos A)(\cos B - \cos A)}{\cos^2 A \cos^2 B}$

8. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$

9. $\cos B(\cos A \cos B + \sin A \sin B) - \sin B(\sin A \cos B - \cos A \sin B) = \cos A$

10. $\sin A \sin B(\cot^2 B - \cot^2 A) = \dfrac{\sin A}{\sin B} - \dfrac{\sin B}{\sin A}$

B 3 Trigonometric Equations

In contrast to trigonometric identities where the solution set is the whole domain, the solution set for a trigonometric equation is a subset of the domain. Thus, just as the quadratic equation

$$x^2 - 5x + 6 = 0, \quad x \in \text{Re}$$

has the solution set $\{2, 3\}$; so the trigonometric equation

$$\sin x = 0, \quad x \in \text{Re}$$

has the solution set

$$\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\} = \{n\pi, n \in \text{I}\},$$

as we saw in Chapter 12.

To find the solution set of a trigonometric equation, we first find the solutions in the interval of the domain from 0 to 2π. We can then state the complete solution set by using the fact that the trigonometric functions are periodic. Thus,

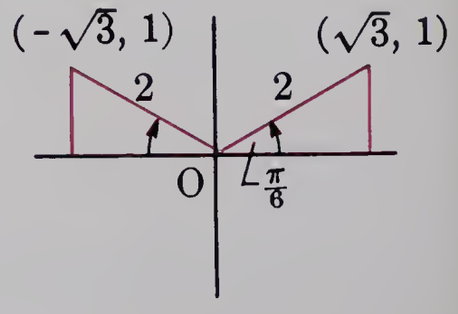
$$\sin x = 0, \text{ if } x = 0 \text{ and if } x = \pi, \ 0 \leq x < 2\pi;$$

therefore, by using the fact that $\sin(x + 2n\pi) = \sin x$, we see that the solution set is

$$\{0 + 2n\pi, \ \pi + 2n\pi, \ n \in \text{I}\} = \{n\pi, \ n \in \text{I}\}.$$

Example 1

Find the solution set of $\sin x = \frac{1}{2}$.



Solution: $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin(\pi - \frac{\pi}{6}) = \sin \frac{5\pi}{6} = \frac{1}{2}$
(angles in first and second quadrants)
$\therefore$ the solution set of $\sin x = \frac{1}{2}$ is

$$\{\tfrac{\pi}{6} + 2n\pi, \ \tfrac{5\pi}{6} + 2n\pi, \ n \in \text{I}.\}$$

Example 2

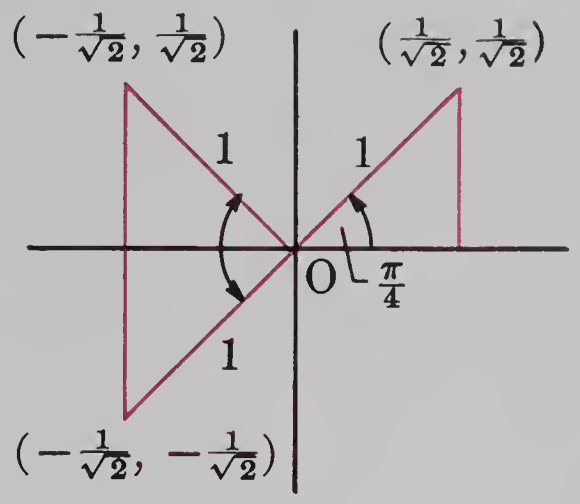
Find the solution set of

$$5\cos x-\sqrt{2}=3\cos x-2\sqrt{2}.$$

Solution: $5\cos x-\sqrt{2}=3\cos x-2\sqrt{2}$ (1)

$\therefore \quad 2\cos x=-\sqrt{2}$

and $\quad \cos x=-\frac{1}{\sqrt{2}}$. (2)



Now $\cos x$ is negative in the second and third quadrants and

$$|\cos x|=\tfrac{1}{\sqrt{2}}.$$

But $\quad \cos\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}}$;

$\therefore \quad x=\pi-\tfrac{\pi}{4}=\tfrac{3\pi}{4}$

and $\quad x=\pi+\tfrac{\pi}{4}=\tfrac{5\pi}{4}$

are two solutions of (2).

$\therefore$ the solution set of (1) is

$$\{\tfrac{3\pi}{4}+2n\pi,\ \tfrac{5\pi}{4}+2n\pi,\ n\in \mathrm{I}\}=\{(2n+\tfrac{3}{4})\pi,\ (2n+\tfrac{5}{4})\pi,\ n\in \mathrm{I}\}.$$

Example 3

Find the solution set of $\tan 2\theta=.6832$, using the tables to express 2θ to the nearest degree.

Solution: .6832 is positive; therefore the two values of 2θ in the interval $0<2\theta<360°$ lie in the first and third quadrants.

From the tables,

$$2\theta=34° \quad \text{(to nearest degree)},$$

which is in the first quadrant.

The second value of $2\theta<360°$ is given by

$$2\theta=34°+180°=214°,$$

which is in the third quadrant.

The solution set for 2θ is

$$\{34°+n(360°),\ 214°+n(360°),\ n\in \mathrm{I}\};$$

$\therefore$ the solution set for θ is

$$\{17°+n(180)°,\ 107°+n(180)°,\ n\in \mathrm{I}\}.$$

Note: This answer could also be written $\{17°+m(90)°,\ m\in \mathrm{I}\}$,

because $\{17°+n(180)°,\ n\in \mathrm{I}\}=\{17°+2n(90)°,\ n\in \mathrm{I}\}$

and $\quad \{107°+n(180°),\ n\in \mathrm{I}\}=\{17°+(2n+1)(90)°,\ n\in \mathrm{I}\}.$

EXERCISE B—3

(Give solutions to the nearest degree.)

Find the solution sets:

1. $\cos \theta = -1$
2. $\sin \theta = \frac{1}{\sqrt{2}}$
3. $\tan \theta = \frac{1}{\sqrt{3}}$
4. $\cot \theta = -1$
5. $\csc \theta = 2$
6. $\sec \theta = 2$
7. $\sin 3\theta = -.9675$
8. $\cos 2\theta = .6145$
9. $\tan 3\theta = -.5100$
10. $\cos 5\theta = -.7660$
11. $4 \sin \theta = 3 \cos \theta$
12. $2 \sec \theta = \csc \theta$
13. $3 \cos \theta - .2317 = \cos \theta + .2532$
14. $4 \tan \theta + .3825 = \tan \theta - .4215$
15. $\sin 2\theta = \sin(90° - 2\theta)$
16. $\tan \theta = 2 \cot \theta$
17. $6 \sin^2 x - 5 \sin x + 1 = 0$
18. $4 \cos^2 x + 3 \sin x - 1 = 0$
19. $\tan^2 x - 2.5 \tan x + 1 = 0$
20. $2 \sin^2 x + \sqrt{3} \sin x - 3 = 0$
21. $\tan^2 x - 2 \tan x + 1 = 0$
22. $\cos^2 x + 5 \cos x + 6 = 0$
23. $\cot^2 x + \cot x + 1 = 0$
24. $\tan x \cos x = 1$
25. (a) Why are there inadmissible values for $\sin x$ in (20)?
    (b) Why are there not two sets of solutions in (21) as there are in (19)?
    (c) Why do (22), (23), and (24) have the solution set ϕ, the empty set?

Appendix C

Angular Velocity and Simple Harmonic Motion

C 1 Angular Velocity

In Section 12-2 we pointed out that rotating objects like the earth or a merry-go-round turn through increasingly large angles as time progresses. Such rotating objects are said to rotate with an *angular velocity.* This corresponds to motion in a straight line, when a body which moves x miles in t hours is travelling with an average *linear velocity* of $\frac{x}{t}$ miles per hour.

In the case of angular motion (rotation), if the body rotates through an angle of θ radians in t minutes, then it rotates with an average angular velocity of $\frac{\theta}{t}$ radians per minute. Frequently angular velocity is stated in revolutions per minute (abbreviation, *r.p.m.*). When we note that one complete revolution is 360° or 2π radians, we obtain the following result:

n revolutions per minute $= 2\pi n$ radians per minute.

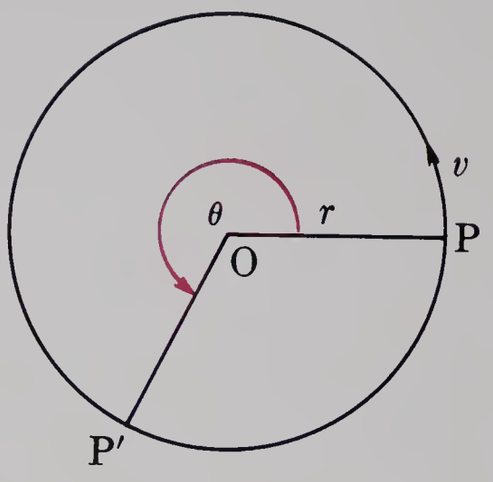
The Greek letter ω (omega) is frequently used to represent the angular velocity.

Example 1

A gyroscope rotates at 500 r.p.m. What is its angular velocity in radians per minute?

Solution:

$$1 \text{ r.p.m.} = 2\pi \text{ rad/min.}$$
$$\therefore 500 \text{ r.p.m.} = 1000\pi \text{ rad/min.}$$
$$\simeq 3142 \text{ rad/min.}$$



If we consider a body starting at P and moving in a counterclockwise direction on a circle, centre O, of radius r feet, with a *constant* speed of v feet per second, then in t seconds it moves from P to P′, a distance vt feet along the circle where the angle POP′ is θ.

$$\text{Arc length} = r\theta = \text{distance travelled} = vt$$

$$\therefore \theta = \frac{v}{r}t$$

and we find $\frac{\theta}{t} = \frac{v}{r}$.

Now $\frac{\theta}{t}$ is equal to the *constant* angular velocity, ω, in radians per second, and so $\frac{v}{r}$ is also the angular velocity, ω, in radians per second; that is,

$$\omega = \frac{\theta}{t} = \frac{v}{r}.$$

Example 2

A car travels around a circular race track at 60 miles per hour. The radius of the track is 1000 feet. What is the angular velocity of the car about the centre of the track in radians per minute?

Solution: A speed of 60 miles per hour is equivalent to a speed of 88 feet per second, or 88×60 feet per minute.

Let ω represent the angular velocity in radians per minute.

$$\text{Then } \omega = \frac{88 \times 60}{1000}$$
$$= 5.28.$$

The angular velocity is 5.28 radians per minute.

EXERCISE C—1

(If the result is not exact, then round off to 2 figures.)

1. A 36″-diameter wheel rotates at 10 revolutions per minute. At what speed is a point on the rim moving in feet per second?

2. A weight is moving in a horizontal circle of radius 6 feet with a constant speed of 66 feet per second. What is its angular velocity in radians per second? in revolutions per minute?

3. A car wheel is 18 inches in radius, and the car travels at 60 miles per hour. What is the angular velocity of the wheel (a) in radians per second, (b) in revolutions per minute? (The speed of a point on the rim of the wheel relative to the axle is equal to the speed of the car.)

4. A pulley which is 18 inches in diameter rotates at 300 revolutions per minute. What are the angular velocities of a point on the rim and of a point 6 inches from the centre? What is the linear speed of the point on the rim of the pulley? of the point 6 inches from the centre?

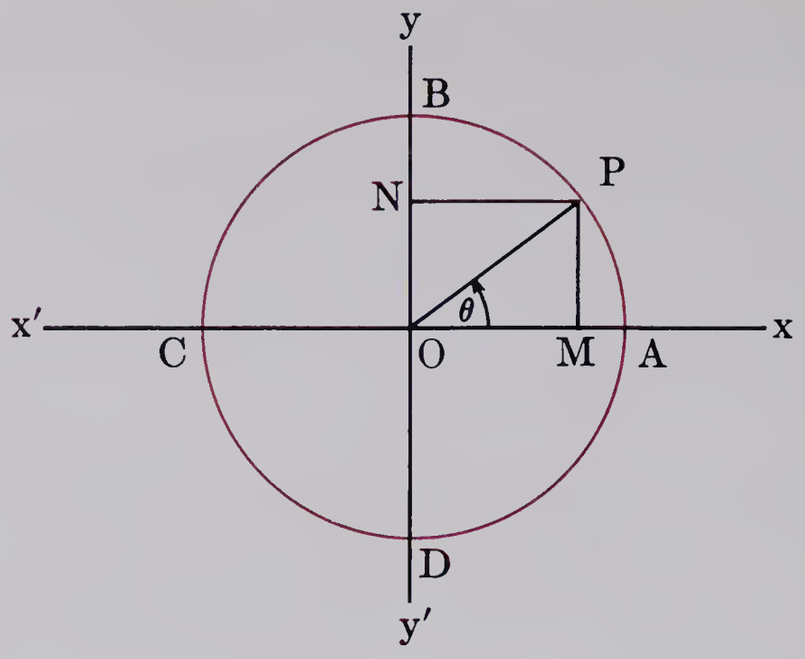
5. A wheel which is 12 inches in diameter is turned by a belt drive which moves with a linear speed of 540 feet per minute. Assuming that the belt does not slip, what is the number of revolutions per minute of the wheel?

6. A wheel with diameter 5 feet drives a second wheel with diameter 3 feet by means of a nonslipping belt. If the angular velocity of the large wheel is 10 revolutions per minute, what is (a) the speed of the belt in feet per minute and (b) the angular velocity of the driven wheel?

7. Two gears with *small* teeth have radii of 3 inches and 6 inches respectively. The smaller gear is driven at 200 revolutions per minute and it drives the second gear. What is the linear speed of (a) a point on the rim of the first gear, (b) a point on the rim of the second gear? What is the angular velocity of the second gear? (Assume that the gears can be considered as circles in contact without teeth and with no slipping between them.)

8. A satellite is in orbit 400 miles above the earth's surface. The orbit is completed in 1.5 hours. What is the satellite's angular velocity in degrees per minute and its orbital speed in miles per hour? The earth's radius can be taken as 4,000 miles.

C 2 Simple Harmonic Motion



Let P be any point on the circle ABCD, centre at the origin O, and with unit radius; the equation of the circle is $x^2+y^2=1$.

If P represents the position of a particle moving along the circle, then at any time t, $\angle AOP=\theta$, and, if the particle moves at constant speed, the angular velocity ω of OP is constant; therefore

$$\omega=\frac{\theta}{t} \text{ and } \theta=\omega t.$$

Now consider the point M:

$$OM=OP\cos\theta.$$
$$\therefore x=1\times\cos\theta.$$
$$x=\cos\omega t.$$

Thus the distance x is a function of the angle $\theta=\omega t$ and so of the time t. Note that $\omega=\frac{\theta}{t}$ is the angular velocity of OP in radians per unit time. The motion of a particle represented by the point M is said to be *simple harmonic* on the line x′Ox. Note OM is the projection of OP on x′Ox; that is, the length of the projection of OP is given by the value of the trigonometric function.

If we consider the point N in the same way, we then find that

$$ON=y=\sin\omega t.$$

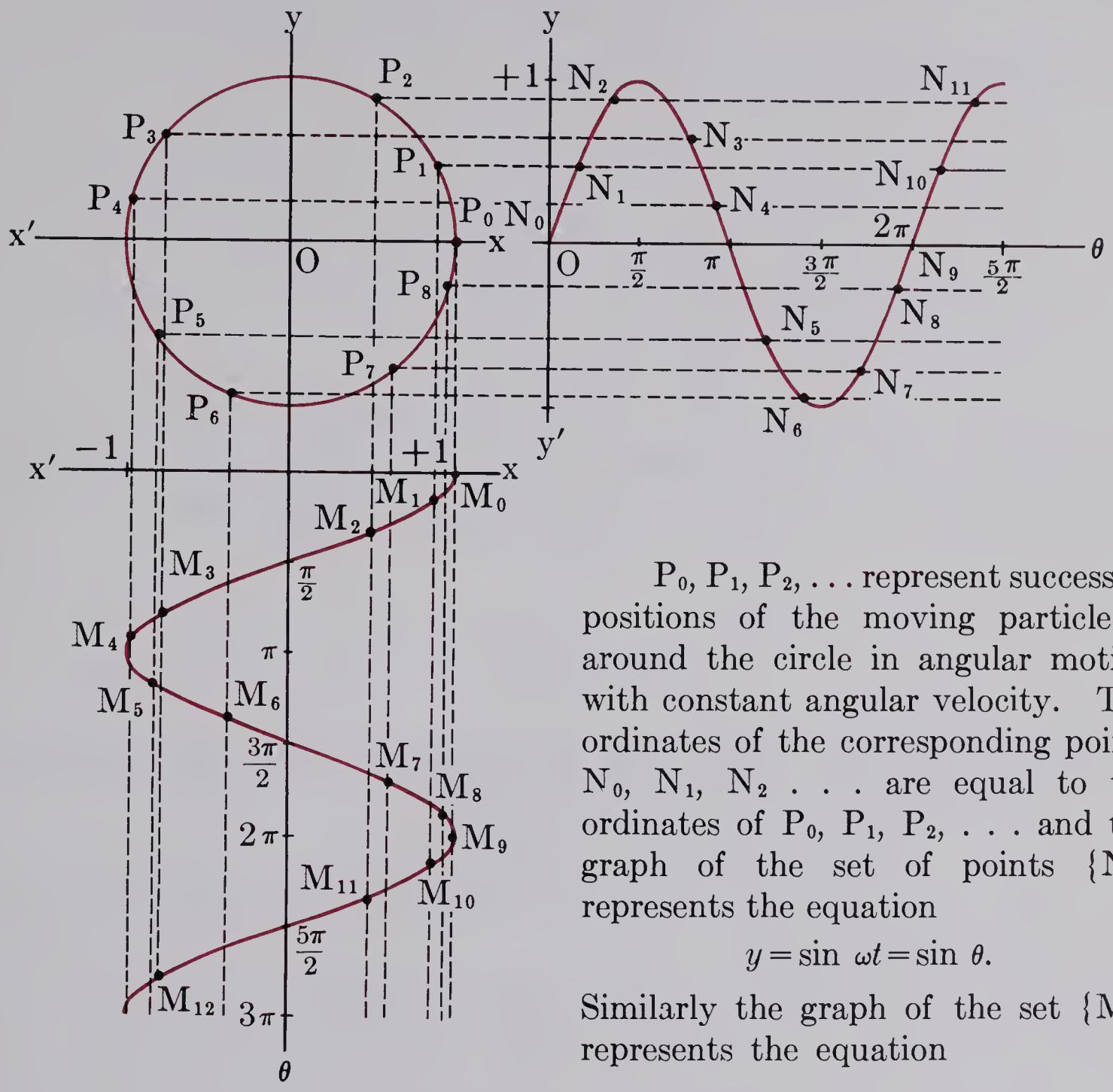
The motion of a particle represented by N is said to be *simple harmonic* on the line y′Oy, and ON is the projection of OP on y′Oy.

We note that the expressions for x and y are functions of t and that x leads y by a phase angle of $\frac{\pi}{2}$ or equivalently y lags x by a phase angle $\frac{\pi}{2}$;

$$x=\cos\omega t=\sin\left(\omega t+\tfrac{\pi}{2}\right) \text{ and } y=\cos\left(\omega t-\tfrac{\pi}{2}\right)=\sin\omega t.$$

We can illustrate the interrelation of the angular motion of a point P about O and the simple harmonic motion of its projections graphically as follows:

Figure C-1



$P_0, P_1, P_2, \ldots$ represent successive positions of the moving particle P around the circle in angular motion with constant angular velocity. The ordinates of the corresponding points $N_0, N_1, N_2 \ldots$ are equal to the ordinates of $P_0, P_1, P_2, \ldots$ and the graph of the set of points $\{N_i\}$ represents the equation

$$y = \sin\ \omega t = \sin\ \theta.$$

Similarly the graph of the set $\{M_i\}$ represents the equation

$$x = \cos\omega t = \cos\ \theta.$$

DEFINITION: If x is the distance of a particle on a fixed line from a fixed point on the line and

$$x = a\ \sin\ \omega(t+d),$$

where a, d, and ω are constants and t is the time measured in suitable units, then the motion of the particle is a *simple harmonic motion* with

$$\text{amplitude} = a$$

$$\text{period} = \frac{2\pi}{\omega}$$

and phase shift $= d$ relative to $\sin\ \omega t$.

In the example (Figure C-1) we have two special cases:

the motion of ON has amplitude $=1$

$$\text{period} = \frac{2\pi}{\omega}$$

$$\text{phase shift} = 0 \quad \text{relative to } \sin\omega t;$$

the motion of OM has amplitude $=1$

$$\text{period} = \frac{2\pi}{\omega}$$

$$\text{phase shift} = \frac{\pi}{2\omega} \quad \text{relative to } \sin\omega t.$$

Note that the phase angle is $\frac{\pi}{2}$.

We note that, in the case of the particle represented by M, the motion is on the line x′Ox and is confined between the points C and A, and the amplitude is the maximum distance of M from the central point O of the motion and is equal to either of the distances OA or OC.

EXERCISE C—2

1. Sketch a diagram like Figure C-1 to illustrate the simple harmonic motions given by $x = 2\cos\pi t$ and $y = 2\sin\pi t$.

2. Show by a diagram that if the angular motion starts at (0, 1) on a circle with angular velocity ω in the xy-plane, then the simple harmonic motion equations are

$$x = \cos\left(\omega t + \frac{\pi}{2}\right) = -\sin\omega t,$$

$$y = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos\omega t.$$

What is the phase angle relative to $\sin\omega t$ and the phase shift relative to $\sin\ \omega t$?

3. If the angular motion on a unit circle (i.e., a circle of unit radius) in the xy-plane starts at a point P_0 such that the angle between OP_0 and the x-axis is α, show that

$$x = \cos(\omega t + \alpha) \text{ and } y = \sin(\omega t + \alpha).$$

What is the phase angle relative to $\cos\ \omega t$ and the phase shift relative to $\cos\ \omega t$?

REVIEW TEST 1

(Chapters 1 and 2)

1. Which of the eleven properties of the real numbers listed in Chapter 1 are possessed by the set of integers I?

2. If $S = ut + \frac{1}{2}ut^2$, solve for u.

3. Find $\{x \in \text{Re} | 5 > |5x - 4|\}$.
   Graph the set on a number line.

4. Graph the ordered pairs in the Cartesian product $R \times S$
   if $R = \{x \in I | 5 > |x|\}$
   and $S = \{x \in I | 5 \leq x \leq 8\}$.

5. (a) From $R \times S$ in (4), select the ordered pairs in the relation whose rule of formation is
   $$y = x + 4.$$
   (b) State the domain and range for the relation in (a).
   (c) Graph the relation.

6. (a) Under the headings intercepts, domain, range, and symmetry, discuss the relation denoted by
   $$T = \{(x, y) | y^2 = -x;\ x \geq -16,\ x \in \text{Re}\}.$$
   (b) Draw the graph of T.

7. Classify the following systems of linear equations as dependent, independent, or inconsistent. Find a unique solution where possible. $(x, y \in \text{Re})$
   (a) $5x + 4y = 7,\ x - 3y = 9$
   (b) $5x + 2y = 0,\ 15x + 6y = 0$
   (c) $x - 3y = 4,\ x = 3y + 2$
   (d) $5x - 3y = \frac{7}{4},\ 2x + 8y = 3$

8. Given that $x, y \in \text{Re}$, graph the ordered pairs in the set
   $$\{(x, y) | x - y + 4 > 0\} \cap \{(x, y) | x + 2y - 11 < 0\} \cap \{(x, y) | y \geq 0\}.$$

9. Under the headings intercepts, domain, and range, discuss the function f defined by
   $$f(x) = -3(x - 1)^2,\ -1 \leq x \leq 3,\ x \in I.$$
   Draw the graph of the function.

10. (a) If function g is defined by

$$g(x)=\frac{1}{x^2+1},\ x\in \mathrm{Re},\ \text{find}$$

(1) $g(0)$ (2) $g(1)$ (3) $g(-1)$

(4) $g(3)$ (5) $g(-3)$ (6) $g(7)$

(b) What is the range of g?

(c) What appears to be the maximum value of $g\ (x)$?

11. State the slope and intercepts for lines represented by the following equations:

(a) $5x-2y=10$ (b) $y=3x-4$

(c) $x-y=2$ (d) $5x+y=0$

12. Find equations for lines with

(a) $m=3,\ b=\frac{1}{2}$ (b) $m=-\frac{5}{2},\ a=\frac{4}{5}$

(c) $\theta=35°$, (1, 1) on line (d) $\theta=135°,\ b=-2.$

13. (a) Write an equation for the family of lines with slope -5.

(b) From the family in (a), select the line passing through $(-5,\ -8)$.

14. (a) For the function defined by

$$f(x)=(x-2)(x-3)(x-5),\ x\in \mathrm{Re},$$

find $f(2)$, $f(3)$, and $f(5)$.

(b) What are the x-intercepts of the graph?

(c) What is the sign of $f(x)$ for values of x

(i) >5? (ii) <2?

(d) Indicate in a sketch the probable graph of this function.

15. The volume of a sphere is given by $v=\frac{4}{3}\pi r^3$, where v is the volume in cubic inches and r is the radius in inches. Use set-builder notation to represent the function with this rule of formation. If $\pi\simeq 3.14$, find $f(1)$ and $f(3)$. What is the minimum value of $f(r)$?

REVIEW TEST 2

(Chapters 3, 4, and 5)

1. Simplify:

(a) $(a^3b^2)\div(a^2b^5)$ (b) $(ab^2)\times(a^3b^4)$

(c) $(ab^2)^3\times(bc^3)^2$ (d) $(a^2b^{-3})\times(ab^{-2})$

(e) $(p^3q^2)\times(p^{-1}q^{-3})$ (f) $(x^{-1}y^{-2})\div(x^2y^{-1})$

(g) $\dfrac{2^3\times3^4\times7^2}{2\times3^5\times7^3}$ (h) $\dfrac{5^2\times2^{-3}}{3^2\times10^{-1}}$

(i) $(2a^3)^2\times(3b^2)^3$ (j) $(25xy)^2\div(5x^3y)$

2. Simplify:

(a) $p^{\frac{3}{2}}q^{\frac{1}{2}}p^{-\frac{1}{2}}q^3$ (b) $(ab^2)^{\frac{1}{2}}\times(a^2b^3)^{-\frac{1}{2}}$

(c) $(xy)^{\frac{1}{3}}\div(x^2y^3)^{\frac{1}{6}}$ (d) $(c^{\frac{2}{3}}d^{\frac{1}{3}})^2\times(cd^2)^{-\frac{1}{3}}$

(e) $(8)^{\frac{1}{3}}\times(9)^{\frac{1}{2}}\times(625)^{-\frac{1}{4}}$ (f) $(81)^{\frac{1}{2}}\div(32)^{\frac{1}{5}}$

3. Express the following numbers in scientific notation:

(a) 258,000 (b) .0027

(c) .0000865 (d) 184,300,000

(e) 28 million (f) 35 thousandths

(g) 5,178 (h) .04679

Express (a)–(d) in computer-standard notation.

4. Verify the following:

(a) $(x^2-y^2)^{\frac{1}{2}}(x+y)^{\frac{1}{2}}=(x+y)(x-y)^{\frac{1}{2}}$

(b) $(x^2-y^2)^{\frac{3}{2}}\div(x-y)^{\frac{1}{2}}=(x-y)(x+y)^{\frac{3}{2}}$

(c) $\dfrac{(x^2-y^2)(x+y)^{-2}}{(x-y)(x+y)^2}=\dfrac{1}{(x+y)^3}$

(d) $\dfrac{x^{\frac{3}{2}}-y^{\frac{3}{2}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}=x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$

5. Sketch the graphs of

(a) $y=(1.5)^x$, (b) $y=4^x$, (c) $y=.5^x$,

in the domain $\{x|-1\leq x\leq2\}$.

For what values of x are

(i) $(1.5)^x<4^x$? (ii) $4^x<.5^x$?

For what values of x are the expressions in (a), (b), and (c) less than 1?

6. Sketch the graph of the function

$$\{(x, y)|y=10^x;\ -1\leq x\leq1,\ x\in\mathrm{Re}\}$$

using the table of the exponential function to base 10 for any values required. Compare it with the graph of

$$x=\log_{10}y \text{ for } 1\leq y\leq10,$$

and so deduce that $y=10^{\log_{10}y}$ and $x=\log_{10}(10^x)$.

7. Use the laws of logarithms to show that

   (a) $\log_{10} s + \log_{10} t - \log_{10} v = \log_{10} \frac{st}{v}$

   (b) $\frac{1}{2}(4 \log_{10} a + 3 \log_{10} b) = \log_{10}(a^4 b^3)^{\frac{1}{2}}$.

8. Given that

$$\log_{10} 2 = .3010, \quad \log_{10} 3 = .4771, \quad \log_{10} 7 = .8451$$

   find, without the use of the tables, the values of:

   (a) $\log_{10} 42$ (b) $\log_{10} .35$

   (c) $\log_{10} 1.5$ (d) $\log_{10} 120$

   (e) $10^{2.6010}$ (f) $10^{-.6990}$

   (g) $10^{1.3222}$ (h) $10^{.7781} \div 10^{.8451}$

9. Evaluate the following using tables:

   (a) 27.50×1.305 (b) $46.34 \div 63.85$

   (c) $\dfrac{8.760 \times 355.5}{881.5 \times 47.00}$ (d) $\dfrac{50.92 \times 57.57 \times 62.28}{410.9 \times .6601}$

   (e) $.06988 \times .9283$ (f) $.3570 \div 76.59$

   (g) $\dfrac{25.8 \times .3743}{1.283 \times .06639}$ (h) $\dfrac{7.733 \times 6326}{53.99 \times .4865}$

10. Evaluate the following using tables:

    (a) $\sqrt{486.5}$ (b) $\sqrt[3]{23.06}$

    (c) $\sqrt[4]{.09429}$ (d) $\sqrt[5]{5.304}$

    (e) $(83.56)^{\frac{4}{3}}$ (f) $(.6554)^{\frac{3}{2}}$

    (g) $(19.33)^{\frac{1}{2}} \times (3.542)^{\frac{3}{5}}$ (h) $(9.212)^{\frac{3}{4}} \div (37.14)^{\frac{1}{3}}$

11. Find the amount at (a) simple interest and (b) compound interest of \$450 for 8 years at 3% annually.

12. Find the amount of \$1,200 for 12 years at 6% (a) compounded annually, (b) compounded quarterly.

13. If the interest rate is 8% compounded quarterly, find the equivalent annual rate.

14. Find the present value of \$800 due in 10 years at 5%.

15. Find the present value of \$6,500 due in $6\frac{1}{2}$ years at 5% compounded semi-annually.

REVIEW TEST 3

(Chapters 6, 7, and 8)

1. (a) Give the equation of the parabola of the form $Y = aX^2$ that you would use as an aid in sketching the parabola whose equation is
$$y+2 = -3(x-4)^2.$$
   (b) State X and Y in terms of x and y.
   (c) Sketch the two parabolas.
   (d) State the vertex and axis of symmetry of each parabola and whether the parabolas are concave upwards or downwards.
   (e) State whether the parabolas have minimum or maximum points at their vertex.
   (f) State the range of each parabola.

2. (a) For what value of x will the function defined by
$$f(x) = -x^2-3x+6$$
   have a maximum or minimum value? Specify which.
   (b) What is this maximum or minimum value?

3. Find any real roots of the equation $3x^2-4x-5=0$.

4. (a) Find $\cos x$ if $2\cos^2 x - 5\cos x + 2 = 0$.
   (b) Find any real numbers x in the range $0 < x < \frac{\pi}{2}$ which satisfy this equation.

5. Solve for x: (If the roots are not real, write them in the form $a+ib$.)
$$4x(x-1)+x(4x-1) = (4x-1)(x-2)$$

6. (a) What are the only *possible* linear factors of the polynomial
$$P(x) = x^3-6x^2+3x+6?$$
   (b) Use the Factor Theorem to decide whether $x+3$ is a factor of $P(x)$.

7. (a) Find (without solving) the sum and product of the roots of the equation
$$4x^2-9x-5=0.$$
   (b) Find k if one root of the equation
$$3x^2-7x+k=0$$
   is greater by 4 than the other.

8. The sum of the squares of two integers differing by 4 is 170. Find the integers.

9. Honest Otto can sell 200 cars at \$3000 each. For each car over 200 that he sells, he is willing to lower the price to all by \$10 a car. What number of cars sold will give him maximum receipts?

10. (a) Sketch the parabola whose equation is
$$y+5=4(x-2)^2$$
labelling clearly the vertex and axis of symmetry.

(b) Is the vertex a maximum point?

(c) Find the points where the parabola in (a) crosses the x- and y-axes.

(d) What is the range of y?

11. (a) Complete the square to write $-3x^2+2x-6$ in the form $a(x-m)^2+d$.

(b) For what value of x will the function $f(x)=-3x^2+2x-6$ have a maximum or minimum value?

(c) Find the maximum or minimum value (specify which) of the function.

12. A rectangular pigpen is surrounded by a fence and divided into two equal parts by a fence parallel to one side. Its perimeter is 56 rods and its area is a maximum. Find the dimensions of the two equal parts.

13. Describe (without solving) the roots of the equations

(a) $2x^2+4x-5=0$ (b) $2x^2+4x+5=0$.

14. Find any real values of x for which
$$\sqrt{x-3}+\sqrt{3x+4}=\sqrt{7+6x}.$$

15. Solve for x:
$$2^{2x}-2^x-12=0$$

16. Show that $x+1$ is a factor of $P(x)=4937x^{1837}+2285x^{1836}+2652$.

17. (a) Find, using two different methods, a monic quadratic equation whose roots are $\frac{4}{3}$ and $-\frac{1}{3}$.

(b) Find k if one root of the equation
$$4x^2-kx+6=0$$
is twice the other.

18. Honest Otto agreed to supply a certain number of cruisers to the police department for a total of \$120,000. The next year the price he charged per car rose by \$100 and the city ordered 2 fewer cars for the same amount of money. How many cars were ordered each year?

REVIEW TEST 4

(Chapters 10 and 11)

1. A chord 30 inches long is drawn in a circle of radius 17 inches. Calculate the distance of the chord from the centre.

2. PA is a tangent drawn to a circle from an external point P. If the radius of the circle is 12 cm. and PA = 16 cm., calculate the distance from P to the centre of the circle.

3. BDEF is a cyclic quadrilateral with BD = BF. ABC is the tangent at B and $\angle ABD = 40°$. Calculate the size of $\angle DEF$.

4. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

5. ABCD is a cyclic quadrilateral in which AD = CD. If $\angle BAC = 30°$ and $\angle ADB = 40°$, calculate the size of

   (i) $\angle DAC$, (ii) $\angle DCB$, (iii) $\angle ABC$.

6. PQ is a chord of a circle centre O. If the circle on OP as diameter intersects PQ at A, prove that A is the midpoint of PQ.

7. BC is a tangent to a circle with C the point of contact. A is any other point on the circle and AD is a chord parallel to BC. Prove that $\triangle CDA$ is isosceles.

8. PX and PY are tangents touching a circle, centre O, at X and Y. XO and PY are produced to meet at R and YO and PX are produced to meet at Q. Prove XY $\parallel$ QR.

9. Write equations of circles, centre the origin, with the following radii:

   (i) 9 (ii) $2\sqrt{5}$ (iii) $\frac{3}{2}$

10. Sketch the graphs of the regions determined by the following inequalities:

    (i) $x^2+y^2<16$, $x \geq 0$

    (ii) $x^2+y^2 \geq 25$, $0 \leq x < 6$, $0 \leq y < 6$

11. Find the length of the tangent drawn to the circle $x^2+y^2=13$ from the point $(5, -2)$.

12. Find the equation of the tangent at the point (3, 5) on the circle represented by $x^2+y^2=34$.

13. P(1, 2) is the midpoint of a chord of the circle represented by $x^2+y^2=25$. Find the equation of the chord and the length of the chord.

14. By writing the equation $x^2+y^2-2x-4y=20$ in the form $(x-a)^2+(y-b)^2=x^2$ and letting $X=x-a$ and $Y=y-b$, prove that the given equation is the equation of a circle, centre (1, 2) and radius 1 unit. Find the equation of the tangent at the point (4, 6) on the circle.

REVIEW TEST 5

(Chapters 12 and 13)

1. Find the values of the sine, cosine, and tangent of the following:

(a) $36°$	(b) $148°$	(c) $-159°$	(d) $249°$
(e) $-325°$	(f) $293°$	(g) $584°$	(h) $-407°$
(i) .2793	(j) $-.8901$	(k) -1.3439	(l) 2.0246
(m) $\frac{\pi}{12}$	(n) $\frac{17\pi}{6}$	(o) $-\frac{5\pi}{18}$	(p) $-\frac{22\pi}{9}$

2. Sketch the graphs of the following for $-4\pi \leq \theta \leq 4\pi$:

(a) $y=2\sin\theta$	(b) $y=\frac{1}{2}\cos\theta$	(c) $y=\sin 2\theta$
(d) $y=3\sin .5\theta$	(e) $y=\sin(\theta+\frac{2\pi}{3})$	(f) $y=4\sin(\theta-\frac{\pi}{4})$

List four values of θ for which (i) $y=\theta$, (ii) y has a maximum value, (iii) y has a minimum value.

3. State the amplitude and period of each of the following:

(a) $y=3.5\sin 2\theta$	(b) $y=.25\sin\frac{\theta}{4}$	(c) $y=\frac{1}{4}\sin\frac{3}{4}\theta$
(d) $y=120\sin 6\theta$	(e) $y=.05\sin 2\pi\theta$	(f) $y=2\pi\sin 3\theta$

4. State the amplitude and the phase angle relative to $\sin\theta$ of each of the following:

(a) $y=4\sin(\theta+\frac{\pi}{3})$	(b) $y=\sqrt{3}\sin(\theta-25°)$
(c) $y=.1\sin(\theta+\frac{5\pi}{6})$	(d) $y=\frac{2}{5}\sin(\theta+135°)$
(e) $y=350\sin(\theta-\frac{3\pi}{4})$	(f) $y=\frac{\pi}{3}\sin(\theta+90°)$

5. State the phase shift relative to $\sin\theta$ of each of the following:

(a) $y=\sin(\theta-40°)$	(b) $y=\sin(\theta+65°)$	(c) $y=\sin(\theta+\frac{3\pi}{2})$
(d) $y=\sin(\theta+\pi)$	(e) $y=\cos\theta$	(f) $y=-\cos\theta$

6. Solve the following triangles (if possible):

(a)	$a=25$,	$b=35$,	$c=45$	(b)	$a=25$,	$b=45$,	$C=50°$
(c)	$A=35°$,	$B=75°$,	$c=60$	(d)	$b=36$,	$c=67$,	$A=125°$
(e)	$a=21$,	$c=33$,	$C=48°$	(f)	$b=12$,	$a=10$,	$A=65°$
(g)	$a=26$,	$b=17$,	$c=45$	(h)	$c=17$,	$b=13$,	$B=115°$
(i)	$a=43$,	$b=71$,	$B=143°$	(j)	$A=27°$,	$C=117°$,	$c=37$
(k)	$p=3.5$,	$R=57°$	$Q=41°$	(l)	$x=1.4$,	$y=.9$,	$z=2.1$
(m)	$e=43$,	$f=35$,	$G=61°$	(n)	$d=12$,	$e=17$,	$F=102°$
(o)	$p=7.3$,	$q=4.1$,	$Q=74°$	(p)	$e=5.3$,	$f=2.4$,	$F=117°$
(q)	$a=4.7$,	$b=6.7$,	$B=133°$	(r)	$x=3.5$,	$y=8.9$,	$Z=32°$
(s)	$r=7.3$,	$s=2.7$,	$t=4.1$	(t)	$a=1.7$,	$b=2.4$,	$B=42°$

7. An airplane flies due north for one hour at a ground speed of 300 miles per hour. It then heads N60°E at the same speed for 40 minutes. How far is it from the starting point and what is its bearing from its starting point?

8. Two radar stations are 100 miles apart on a northbound coastline. The northern station reports a ship at 150 miles and the southern station reports the same ship at 120 miles. What is the bearing of the ship from each station? (Assume a flat earth.)

9. Two fire ranger stations sight a fire at bearings of S35°W and S43°E respectively. The stations lie on a due east-west line and are 15 miles apart. How far is the fire from each station? (Assume a flat earth.)

10. A surveyor measures a line of 2 miles due east from a previously fixed point. He is then out of sight of the fixed point, but he turns through an acute angle and after walking 1.5 miles in a line he is seen from the fixed point at a bearing of 123° through East from North. How far is he from the fixed point? (Assume negligible changes in heights.)

REVIEW TEST 6

(Chapter 14)

1. For the sequence whose nth term is $(n+1)^2$, write

(a) t_1	(b) t_{99}	(c) t_{10}
(d) t_{n-1}	(e) t_{2k}	(f) t_{2k-5}

2. Discover a probable rule of formation for the sequence

$$1\tfrac{1}{4},\ 2,\ 2\tfrac{3}{4},\ 3\tfrac{1}{2},\ \ldots$$

Use the rule to write the next two terms and the nth term.

3. Find the sum of the first four terms of the series for which $t_n = n + \frac{2}{n}$.

4. In an arithmetic sequence, $a=2$, $d=3$.
   (a) Find n if $t_n = 110$.
   (b) Find the sum of 100 terms of the corresponding series.

5. Find p so that $4p-3$, $4p+5$, and $8p+7$ form an arithmetic sequence.

6. In the geometric sequence $4, -2, 1, \ldots$, which is $\frac{1}{64}$?

7. For the geometric series $9, -3, 1, -\frac{1}{3}, \ldots$, find n if $S_n = \frac{182}{27}$.

8. What amount invested at the end of each year for 10 years will accumulate to \$1000 immediately after the last deposit has been made if the rate of interest is 5% per annum compounded semi-annually?

9. Find the present value of an annuity of \$600 which begins now and runs for 7 years if money is worth 6% per annum compounded quarterly.

10. A man plans to repay a loan by making 15 semi-annual payments of \$500 starting on January 1st. 1969. Find the equivalent cash value of these payments on January 1st, 1966 if the prevailing rate of interest is 4% per annum compounded semi-annually.

POWERS, ROOTS, AND RECIPROCALS 1–100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3,420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

TABLE OF SINES, COSINES, TANGENTS

Angle		Sine	Cosine	Tangent
Degrees	Radians			
0°	.0000	.0000	1.0000	.0000
1°	.0175	.0175	.9998	.0175
2°	.0349	.0349	.9994	.0349
3°	.0524	.0523	.9986	.0524
4°	.0698	.0698	.9976	.0699
5°	.0873	.0872	.9962	.0875
6°	.1047	.1045	.9945	.1051
7°	.1222	.1219	.9925	.1228
8°	.1396	.1392	.9903	.1405
9°	.1571	.1564	.9877	.1584
10°	.1745	.1736	.9848	.1763
11°	.1920	.1908	.9816	.1944
12°	.2094	.2079	.9781	.2126
13°	.2269	.2250	.9744	.2309
14°	.2443	.2419	.9703	.2493
15°	.2618	.2588	.9659	.2679
16°	.2793	.2756	.9613	.2867
17°	.2967	.2924	.9563	.3057
18°	.3142	.3090	.9511	.3249
19°	.3316	.3256	.9455	.3443
20°	.3491	.3420	.9397	.3640
21°	.3665	.3584	.9336	.3839
22°	.3840	.3746	.9272	.4040
23°	.4014	.3907	.9205	.4245
24°	.4189	.4067	.9135	.4452
25°	.4363	.4226	.9063	.4663
26°	.4538	.4384	.8988	.4877
27°	.4712	.4540	.8910	.5095
28°	.4887	.4695	.8829	.5317
29°	.5061	.4848	.8746	.5543
30°	.5236	.5000	.8660	.5774
31°	.5411	.5150	.8572	.6009
32°	.5585	.5299	.8480	.6249
33°	.5760	.5446	.8387	.6494
34°	.5934	.5592	.8290	.6745
35°	.6109	.5736	.8192	.7002
36°	.6283	.5878	.8090	.7265
37°	.6458	.6018	.7986	.7536
38°	.6632	.6157	.7880	.7813
39°	.6807	.6293	.7771	.8098
40°	.6981	.6428	.7660	.8391
41°	.7156	.6561	.7547	.8693
42°	.7330	.6691	.7431	.9004
43°	.7505	.6820	.7314	.9325
44°	.7679	.6947	.7193	.9657
45°	.7854	.7071	.7071	1.0000

TABLE OF SINES, COSINES, TANGENTS

Angle		Sine	Cosine	Tangent
Degrees	Radians			
45°	.7854	.7071	.7071	1.0000
46°	.8029	.7193	.6947	1.0355
47°	.8203	.7314	.6820	1.0724
48°	.8378	.7431	.6691	1.1106
49°	.8552	.7547	.6561	1.1504
50°	.8727	.7660	.6428	1.1918
51°	.8901	.7771	.6293	1.2349
52°	.9076	.7880	.6157	1.2799
53°	.9250	.7986	.6018	1.3270
54°	.9425	.8090	.5878	1.3764
55°	.9599	.8192	.5736	1.4281
56°	.9774	.8290	.5592	1.4826
57°	.9948	.8387	.5446	1.5399
58°	1.0123	.8480	.5299	1.6003
59°	1.0297	.8572	.5150	1.6643
60°	1.0472	.8660	.5000	1.7321
61°	1.0647	.8746	.4848	1.8040
62°	1.0821	.8829	.4695	1.8807
63°	1.0996	.8910	.4540	1.9626
64°	1.1170	.8988	.4384	2.0503
65°	1.1345	.9063	.4226	2.1445
66°	1.1519	.9135	.4067	2.2460
67°	1.1694	.9205	.3907	2.3559
68°	1.1868	.9272	.3746	2.4751
69°	1.2043	.9336	.3584	2.6051
70°	1.2217	.9397	.3420	2.7475
71°	1.2392	.9455	.3256	2.9042
72°	1.2566	.9511	.3090	3.0777
73°	1.2741	.9563	.2924	3.2709
74°	1.2915	.9613	.2756	3.4874
75°	1.3090	.9659	.2588	3.7321
76°	1.3265	.9703	.2419	4.0108
77°	1.3439	.9744	.2250	4.3315
78°	1.3614	.9781	.2079	4.7046
79°	1.3788	.9816	.1908	5.1446
80°	1.3963	.9848	.1736	5.6713
81°	1.4137	.9877	.1564	6.3138
82°	1.4312	.9903	.1392	7.1154
83°	1.4486	.9925	.1219	8.1443
84°	1.4661	.9945	.1045	9.5144
85°	1.4835	.9962	.0872	11.4301
86°	1.5010	.9976	.0698	14.3007
87°	1.5184	.9986	.0523	19.0811
88°	1.5359	.9994	.0349	28.6363
89°	1.5533	.9998	.0175	57.2900
90°	1.5708	1.0000	.0000	undefined

TABLE OF COTANGENTS, SECANTS, COSECANTS

Angle		Cotangent	Secant	Cosecant
Degrees	Radians			
0°	.0000	undefined	1.000	undefined
1°	.0175	57.2900	1.000	57.30
2°	.0349	28.6363	1.001	28.65
3°	.0524	19.0811	1.001	19.11
4°	.0698	14.3007	1.002	14.34
5°	.0873	11.4301	1.004	11.47
6°	.1047	9.5144	1.006	9.567
7°	.1222	8.1443	1.008	8.206
8°	.1396	7.1154	1.010	7.185
9°	.1571	6.3138	1.012	6.392
10°	.1745	5.6713	1.015	5.759
11°	.1920	5.1446	1.019	5.241
12°	.2094	5.7046	1.022	4.810
13°	.2269	4.3315	1.026	4.445
14°	.2443	4.0108	1.031	4.134
15°	.2618	3.7321	1.035	3.864
16°	.2793	3.4874	1.040	3.628
17°	.2967	3.2709	1.046	3.420
18°	.3142	3.0777	1.051	3.236
19°	.3316	2.9042	1.058	3.072
20°	.3491	2.7475	1.064	2.924
21°	.3665	2.6051	1.071	2.790
22°	.3840	2.4751	1.079	2.669
23°	.4014	2.3559	1.086	2.559
24°	.4189	2.2460	1.095	2.459
25°	.4363	2.1445	1.103	2.366
26°	.4538	2.0503	1.113	2.281
27°	.4712	1.9626	1.122	2.203
28°	.4887	1.8807	1.133	2.130
29°	.5061	1.8040	1.143	2.063
30°	.5236	1.7321	1.155	2.000
31°	.5411	1.6643	1.167	1.942
32°	.5585	1.6003	1.179	1.887
33°	.5760	1.5399	1.192	1.836
34°	.5934	1.4826	1.206	1.788
35°	.6109	1.4281	1.221	1.743
36°	.6283	1.3764	1.236	1.701
37°	.6458	1.3270	1.252	1.662
38°	.6632	1.2799	1.269	1.624
39°	.6807	1.2349	1.287	1.589
40°	.6981	1.1918	1.305	1.556
41°	.7156	1.1504	1.325	1.524
42°	.7330	1.1106	1.346	1.494
43°	.7505	1.0724	1.367	1.466
44°	.7679	1.0355	1.390	1.440
45°	.7854	1.0000	1.414	1.414

TABLE OF COTANGENTS, SECANTS, COSECANTS

Angle		Cotangent	Secant	Cosecant
Degrees	Radians			
45°	.7854	1.0000	1.414	1.414
46°	.8029	.9657	1.440	1.390
47°	.8203	.9325	1.466	1.367
48°	.8378	.9004	1.494	1.346
49°	.8552	.8693	1.524	1.325
50°	.8727	.8391	1.556	1.305
51°	.8901	.8098	1.589	1.287
52°	.9076	.7813	1.624	1.269
53°	.9250	.7536	1.662	1.252
54°	.9425	.7265	1.701	1.236
55°	.9599	.7002	1.743	1.221
56°	.9774	.6745	1.788	1.206
57°	.9948	.6494	1.836	1.192
58°	1.0123	.6249	1.887	1.179
59°	1.0297	.6009	1.942	1.167
60°	1.0472	.5774	2.000	1.155
61°	1.0647	.5543	2.063	1.143
62°	1.0821	.5317	2.130	1.133
63°	1.0996	.5095	2.203	1.122
64°	1.1170	.4877	2.281	1.113
65°	1.1345	.4663	2.366	1.103
66°	1.1519	.4452	2.459	1.095
67°	1.1694	.4245	2.559	1.086
68°	1.1868	.4040	2.669	1.079
69°	1.2043	.3839	2.790	1.071
70°	1.2217	.3640	2.924	1.064
71°	1.2392	.3443	3.072	1.058
72°	1.2566	.3249	3.236	1.051
73°	1.2741	.3057	3.420	1.046
74°	1.2915	.2867	3.628	1.040
75°	1.3090	.2679	3.864	1.035
76°	1.3265	.2493	4.134	1.031
77°	1.3439	.2309	4.445	1.026
78°	1.3614	.2126	4.810	1.022
79°	1.3788	.1944	5.241	1.019
80°	1.3963	.1763	5.759	1.015
81°	1.4137	.1584	6.392	1.012
82°	1.4312	.1405	7.185	1.010
83°	1.4486	.1228	8.206	1.008
84°	1.4661	.1051	9.567	1.006
85°	1.4835	.0875	11.47	1.004
86°	1.5010	.0699	14.34	1.002
87°	1.5184	.0524	19.11	1.001
88°	1.5359	.0349	28.65	1.001
89°	1.5533	.0175	57.30	1.000
90°	1.5708	.0000	undefined	1.000

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

THE EXPONENTIAL FUNCTION TO BASE 10

	0	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155

THE EXPONENTIAL FUNCTION TO BASE 10

	0	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977

COMPOUND AMOUNT OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	Yrs.
1	1.01000	1.01250	1.01500	1.01750	1.02000	1.02500	1.03000	1.03500	1
2	1.02010	1.02516	1.03023	1.03531	1.04040	1.05063	1.06090	1.07123	2
3	1.03030	1.03797	1.04568	1.05342	1.06121	1.07689	1.09273	1.10872	3
4	1.04060	1.05094	1.06136	1.07185	1.08243	1.10381	1.12551	1.14752	4
5	1.05101	1.06408	1.07728	1.09062	1.10408	1.13141	1.15927	1.18769	5
6	1.06152	1.07738	1.09344	1.10970	1.12616	1.15969	1.19405	1.22926	6
7	1.07214	1.09085	1.10985	1.12912	1.14869	1.18869	1.22987	1.27228	7
8	1.08286	1.10449	1.12649	1.14888	1.17166	1.21840	1.26677	1.31681	8
9	1.09369	1.11829	1.14339	1.16899	1.19509	1.24886	1.30477	1.36290	9
10	1.10462	1.13227	1.16054	1.18944	1.21899	1.28008	1.34392	1.41060	10
11	1.11567	1.14642	1.17795	1.21026	1.24337	1.31209	1.38423	1.45997	11
12	1.12683	1.16075	1.19562	1.23144	1.26824	1.34489	1.42576	1.51107	12
13	1.13809	1.17526	1.21355	1.25299	1.29361	1.37851	1.46853	1.56396	13
14	1.14947	1.18995	1.23176	1.27492	1.31948	1.41297	1.51259	1.61869	14
15	1.16097	1.20483	1.25023	1.29723	1.34587	1.44830	1.55797	1.67535	15
16	1.17258	1.21989	1.26899	1.31993	1.37279	1.48451	1.60471	1.73399	16
17	1.18430	1.23514	1.28802	1.34303	1.40024	1.52162	1.65285	1.79468	17
18	1.19615	1.25058	1.30734	1.36653	1.42825	1.55966	1.70243	1.85749	18
19	1.20811	1.26621	1.32695	1.39044	1.45681	1.59865	1.75351	1.92250	19
20	1.22019	1.28204	1.34686	1.41478	1.48595	1.63862	1.80611	1.98979	20
21	1.23239	1.29806	1.36706	1.43954	1.51567	1.67958	1.86029	2.05943	21
22	1.24472	1.31429	1.38756	1.46473	1.54598	1.72157	1.91610	2.13151	22
23	1.25716	1.33072	1.40838	1.49036	1.57690	1.76461	1.97359	2.20611	23
24	1.26973	1.34735	1.42950	1.51644	1.60844	1.80873	2.03279	2.28333	24
25	1.28243	1.36419	1.45095	1.54298	1.64060	1.85394	2.09378	2.36324	25
26	1.29526	1.38124	1.47271	1.56998	1.67342	1.90029	2.15659	2.44596	26
27	1.30821	1.39851	1.49480	1.59746	1.70689	1.94780	2.22129	2.53157	27
28	1.32129	1.41599	1.51722	1.62541	1.74102	1.99650	2.28793	2.62017	28
29	1.33450	1.43369	1.53998	1.65386	1.77584	2.04641	2.35657	2.71188	29
30	1.34785	1.45161	1.56308	1.68280	1.81136	2.09757	2.42726	2.80679	30
31	1.36133	1.46976	1.58653	1.71225	1.84759	2.15001	2.50008	2.90503	31
32	1.37494	1.48813	1.61032	1.74221	1.88454	2.20376	2.57508	3.00671	32
33	1.38869	1.50673	1.63448	1.77270	1.92223	2.25885	2.65234	3.11194	33
34	1.40258	1.52557	1.65900	1.80372	1.96068	2.31532	2.73191	3.22086	34
35	1.41660	1.54464	1.68388	1.83529	1.99988	2.37320	2.81386	3.33359	35
36	1.43077	1.56394	1.70914	1.86741	2.03988	2.43253	2.89827	3.45026	36
37	1.44508	1.58349	1.73478	1.90009	2.08068	2.49334	2.98522	3.57102	37
38	1.45953	1.60329	1.76080	1.93334	2.12229	2.55568	3.07478	3.69601	38
39	1.47412	1.62333	1.78721	1.96717	2.16474	2.61957	3.16702	3.82537	39
40	1.48886	1.64362	1.81402	2.00160	2.20803	2.68506	3.26203	3.95925	40

COMPOUND AMOUNT OF 1

Yrs.	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	1.04000	1.04500	1.05000	1.06000	1.07000	1.08000	1.09000	1.10000	1
2	1.08160	1.09203	1.10250	1.12360	1.14490	1.16640	1.18810	1.21000	2
3	1.12486	1.14117	1.15763	1.19102	1.22504	1.25971	1.29503	1.33100	3
4	1.16986	1.19252	1.21551	1.26248	1.31080	1.36049	1.41158	1.46410	4
5	1.21665	1.24618	1.27628	1.33823	1.40255	1.46933	1.53862	1.61051	5
6	1.26532	1.30226	1.34010	1.41852	1.50073	1.58687	1.67710	1.77156	6
7	1.31593	1.36086	1.40710	1.50363	1.60578	1.71382	1.82804	1.94872	7
8	1.36857	1.42210	1.47746	1.59385	1.71819	1.85093	1.99256	2.14359	8
9	1.42331	1.48610	1.55133	1.68948	1.83846	1.99900	2.17189	2.35795	9
10	1.48024	1.55297	1.62889	1.79085	1.96715	2.15893	2.36736	2.59374	10
11	1.53945	1.62285	1.71034	1.89830	2.10485	2.33164	2.58043	2.85312	11
12	1.60103	1.69588	1.79586	2.01220	2.25219	2.51817	2.81266	3.13843	12
13	1.66507	1.77220	1.88565	2.13293	2.40985	2.71962	3.06580	3.45227	13
14	1.73168	1.85194	1.97993	2.26090	2.57853	2.93719	3.34173	3.79750	14
15	1.80094	1.93528	2.07893	2.39656	2.75903	3.17217	3.64248	4.17725	15
16	1.87298	2.02237	2.18287	2.54035	2.95126	3.42594	3.97031	4.59497	16
17	1.94790	2.11338	2.29202	2.69277	3.15882	3.70002	4.32763	5.05447	17
18	2.02582	2.20848	2.40662	2.85434	3.37993	3.99602	4.71712	5.55992	18
19	2.10685	2.30786	2.52695	3.02560	3.61653	4.31570	5.14166	6.11591	19
20	2.19112	2.41171	2.65330	3.20714	3.86968	4.66096	5.60441	6.72750	20
21	2.27877	2.52024	2.78596	3.39956	4.14056	5.03383	6.10881	7.40025	21
22	2.36992	2.63365	2.92526	3.60354	4.43040	5.43654	6.65860	8.14027	22
23	2.46472	2.75217	3.07152	3.81975	4.74053	5.87146	7.25787	8.95430	23
24	2.56330	2.87601	3.22510	4.04893	5.07237	6.34118	7.91108	9.84973	24
25	2.66584	3.00543	3.38635	4.29187	5.42743	6.84848	8.62308	10.83471	25
26	2.77247	3.14068	3.55567	4.54938	5.80735	7.39635	9.39916	11.91818	26
27	2.88337	3.28201	3.73346	4.82235	6.21387	7.98806	10.24508	13.10999	27
28	2.99870	3.42970	3.92013	5.11169	6.64884	8.62711	11.16714	14.42099	28
29	3.11865	3.58404	4.11614	5.41839	7.11426	9.31727	12.17218	15.86309	29
30	3.24340	3.74532	4.32194	5.74349	7.61226	10.06266	13.26768	17.44940	30
31	3.37313	3.91386	4.53804	6.08810	8.14511	10.86767	14.46177	19.19434	31
32	3.50806	4.08998	4.76494	6.45339	8.71527	11.73708	15.76333	21.11378	32
33	3.64838	4.27403	5.00319	6.84059	9.32534	12.67605	17.18203	23.22515	33
34	3.79432	4.46636	5.25335	7.25103	9.97811	13.69013	18.72841	25.54767	34
35	3.94608	4.66734	5.51601	7.68609	10.67658	14.78534	20.41397	28.10244	35
36	4.10393	4.87737	5.79181	8.14725	11.42394	15.96817	22.25122	30.91268	36
37	4.26808	5.09686	6.08140	8.63609	12.22362	17.24562	24.25383	34.00395	37
38	4.43881	5.32621	6.38547	9.15425	13.07927	18.62527	26.43668	37.40434	38
39	4.61636	5.56589	6.70475	9.70351	13.99482	20.11529	28.81598	41.14478	39
40	4.80102	5.81636	7.03998	10.28572	14.97446	21.72452	31.40942	45.25926	40

PRESENT VALUE OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	Yrs.
1	0.99010	0.98765	0.98522	0.98280	0.98039	0.97561	0.97087	0.96618	1
2	0.98030	0.97546	0.97066	0.96590	0.96117	0.95181	0.94260	0.93351	2
3	0.97059	0.96342	0.95632	0.94928	0.94232	0.92860	0.91514	0.90194	3
4	0.96098	0.95152	0.94218	0.93296	0.92385	0.90595	0.88849	0.87144	4
5	0.95147	0.93978	0.92826	0.91691	0.90573	0.88385	0.86261	0.84197	5
6	0.94204	0.92817	0.91454	0.90114	0.88797	0.86230	0.83748	0.81350	6
7	0.93272	0.91672	0.90103	0.88565	0.87056	0.84127	0.81309	0.78599	7
8	0.92348	0.90540	0.88771	0.87041	0.85349	0.82075	0.78941	0.75941	8
9	0.91434	0.89422	0.87459	0.85544	0.83675	0.80073	0.76642	0.73373	9
10	0.90529	0.88318	0.86167	0.84073	0.82035	0.78120	0.74409	0.70892	10
11	0.89632	0.87228	0.84893	0.82627	0.80426	0.76214	0.72242	0.68495	11
12	0.88745	0.86151	0.83639	0.81206	0.78849	0.74356	0.70138	0.66178	12
13	0.87866	0.85087	0.82403	0.79809	0.77303	0.72542	0.68095	0.63940	13
14	0.86996	0.84037	0.81185	0.78436	0.75788	0.70773	0.66112	0.61778	14
15	0.86135	0.82999	0.79985	0.77087	0.74302	0.69047	0.64186	0.59869	15
16	0.85282	0.81975	0.78803	0.75762	0.72845	0.67363	0.62317	0.57671	16
17	0.84438	0.80963	0.77639	0.74459	0.71416	0.65720	0.60502	0.55720	17
18	0.83602	0.79963	0.76491	0.73178	0.70016	0.64117	0.58739	0.53836	18
19	0.82774	0.78976	0.75361	0.71919	0.68643	0.62553	0.57029	0.52016	19
20	0.81954	0.78001	0.74247	0.70682	0.67297	0.61027	0.55368	0.50257	20
21	0.81143	0.77038	0.73150	0.69467	0.65978	0.59539	0.53755	0.48557	21
22	0.80340	0.76087	0.72069	0.68272	0.64684	0.58087	0.52189	0.46915	22
23	0.79544	0.75147	0.71004	0.67098	0.63416	0.56670	0.50669	0.45329	23
24	0.78757	0.74220	0.69954	0.65944	0.62172	0.55288	0.49193	0.43796	24
25	0.77977	0.73303	0.68921	0.64810	0.60953	0.53939	0.47761	0.42315	25
26	0.77205	0.72398	0.67902	0.63695	0.59758	0.52624	0.46369	0.40884	26
27	0.76440	0.71505	0.66899	0.62599	0.58586	0.51340	0.45019	0.39501	27
28	0.75684	0.70622	0.65910	0.61523	0.57438	0.50088	0.43708	0.38165	28
29	0.74934	0.69750	0.64936	0.60465	0.56311	0.48866	0.42435	0.36875	29
30	0.74192	0.68889	0.63976	0.59425	0.55207	0.47674	0.41199	0.35628	30
31	0.73458	0.68004	0.63031	0.58403	0.54125	0.46512	0.39999	0.34423	31
32	0.72730	0.67198	0.62099	0.57398	0.53063	0.45377	0.38834	0.33259	32
33	0.72010	0.66369	0.61182	0.56411	0.52023	0.44270	0.37703	0.32134	33
34	0.71297	0.65549	0.60277	0.55441	0.51003	0.43191	0.36605	0.31048	34
35	0.70591	0.64740	0.59387	0.54487	0.50003	0.42137	0.35538	0.29998	35
36	0.69893	0.63941	0.58509	0.53550	0.49022	0.41109	0.34503	0.28983	36
37	0.69201	0.63151	0.57644	0.52629	0.48061	0.40107	0.33498	0.28003	37
38	0.68515	0.62372	0.56792	0.51724	0.47119	0.39129	0.32523	0.27057	38
39	0.67837	0.61602	0.55953	0.50834	0.46195	0.38174	0.31575	0.26141	39
40	0.67165	0.60841	0.55126	0.49960	0.45289	0.37243	0.30656	0.25257	40

PRESENT VALUE OF 1

Yrs.	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	0.96154	0.95694	0.95238	0.94340	0.93458	0.92593	0.91743	0.90909	1
2	0.92456	0.91573	0.90703	0.89000	0.87344	0.85734	0.84168	0.82645	2
3	0.88900	0.87630	0.86384	0.83962	0.81630	0.79383	0.77218	0.75131	3
4	0.85480	0.83856	0.82270	0.79209	0.76290	0.73503	0.70842	0.68301	4
5	0.82193	0.80245	0.78353	0.74726	0.71299	0.68058	0.64993	0.62092	5
6	0.79031	0.76790	0.74622	0.70496	0.66634	0.63017	0.59627	0.56447	6
7	0.75992	0.73483	0.71068	0.66506	0.62275	0.58349	0.54703	0.51316	7
8	0.73069	0.70319	0.67684	0.62741	0.58201	0.54027	0.50187	0.46651	8
9	0.70259	0.67290	0.64461	0.59190	0.54393	0.50025	0.46043	0.42410	9
10	0.67556	0.64393	0.61391	0.55840	0.50835	0.46319	0.42241	0.38554	10
11	0.64958	0.61620	0.58468	0.52679	0.47509	0.42888	0.38753	0.35049	11
12	0.62460	0.58966	0.55684	0.49697	0.44401	0.39711	0.35553	0.31863	12
13	0.60057	0.56427	0.53032	0.46884	0.41496	0.36770	0.32618	0.28966	13
14	0.57748	0.53997	0.50507	0.44230	0.38782	0.34046	0.29925	0.26333	14
15	0.55526	0.51672	0.48102	0.41727	0.36245	0.31524	0.27454	0.23939	15
16	0.53391	0.49447	0.45811	0.39365	0.33874	0.29189	0.25187	0.21763	16
17	0.51337	0.47318	0.43630	0.37136	0.31657	0.27027	0.23107	0.19784	17
18	0.49363	0.45280	0.41552	0.35034	0.29586	0.25025	0.21199	0.17986	18
19	0.47464	0.43330	0.39573	0.33051	0.27651	0.23171	0.19449	0.16351	19
20	0.45639	0.41464	0.37689	0.31181	0.25842	0.21455	0.17843	0.14864	20
21	0.43883	0.39679	0.35894	0.29416	0.24151	0.19866	0.16370	0.13513	21
22	0.42196	0.37970	0.34185	0.27751	0.22571	0.18394	0.15018	0.12285	22
23	0.40573	0.36335	0.32557	0.26180	0.21095	0.17032	0.13778	0.11168	23
24	0.39012	0.34770	0.31007	0.24698	0.19715	0.15770	0.12640	0.10153	24
25	0.37512	0.33273	0.29530	0.23300	0.18425	0.14602	0.11597	0.09230	25
26	0.36069	0.31840	0.28124	0.21981	0.17220	0.13520	0.10639	0.08390	26
27	0.34682	0.30469	0.26785	0.20737	0.16093	0.12519	0.09761	0.07628	27
28	0.33348	0.29157	0.25509	0.19563	0.15040	0.11591	0.08955	0.06934	28
29	0.32065	0.27902	0.24295	0.18456	0.14056	0.10733	0.08215	0.06304	29
30	0.30832	0.26700	0.23138	0.17411	0.13137	0.09938	0.07537	0.05731	30
31	0.29646	0.25550	0.22036	0.16426	0.12277	0.09202	0.06915	0.05210	31
32	0.28506	0.24450	0.20987	0.15496	0.11474	0.08520	0.06344	0.04736	32
33	0.27409	0.23397	0.19987	0.14619	0.10724	0.07889	0.05820	0.04306	33
34	0.26355	0.22390	0.19036	0.13791	0.10022	0.07305	0.05339	0.03914	34
35	0.25341	0.21425	0.18129	0.13011	0.09366	0.06763	0.04899	0.03558	35
36	0.24367	0.20503	0.17266	0.12274	0.08754	0.06263	0.04494	0.03235	36
37	0.23430	0.19620	0.16444	0.11579	0.08181	0.05799	0.04123	0.02941	37
38	0.22529	0.18775	0.15661	0.10924	0.07646	0.05369	0.03783	0.02673	38
39	0.21662	0.17967	0.14915	0.10306	0.07146	0.04971	0.03470	0.02430	39
40	0.20829	0.17193	0.14205	0.09722	0.06678	0.04603	0.03184	0.02209	40

ANSWERS

CHAPTER 1

Exercise 1-1 (Page 2)

1. (i) N, W do not (ii) I, N, W do not
2. All except multiplicative inverse and multiplicative identity.
3. O has no addition defined, hence no additive properties; also no multiplicative identity.
4. $3x+4=6$
5. $\frac{a+b}{2}$
6. 0.71524101001000100001 . . . with the dots indicating that the pattern of digits after the '4' is to continue.
7. $\frac{3709}{999}$
8. If $\alpha+\frac{a}{b}$ were rational, then $\alpha+\frac{a}{b}=\frac{c}{d}$, $d\neq 0$, and $\alpha=\frac{c}{d}-\frac{a}{b}=\frac{bc-ad}{bd}$, a rational number. Contradiction!
9. If $\frac{a}{b}\times\alpha=\frac{c}{d}$, then $\alpha=\frac{bc}{ad}$ if $a\neq 0$. Contradiction! The exception is $\frac{a}{b}=0$.

Exercise 1-2 (Page 5)

1. (a) 6 (b) -3.5 (c) $-\frac{43}{8}$ (d) $-\frac{8}{15}$ (e) $\frac{31}{10}$ (f) -2
3. $\frac{1}{3}<x<1$
4. (a) $-\frac{7}{2}<x<1$ (b) $1<x<4$ (c) $-4\leq x\leq 3$ (d) ϕ
6. (a) $\{3, -7\}$ (b) $\{1, -\frac{1}{3}\}$ (c) $\{3, 4\}$ (d) ϕ
   (e) $\{x\in \mathrm{Re} | 2<x<3\}$ (f) $\{2, -2\}$, $\{x\in \mathrm{Re} | -3<x<3\}$
8. (a) $-1<x<\frac{11}{3}$ (b) $-1\leq x\leq\frac{29}{5}$ (c) ϕ (d) $a\leq -1$ or $a\geq\frac{1}{2}$
   (e) $a<-1$ or $a>\frac{17}{3}$ (f) $x>-5$
9. (a) $-2<x<\frac{2}{3}$ (b) $x\leq\frac{1}{3}$ or $x\geq 5$ (c) ϕ (d) $\{x\in \mathrm{Re} | x\neq\frac{9}{4}\}$
   (e) $x>\frac{1}{2}$ (f) -3

Exercise 1-3 (Page 7)

1. 12
2. 16
3. 1
4. $-\frac{20}{3}$
5. $\{x\in \mathrm{Re} | x>-2\}$
6. $\{70, -60\}$
7. $\{\frac{35}{18}, -\frac{35}{6}\}$
8. $\frac{55}{2}$
9. 24 minutes

Exercise 1-4 (Page 9)

1. $t=\frac{d}{v}$
2. $w=\frac{\mathrm{A}}{b}$
3. $\pi=\frac{c}{d}$
4. $r=\frac{i}{pt}$
5. $u=v-at$
6. $a=\frac{v-u}{t}$
7. $a=\frac{2s}{t^2}$
8. $r=\sqrt{\frac{\mathrm{A}}{\pi}}$
9. $l=\frac{2s}{n}-a$
10. $h=\frac{\mathrm{A}}{\pi r^2}$
11. $r=\sqrt[3]{\frac{3\mathrm{V}}{4\pi}}$
12. $h=\frac{3\mathrm{V}}{\pi r^2}$
13. $\mathrm{F}=\frac{9}{5}\mathrm{C}+32>68°$
14. $n=\frac{\mathrm{IR}}{\mathrm{E}-r\mathrm{I}}$
15. $x=\frac{y-7}{5}$
16. $y=\frac{15-5x}{3}$
17. $y=\frac{c-ax}{b}$
18. $\mathrm{R}=1$
19. $r_3=8$
20. $\mathrm{R}=\frac{72}{17}$
21. $x=\frac{y-17}{6}$
22. $x=\frac{r-qx_2}{p}$

Exercise 1-5 (Page 11)

1. (a) $\{(-2, 3), (-2, 4), (-2, 5), (0, 3), (0, 4), (0, 5), (2, 3), (2, 4), (2, 5)\}$
   (b) $\{(-3, -1), (-3, 0), (-3, 1), (-2, -1), (-2, 0), (-2, 1), (-1, -1), (-1, 0), (-1, 1)\}$
   (c) $\{(1, 2), (1, -2), (1, 3), (1, -3), (2, 2), (2, -2), (2, 3), (2, -3), (3, 2), (3, -2), (3, 3), (3, -3)\}$
   (d) $\{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5)\}$
   (e) $\{(2, -3), (2, 0), (2, 3), (4, -3), (4, 0), (4, 3), (6, -3), (6, 0), (6, 3), (8, -3), (8, 0), (8, 3)\}$
3. $\{(-3, -3), (-3, -2), (-3, -1), (-2, -3), (-2, -2), (-2, -1), (-1, -3), (-1, -2), (-1, -1)\}$
4. (a) $\{(\frac{1}{3}, 0), (\frac{1}{3}, 2), (\frac{1}{4}, 0), (\frac{1}{4}, 2), (\frac{1}{5}, 0), (\frac{1}{5}, 2)\}$
   (b) $\{(a, c), (a, d), (a, e), (a, f), (b, c), (b, d), (b, e), (b, f), (c, c), (c, d), (c, e), (c, f)\}$
   (c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$
   (d) $\{(0, -2), (0, -1), (0, 0), (0, 1), (0, 2), (1, -2), (1, -1), (1, 0), (1, 1), (1, 2), (2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}$
5. (i) $x=y$ (ii) yes
6. ϕ
7. (i) $\{(2, 3), (2, 5), (2, 7), (4, 3), (4, 5), (4, 7), (6, 3), (6, 5), (6, 7)\}$
   (ii) $\{(2, 3), (4, 5), (6, 7)\}$

Exercise 1-6 (Page 15)

2. (a) Domain is Re. Range is $\{-3\}$.
   (b) Domain is $\{5\}$. Range is Re.
   (c) Domain is Re. Range is $\{0\}$.
   (d) Domain is $\{0\}$. Range is Re.
3. (a) Re (b) Re (c) $x=-\frac{2}{3}$, x-intercept (d) $y=2$, y-intercept
4. (a) is not a member. (b), (c), (d) are members.

Exercise 1-7 (Page 17)

1. x-intercept is 3; y-intercept is 5.
   $(6, -5)$, $(-6, 15)$ are on the line; $(2, 3)$, $(5, -3)$ are not.
2. lines through the following points:
   (a) $(-2, 0)$, $(0, -\frac{4}{3})$ (b) $(\frac{16}{3}, 0)$, $(0, 2)$ (c) $(2, 0)$, $(0, 2)$
   (d) $(3, 0)$, $(0, -4)$ (e) $(\frac{5}{3}, 0)$, $(0, -5)$ (f) $(0, 0)$, $(1, 3)$
3. These lines belong to a family of parallel lines.
4. The family of parallel lines is $3x-4y=k$.
5. The family of lines is perpendicular to the family in (4).
6. (a) No. If x and y are integers, then $4x+4y$ is an integer divisible by 4 and hence cannot equal 7.
   (b) No

Exercise 1-8 (Page 21)

1. inconsistent; two parallel lines.
2. dependent; two coincident lines
3. consistent; two distinct intersecting lines
4. consistent; two distinct intersecting lines
5. consistent; two distinct intersecting lines
6. dependent; two coincident lines
7. -2

(In the following, the first member in each ordered pair corresponds to the first variable in the equation.)

8. $(3, -1)$ 9. $(\frac{5}{2}, \frac{5}{2})$ 10. inconsistent (no solution) 11. $(\frac{3}{2}, \frac{5}{2})$
12. $(-1, 2)$ 13. $(\frac{41}{7}, \frac{2}{7})$ 14. $(5, 1)$
15. dependent (infinite number of solutions)
16. $(2, 1)$ 17. inconsistent (no solution) 19. $(4, 5)$
20. $(20, -21)$ 21. inconsistent (no solution) 22. $(-8, -12)$
23. $(5, 4)$ 24. $(10, 3)$ 25. $(5, 0)$ 26. $(\frac{1044}{2655}, \frac{259}{53})$
27. $(4, 3)$ 28. dependent (infinite number of solutions) 30. $(4, 5)$
31. $(\frac{1}{6}, -\frac{1}{2})$ 32. $(\frac{21}{37}, \frac{7}{13})$ 33. 10 34. $\{k \in \mathrm{Ra} | k \neq 10\}$ 35. ϕ

Exercise 1-9 (Page 23)

1. $\left(\frac{4}{p}, -\frac{2}{q}\right)$, $p \neq 0$, $q \neq 0$
2. $\left(\frac{k}{a-b}, \frac{-k}{a-b}\right)$, $a \neq b$
3. $\left(\frac{q+2n}{mq+np}, \frac{p-2m}{mq+np}\right)$, $mq+np \neq 0$
4. $\left(\frac{2}{a}, -\frac{2}{b}\right)$, $a \neq 0$, $b \neq 0$
5. $(b+1, a)$, $a \neq ab$
6. $\left(\frac{a_2b_1+a_1b_2}{a_1+a_2}, \frac{b_1-b_2}{a_1+a_2}\right)$, $a_1+a_2 \neq 0$

7. (a, b), $a \neq b$

8. $(\frac{1}{2}m(a+b), \frac{1}{2}n(a-b))$, $m \neq 0$, $n \neq 0$

9. (p, q), $p \neq q$, $q \neq 0$, $q \neq -1$

10. $\left(\frac{ad-bc+be}{b+d}, \frac{de-ad+bc}{b+d}\right)$, $d \neq -b$

Exercise 1-10 (Page 26)

1. (a) All points to the right of the y-axis.
   (b) All points below the x-axis.
   (c) All points below the line $3x-4y=-12$.
   (d) All points on and above the line $3x-4y=-12$.
   (e) All points below the line $3x-4y=-12$ and to the left of the y-axis.
   (f) All points above the line $3x-4y=-12$ and below the x-axis.
   (g) All points within the triangle bounded by $3x-4y=-12$, $x=0$, $y=0$.

6. No. The region, $x<0$, $y<0$, is the third quadrant and the region $3x+2y>13$ involves only the first, second, and fourth quadrants.

Review Exercise 1-11 (Page 28)

1. All except multiplicative identity and, hence, the multiplicative inverse.

2. $\{x|x>2\}$

3. $\{x|x<-2\} \cup \{x|x>5\}$

4. $\frac{b}{4a}$

5. $a=\frac{v^2-u^2}{2s}$, $s \neq 0$

6. (a) $\{(2, 3), (2, 5), (2, 7), (4, 3), (4, 5), (4, 7), (6, 3), (6, 5), (6, 7)\}$
   (b) $\{(3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6), (7, 2), (7, 4), (7, 6)\}$

7. Lines parallel to the line represented by $x-2y=0$ (line through points $(0, 0)$, $(2, 1)$).

8. $(3, -4)$

9. $(-\frac{105}{19}, \frac{81}{19})$

10. $\left(\frac{3(n-3m)}{n^2-m^2}, \frac{3(m-3n)}{m^2-n^2}\right)$, $m \neq \pm n$

11. $\{(1, 1), (2, 3), (3, 5)\}$; domain $\{1, 2, 3\}$; range $\{1, 3, 5\}$.

CHAPTER 2

Exercise 2-1 (Page 34)

1. B, D, E, F

2. R_2, R_4

3. domain $=\{2, 0, -2\}$, $=$range $\{0, 1, 2\}$, ordered pairs $=\{(2, 0), (0, 1), (-2, 2)\}$

4. b; domain $=\{3, 4, 5\}$, range $=\{2\}$

5. a, c, e.

6. (a) $w=10-l$
   (b) yes
   (c) domain $=\{l \in \mathrm{Re} | 0<l \leq 8\}$
   range $=\{w \in \mathrm{Re} | 2 \leq w<10\}$

7. (a) $s=200n$ (b) yes; domain $=\{n\in \mathrm{I}\mid 0\leq n\leq 8\}$
range $=\{s\in \mathrm{I}\mid 0\leq s\leq 1600\}$
ordered pairs $=\{(0,0), (1, 200), (2, 400), (3, 600), (4, 800), (5, 1000), (6, 1200), (7, 1400), (8, 1600)\}$

8. (a) yes (b) 196 ft. (c) $\frac{9}{4}$ sec.

9. (b) Those inverse relations obtained from 5(a), (d), (f).

10. (a) yes (b) no (c) yes.

Exercise 2-2 (Page 37)

1. (a) 8 (b) 5 (c) $\frac{11}{2}$ (d) 0
(e) $5+\frac{1}{x}$ (f) $x+10$ (g) $k+6$ (h) $2r$

2. (a) $3x+3$ (b) x^2+2x-1 (c) 0
(d) $3x^2-6$ (e) 7 (f) $9x^2+18x+7$

3. In the sense that the function f determines the set of ordered pairs $\{(x, y)\mid y=3x\}$

4. (a) 2 (b) 46 (c) -18 (d) 1
ordered pairs: (0, 2), (2, 46), $(-2, -18)$, $(-1, 1)$.

6. (b) $f(2)=f(3)=0$. (c) The zeros of the function are the x-intercepts of the graph.

7. (a) range $=\{f(x)\mid -5f(x)\leq 5\}$ $\{f(x)\mid -5\leq f(x)\leq 5\}$
(b) range $=\{g(x)\mid 0\leq g(x)\leq 9\}$ (c) range $=\{h(x)\mid 0\leq h(x)\leq 9\}$

8. Only the inverse relation of (a).

Exercise 2-3 (Page 38)

1. (Jerry, Chris); (Paul, Geoff).
domain $=\{$Jerry, Paul$\}$, range $=\{$Chris, Geoff$\}$

2. $t=10w+50$
$\{(0, 50), (1, 60), (2, 70), \ldots, (50, 550), (51, 560), (52, 570)\}$
domain $=\{w\in \mathrm{I}\mid 0\leq w\leq 52\}$
range $=\{t\in \mathrm{I}\mid 50\leq t\leq 570\}$

4. (b) (3, 120), 2 hours (c) 40 miles

5. First car: $f(t)=40t+80$; Second car: $g(t)=60t$; 4 hours.

6. (a) $l=\frac{1}{4}t+10$ (c) $13\frac{3}{4}$ inches

7. $\left\{(h, w)\,\middle|\, w=\frac{11(h-40)}{2},\ h\in \mathrm{Re}^+\right\}$; (66, 121), (70, 165)

8. (a) $\mathrm{C}=\frac{1}{2}n+200$ (b) $\{(n, \mathrm{C})\mid \mathrm{C}=\frac{1}{2}n+200,\ n\in \mathrm{I},\ n\geq 0\}$ (c) \$62.50 profit

Exercise 2-4 (Page 43)

1. (a) 2 (b) $-\frac{4}{3}$ (c) 1 (d) 12 (e) 2 (f) 0 (g) 0
(h) undefined

3. (a) 0.700 (b) 1.73 (c) -0.839

4. (a) 45° (b) 0° (c) 39° (d) 100° (e) 13° (f) 142°

5. (a), (c) collinear 6. $-\frac{1}{2}$ 7. $\frac{7}{2}$ 8. -1 9. $4\frac{1}{4}$ 10. $x_1=\dfrac{31+8y_1}{5}$

11. $\frac{1}{2}$ 12. (a) 40° (b) 45°

Exercise 2-5 (Page 45)

1. (a) $y=3x+2$ (b) $y=x+1$ (c) $y=-3x+4$ (d) $y=4x-5$

2. (a) $m=5,\ b=3$ (b) $m=1,\ b=-4$ (c) $m=-3,\ b=-2$
   (d) $m=-1,\ b=7$ (e) $m=-2,\ b=4$ (f) $m=-3,\ b=7$

3. (a) $4y=x+5$ (b) $y=x-1$ (c) $5y=3x+31$ (d) $y=4$

4. (a) $y=2x-2$ (b) $y=-3x-9$ (c) $2y=2x-5$
   (d) $y=5.6713x-3.7809$

6. $4y=12x-7$ 7. $y=2-x$ 8. (a) $y=-3x+b$ (b) $y=5-3x$

9. (a) $y+5=m(x-2)$ (b) $y=-4x+3$

10. $(-7,\ -5),\ (5,\ 11)$ 11. $m=-1$ or $m=-\frac{5}{2}$

Exercise 2-6 (Page 47)

1. (a) $\frac{1}{5}$ (b) $-\frac{3}{2}$ (c) -1 (d) 1 (e) $\frac{3}{7}$ (f) $\dfrac{m}{2k}$ (g) -1 (h) $h+1$

2. (a) $y=-\frac{3}{5}x+4$ (b) $y=\frac{2}{3}x-\frac{7}{6}$ (c) $y=x-\frac{9}{4}$ (d) $y=\frac{3}{4}x-3$
   (e) $y=x-1$ (f) $y=\dfrac{p}{q}x+\dfrac{r}{q}$ (g) $y=-2x+11$
   (h) $y=\left(\dfrac{r+s}{t}\right)x+\dfrac{w}{t}$

3. (a) $-\frac{3}{5}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{3}{4}$ (e) 1 (f) $\dfrac{p}{q}$ (g) -2 (h) $\dfrac{r+s}{t}$

4. (a) $-\frac{1}{3}$ (b) 2 (c) $\frac{3}{5}$ 5. (a) $\frac{6}{5}$ (b) $-\frac{3}{4}$ (c) 7

6. (a) $\frac{20}{9}$ (b) $\pm\sqrt{10}$ (c) $\pm\sqrt{\frac{7}{3}}$

Review Exercise 2-7 (Page 48)

1. yes 2. $\{(x, y)|x=|y|\}$, no

4. (a) 7 (b) -5 (c) $4x-1$ (d) $16x-25$ (e) $4x^2-5$
   (f) $16x^2-72x+81$

5. (a) 0 (b) -1 (c) x^9 (d) 512 (e) $\left(\dfrac{1}{x-a}\right)^3,\ x\neq a$ (f) $-\frac{1}{3\,4\,3}$

6. (1) $m=\frac{3}{5},\ a=5,\ b=-3$ (2) $m=-7,\ a=\frac{8}{7},\ b=8$

7. (1) 31° (2) 98° 8. yes 9. $-\frac{3}{5}$ 10. -1

11. (a) $y=4x-1$ (b) $y=-x+2$ (c) $5y=5x+2$ (d) $y=-x-3$

12. $y=\frac{7}{11}x-\frac{21}{11}$; $y=\frac{7}{11}x+b$

13. x-intercept is nonexistent; y-intercept $=3$; domain $=$ Re, range $=\{y\in \text{Re}|0<y\leq 3\}$; symmetric about y-axis.

CHAPTER 3

Exercise 3-1 (Page 52)

1. (a) 243 (b) 343 (c) 144 (d) 625 (e) 8 (f) -1
   (g) -27 (h) 125 (i) 81 (j) 256 (k) 9 (l) 1024

2. $6^5 = 7776$

3. $(-3)^5 = -243$

4. $(-2)^7 = -128$

5. $5^5 = 3125$

6. $(\frac{1}{2})^5 = \frac{1}{32}$

7. $\dfrac{1}{(.4)^2} = 6.25$

8. $(.3)^6 = .000729$

9. $(\frac{1}{3})^6 = \frac{1}{729}$

10. $7^2 = 49$

11. $(\frac{2}{3})^3 = \frac{8}{27}$

12. $(\sqrt{2})^{10} = 32$

13. π

14. $(\sqrt{5})^{11} = 3125\sqrt{5}$

15. $\left(\dfrac{1}{\sqrt{3}}\right)^8 = \dfrac{1}{81}$

16. π

17. $(\sqrt{\frac{2}{5}})^6 = \frac{8}{125}$

18. x^4

19. 1

20. 12

21. $(.3)^3 = .027$

22. 25

23. $7^2 = 49$

24. $6^5 = 7776$

25. 5

26. 1

27. 1

28. $\dfrac{16}{27x}$

29. $2^4a^{11} = 16a^{11}$

30. p^{12}

31. $\dfrac{2}{q^5}$

32. $\frac{1}{3}$

33. 1

34. .3

35. $\frac{5}{7}$

Exercise 3-2 (Page 55)

1. 216
2. 400
3. .003375
4. $\frac{16}{625}$
5. $\frac{343}{8}$
6. 64
7. 144
8. -216
9. 225
10. $\frac{1}{216}$
11. $\frac{100}{81}$
12. $\frac{64}{225}$
13. $\frac{27}{8}$
14. $\frac{25}{36}$
15. 1000
16. 900
17. $-\frac{27}{8}$
18. $\dfrac{-4a^2b^2}{c^2}$
19. $\dfrac{27x^3}{y^3}$
20. 25
21. .001728
22. $\dfrac{216p^3q^3}{r^3}$
23. 8
24. .027
25. $\frac{4}{25}$
26. $\dfrac{x^2}{y^2}$
27. $\dfrac{x^3y^3}{z}$
28. x^4y^4
29. $-p^6q^7$
30. $\dfrac{9a}{b^4}$
31. $\dfrac{8}{h^2}$
32. $(x^2-y^2)^3$
33. $(xy^2z)^3$
34. p^2q^2
35. $\dfrac{q^{10}r^2}{p}$
36. $\dfrac{q^3}{p^2}$
37. $c+a$
38. $\dfrac{a^5b^3}{c}$
39. $\dfrac{xz^4}{y^8}$
40. $\dfrac{b^2}{c}$
41. p^3q^3
42. $x^8y^{11}z^9$
43. p^7q^{12}

Exercise 3-3 (Page 58)

1. x
2. x^{-3}
3. 1
4. a^5
5. a
6. a^{-2}
7. p^{-6}
8. p^{-5}
9. p^{-5}
10. q^2
11. 1
12. q^{-8}
13. x^{-6}
14. x^{-6}
15. x^6
16. 1
17. b^5
18. b^{-1}
19. 1
20. 1
21. 4
22. 2
23. 3
24. $\frac{1}{5}$
25. $\frac{1}{2}$
26. 4
27. 1
28. 1
29. 1
30. 1

31. 9 32. 7 33. $\frac{1}{64}$ 34. 1024 35. $\frac{1}{27}$ 36. 1

37. $\frac{1}{25}$ 38. 81 39. $\frac{2}{3}$ 40. $(\frac{5}{6})^8$ 41. .5 42. $(.3)^{-6}$

43. $\frac{5}{3y}$ 44. $\frac{16}{3x}$ 45. $\frac{5x^5}{2y^6}$ 46. $\frac{x^2y}{4}$ 47. $\frac{32y^{12}}{x^6}$ 48. $\frac{1}{144x^4y^2}$

49. y^2 50. $\frac{z^4}{144x^6y^2}$ 51. $\frac{a^4b^2}{c^2}$ 52. $\frac{a^6}{b^3}$ 53. $\frac{x^2+y^2}{xy}$ 54. $\frac{c^3}{b^3}$

55. $xy(1+xy)$ 56. $\frac{a^2}{bc}$ 57. $\frac{y}{5x}$ 58. $\frac{1}{450p^4q^3}$

59. $\frac{z}{y^2}\left(\frac{x^2+y}{x}\right)$ 60. $xy+\frac{1}{xy}$

Exercise 3-4 (Page 61)

1. 16 2. 1 3. $\frac{1}{18}$ 4. $y^{-.2}$ 5. 5 6. a^3

7. $\frac{1}{p^2}$ 8. $a^{\frac{1}{2}}$ 9. $ab^{\frac{1}{2}}$ 10. $xy^{\frac{3}{2}}$ 11. 9 12. $x^{5.2}$

13. $\frac{3}{8}$ 14. $p^{.75}$ 15. 8 16. $\frac{1}{b}$ 17. $q^{-\frac{3}{4}}$ 18. $b^{-\frac{1}{6}}$

19. p^3q 20. xy^2 21. a 22. $5y^{\frac{1}{2}}$ 23. $2x^{\frac{2}{3}}$ 24. $24x^{\frac{3}{2}}y^{\frac{7}{2}}$

25. $9a^{\frac{10}{3}}b^{\frac{8}{3}}$ 26. $4(xy)^{1.2}$ 27. $6xy$ 28. $2x^{.8}y^{2.7}$ 29. $\frac{2}{3}a^{1.9}b^{.3}$ 30. $\sqrt{2}b^{\frac{1}{2}}c^{\frac{1}{4}}$

31. $\frac{1}{2}q^{-\frac{2}{5}}$ 32. $\frac{\sqrt{7}}{y^{\frac{3}{2}}}$ 33. $\frac{3}{4}r^{-\frac{9}{2}}h^{\frac{1}{2}}$

34. $(a+b)^{\frac{3}{2}}(a^2-b^2)^{\frac{1}{2}}=(a+b)^{\frac{3}{2}}(a+b)^{\frac{1}{2}}(a-b)^{\frac{1}{2}}=$ R.S.

35. $(x-y)^{\frac{1}{2}}(x^2-y^2)^{\frac{3}{2}}=(x-y)^{\frac{1}{2}}(x-y)^{\frac{3}{2}}(x+y)^{\frac{3}{2}}=$ R.S.

36. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x-y)=(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}})=$ R.S.

37. $(x^2-y^2)^{\frac{2}{3}}(x+y)^{\frac{1}{2}}=(x-y)^{\frac{2}{3}}(x+y)^{\frac{2}{3}}(x+y)^{\frac{1}{2}}=$ R.S.

38. $\frac{5x^2(x^{\frac{1}{2}}+x^{-\frac{1}{2}})}{xy^2}=\frac{5}{y^2}(xx^{\frac{1}{2}}+xx^{-\frac{1}{2}})=$ R.S.

39. $\frac{2y(z^{\frac{2}{3}}+z^{\frac{5}{3}})}{z^{\frac{2}{3}}\ y^{\frac{3}{2}}}=\frac{2y}{y^{\frac{3}{2}}}\left(\frac{z^{\frac{2}{3}}}{z^{\frac{2}{3}}}+\frac{z^{\frac{5}{3}}}{z^{\frac{2}{3}}}\right)=$ R.S.

42. 5.196 43. $\frac{1}{125}$ 44. 6 45. 864 46. $\frac{1}{4}$ 47. 27

48. 4 49. $\frac{1}{25\sqrt{5}}$ 50. 10.481 51. $\frac{1}{125}$ 52. 5.196 53. 2

Exercise 3-5 (Page 63)

1. -4 2. 3 3. nonexistent 4. 2 5. 3.609

6. 4.899 7. 4.326 8. nonexistent 9. 2.646

10. nonexistent 11. -2.759 12. -3 13. 1.710 14. -2

15. 2.520 16. -2

Exercise 3-6 (Page 65)

1. (a) 2.351×10^{2}, .2351E3 (b) 7.568×10^{0}, .7568E1
   (c) 4.371×10^{-1}, .4371E0
2. (a) 5.614×10^{5}, .5614E6 (b) 3.536×10^{1}, .3536E2
   (c) 1.736×10^{-3}, .1736E−2
3. (a) 3.468×10^{3}, .3468E4 (b) 2.741×10^{1}, .2741E2
   (c) 1.693×10^{0}, .1693E1
4. (a) 5.463×10^{0}, .5463E1 (b) 1.284×10^{3}, .1284E4
   (c) 1.234678×10^{6}, .1234678E7
5. (a) 9.87654321×10^{3}, .987654321E4 (b) 3.0×10^{-5}, .3E−4
   (c) 7.892×10^{-2}, .7892E−1

In the following, S stands for scientific, C for computer.

6. S, 870.9 7. S, 458,000 8. S, 697,500 9. S, 16 10. S, 3.2
11. S, 15.625 12. S, 23.676 13. S, 2500 14. S, .08
15. S, 5,369,000,000,000 16. S, 913,333,333,333
17. S, .000000000000000768 18. C, 63.3333 19. C, 4410
20. C, .000008354375

Exercise 3-7 (Page 69)

1. $64^{\frac{5}{3}}<64^{\sqrt{3}}<64^{\frac{7}{4}}$ or $1024=2^{10}<64^{\sqrt{3}}<2^{10}2^{\frac{1}{2}}\simeq1448$

Exercise 3-8 (Page 70)

4. (a) $y=3^{-x}$ (b) $y=(1.2)^{-x}$
   (c) $(1.2)^{-x}<(1.5)^{-x}<2^{-x}<3^{-x}$ in $2\leq x<0$
   (d) The order is reversed in $0<x<1$. The functions all have the value 1 for $x=0$.
5. $5^{x}<3^{x}<2^{x}<1^{x}<(\frac{1}{2})^{x}<(\frac{1}{3})^{x}$ if $x<0$. The order is reversed if $x>0$.
6. The graph of 3(a) is part of the graph of 2(c).
   The graph of 3(b) is part of the graph of 2(d).
7. $(\frac{1}{2})^{x}=2^{-x}$, hence the graphs of $y=(\frac{1}{2})^{x}$ and $y=2^{-x}$ are the same.
   The graph of $y=2^{x}$ is the reflection in the y-axis of the graph of $y=2^{-x}$.

Exercise 3-9 (Page 73)

NOTE: Any solutions obtained using the large scale graphs should be approximately equal to the answers given. This holds for this Exercise and all following Exercises making use of the graphs.

1. 66 2. 65.6 3. 98 4. 6.6 5. 10.5 6. 7.7
7. 9.34 8. 4.03 9. 2.38 10. 2.49 11. 68.7 12. 11.6

13. .844 14. 4680 15. 13.6 16. 2.43 17. .41 18. 4.81
19. .72 20. 1.44

Review Exercise 3-10 (Page 74)

1. (a) a^2 (b) $\frac{p^{-6}q^9}{r^{12}}$ (c) $p^{\frac{7}{6}}q^{\frac{5}{6}}$ (d) x^{6-m+2n} (e) $\cos\theta \sin\theta$
   (f) 2^{3p-4} (g) $|p^3q^2|$ (h) $\frac{x^2}{y^3}$ (i) x^{s-r} (j) $x^{-\frac{3}{2}}y^{-\frac{4}{3}}$ (k) $a-b$
   (l) $ab^{\frac{1}{3}}+a^{\frac{1}{3}}b$ (m) $|x|\sqrt{2xyz}$ (n) 6 (o) $\frac{1}{3}\sqrt{\frac{6rs}{t}}$ (p) $a^{\frac{1}{2}}-b^{\frac{1}{2}}$
   (q) $p^5q^4r^7$ (r) $\frac{10}{3}x^{-\frac{1}{6}}y$ (s) $6x^{\frac{3}{2}}y^{\frac{3}{5}}$ (t) $27x^{\frac{9}{2}}y^9$

2. (a) 7 (b) 6 (c) -3 (d) nonexistent (e) 3.037
   (f) 6.557 (g) 2 (h) 2.828 (i) -3

3. (a) S, 2.3156×10^2 C, .23156E3
   (b) S, 5.79654×10^5 C, .579654E6
   (c) S, 8.9361×10^0 C, .89361E1
   (d) S, 5.137×10^{-3} C, .5137E−2
   (e) S, 5.167×10^1 C, .5167E2
   (f) S, 5.17683×10^3 C, .517683E4
   (g) S, 8.124×10^{-6} C, .8124E−5
   (h) S, 4.73185×10^3 C, .473185E4
   (i) S, 3.979×10^1 C, .3979E2
   (j) S, 2.3189×10^{-2} C, .23189E−1
   (k) S, 1.7×10^0 C, .17E1
   (l) S, 1.6×10^1 C, .16E2
   (m) S, 4.8×10^{-2} C, .48E−1
   (n) S, 3.75×10^{-1} C, .375E0
   (o) S, 4.0×10^3 C, .4E4
   (p) S, 3.0×10^1 C, .3E2
   (q) S, 2.0×10^1 C, .2E2
   (r) S, 4.0×10^{-3} C, .4E−2
   (s) S, 2.056×10^1 C, .2056E2
   (t) S, 1.6666667×10^0 C, .16666667E1
   (u) S, 3.75×10^{-2} C, .375E−1
   (v) S, 1.3×10^{-2} C, .13E−1
   (w) S, 5.0×10^1 C, .50E2
   (x) S, 3.0×10^1 C, .30E2

4. $(1.4)^x<(.5)^x<(2.5)^{-x}$ if $x<0$. The order is reversed if $x>0$.

5. (a) 57.1 (b) 9.23 (c) 12.0 (d) 9.14 (e) 4.53 (f) 18.3
   (g) .966 (h) 3.52 (i) .308 (j) 21.9

6. (a) 2.48 (b) 3.2 (c) 168.6 (d) 78.4 (e) 1.41 (f) 1.31
   (g) 4.43 (h) 4.29 (i) 4.48 (j) 7.52

CHAPTER 4

Exercise 4-1 (Page 78)

1. 78.2 2. .0818 3. 303 4. .00202 5. 29.8 6. .9899
7. .819 8. 9.634 9. 6.463 10. .9243 11. 2.26 12. 6.18
13. 28.3 14. 2940 15. 1.59 16. 954 17. .00000316
18. 2.25 19. 14.98 20. 1678 21. 1.147 22. .002346 23. .004708
24. 2.713

Exercise 4-2 (Page 80)

2. (a) 1.08 (b) 1.18 (c) 1.20 (d) 1.33 (e) 1.38 (f) 1.44
   (g) 1.50 (h) 1.55
3. (a) (1.04, 11) (b) (1.11, 13) (c) (1.22, 17) (d) (1.27, 19)
   (e) (1.28, 19) (f) (1.37, 23)

Exercise 4-3 (Page 82)

1. (a) 1.25 (b) 1.30 (c) 1.38 (d) 1.45 (e) 1.54 (f) 1.08
2. (a) 1.08 (b) 1.25 (c) 1.30 (d) 1.45 (e) 0.60 (f) 0.85
5. (a) $10^{\log_{10}2} \simeq 10^{.30} \simeq 2$ etc.
   (b) $\log_{10}(10^4) = 4 \log_{10}10 = 4(1) = 4$ etc.

Exercise 4-4 (Page 83)

1. (i) (a) $\log_{10}100 + \log_{10}1000 = 2+3 = 5 = \log_{10}10^5$
   (b) $\log_{10}(.001) + \log_{10}10 = -3+1 = -2 = \log_{10}10^{-2}$
   (c) $\log_{10}2 + \log_{10}3 \simeq .30+.48 = .78 \simeq \log_{10}6$
   (d) $\log_{10}3 + \log_{10}3 \simeq .96 \simeq \log_{10}9$
   (e) $\log_{10}8 + \log_{10}4 \simeq 1.50 \simeq \log_{10}32$
   (f) $\log_{10}6 + \log_{10}3 \simeq 1.26 \simeq \log_{10}18$

   (ii) (a) $\log_{10}100 - \log_{10}1000 = 2-3 = -1 = \log_{10}10^{-1}$
   (b) $\log_{10}(.001) - \log_{10}10 = -3-1 = -4 = \log_{10}10^{-4}$
   (c) $\log_{10}2 - \log_{10}3 \simeq .30-.48 = -.18 \simeq \log_{10}(.667)$
   (d) $\log_{10}3 - \log_{10}3 = 0 = \log_{10}1$
   (e) $\log_{10}8 - \log_{10}4 \simeq .30 \simeq \log_{10}2$
   (f) $\log_{10}6 - \log_{10}3 \simeq .30 \simeq \log_{10}2$

2. (iii) (a) $2\log_{10}100 = 2(2) = 4 = \log_{10}10^4 = \log_{10}(10^2)^2$
   (b) $5\log_{10}100{,}000 = 5(5) = 25 = \log_{10}10^{25} = \log_{10}(10^5)^5$
   (c) $3\log_{10}(.001) = 3(-3) = -9 = \log_{10}10^{-9} = \log_{10}(10^{-3})^3$
   (d) $3\log_{10}2 \simeq .9030 \simeq \log_{10}8$
   (f) $5\log_{10}1 = 5(0) = 0 = \log_{10}1^5$

(iv) (a) $\frac{1}{2}\log_{10}100 = \frac{1}{2}(2) = 1 = \log_{10}10 = \log_{10}\sqrt{100}$
(b) $\frac{1}{5}\log_{10}100{,}000 = \frac{1}{5}(5) = 1 = \log_{10}10 = \log_{10}\sqrt[5]{100{,}000}$
(c) $\frac{1}{3}\log_{10}.001 = \frac{1}{3}(-3) = -1 = \log_{10}10^{-1} = \log_{10}\sqrt[3]{.001}$
(e) $\frac{1}{2}\log_{10}9 \simeq \frac{1}{2}(.96) = .48 \simeq \log_{10}3 = \log_{10}\sqrt{9}$
(f) $\frac{1}{5}\log_{10}1 = \frac{1}{5}(0) = 0 = \log_{10}1 = \log_{10}\sqrt[5]{1}$

Exercise 4-5 (Page 88)

1. (a) 1.5265 (b) 4.2005 (c) 6.9135 (d) 762.25
(e) 1125.4 (f) 0.013104 (g) 13.104 (h) 0.0018591
(i) 65223. (j) 0.21842 (k) 0.012232 (l) 21.276
(m) 0.054088 (n) 0.000044361 (o) 241100000. (p) 57.863
(q) 0.074046 (r) 0.86060 (s) 2.3126 (t) 57.916
(u) 2938.3 (v) 0.26650 (w) 0.0037196 (x) 0.000064625

2. (a) 0.2400 (b) 0.5348 (c) 0.7519 (d) 0.9718
(e) 0.8764 (f) 0.4649 (g) 0.7895 (h) 0.9229
(i) 1.1086 (j) $-1.2489 = -2+0.7511$ (k) 3.5653 (l) 2.9393
(m) $-2.6377 = -3+0.3623$ (n) 1.8289 (o) $-1.0520 = -2+0.9480$ (p) 5.7777
(q) $-.0129 = -1+.9871$ (r) 3.1281 (s) 1.1956 (t) $-1.1685 = -2+0.8315$
(u) 0.6430 (v) $-2.2393 = -3+0.7607$ (w) 2.8531 (x) 5.3399

Exercise 4-6 (Page 91)

1. 47.55	2. 88.50	3. 369.2	4. 43050.	5. 93760.
6. 658.9	7. 1,835,000.	8. 0.05124	9. 1.656	10. 2.017
11. 84.71	12. .0005341	13. .02120	14. .000001686	15. 1412.
16. 54680.	17. 1550	18. .005318	19. 82.05	20. 13.01
21. 195100.	22. 3299	23. 19,404,000.	24. .000002717	25. 16210.
26. 3.403×10^{10}	27. 113.1	28. 307.1	29. 172.4	30. $.6431\times10^{-5}$
31. 5.814	32. 6792.	33. .05057	34. 0.1691	35. 0.8890
36. 17.05	37. 38220	38. 2.813×10^{-6}		39. 1.371×10^{8}
40. 0.7384×10^{-11}				

Exercise 4-7 (Page 94)

1. 9.190	2. 0.8481	3. 14.67	4. 4.683	5. 1.258
6. 1.552	7. 0.8920	8. 25.54	9. 5.252	10. 0.2893
11. 644.1	12. 44,970,000	13. 666.9	14. 83.17	15. .02034

16. 1287 17. 0.9006 18. 0.9202×10^{-8} 19. 0.5774×10^{-5}
20. 18.41 21. 8.751 22. 15.19 23. 5.860 24. 0.2542
25. 1.234 26. 2.825 27. 6.064 28. 4.356 29. 0.2975
30. 0.4075

Exercise 4-8 (Page 96)

1. 3 2. -5 3. 4 4. 3 5. 6
6. 2 7. 3 8. 2 9. 1.8388 10. 1.3473
11. -0.1081 12. -1.4672 13. 1.838 14. 1.068 15. 2.477
16. 0.6254 17. 2.5 18. -1 19. $-\frac{2}{3}$ 20. $-\frac{1}{2}$
21. -2.452 22. -0.9983 23. 0.9590 24. -0.5059 25. 31.62
26. .03727 27. 2.793 28. 12.59 29. 23110 30. .07133
31. 1 32. .008718 33. .02069 34. 0.1014 35. 0 or 0.3020
36. 1000 or 100

Exercise 4-9 (Page 99)

1. $\log_2 210$ 2. $\log_5 8$ 3. $\log_4 175$ 4. $\log_7 675$ 5. $-\log_3 10$
6. $\log_2 15$ 7. $\log_e 6$ 8. $\log_b 1$ 9. $.5\log_e 2$ 10. $\log_u(\frac{1}{4})$
11. $a\log_e(bc^2)$ 12. $\log_{10} 1$ 13. $x+y$ 14. $\log_a \frac{b^v}{c^w}$ 15. $(b+c)\log_a x$
16. $(w-1)\log_x p$ 17. 3 18. 4.5 19. 6 20. $\frac{2}{3}$
21. 4 22. 2 23. .6 24. -1.5 25. 0.8104
26. 1.1656 27. 0.8062 28. 1.2390 29. 2.3028 30. 0.6055
31. 2.0 32. 0.2385 33. 25 34. $\frac{1}{16}$ 35. 2
36. $\sqrt[3]{16}$ 37. $\frac{1}{\pi}$ 38. $10^{\frac{5}{7}}$ 39. 1 40. $3^{\frac{5}{2}}$

Exercise 4-11 (Page 107)

1. 10.44 2. 60.7 3. 42.8 4. 117.0 5. 2100.
6. 33.4 7. 2.74 8. 0.1659 9. 1.178 10. 0.1198
11. .0244 12. 37.9 13. 8.41 14. 2030. 15. .0105
16. 2.80 17. 23.2 17. 0.922 19. 3.72 20. 0.153
21. 5.78 22. 0.603 23. 13.8

Review Exercise 4-12 (Page 108)

2. (a) 14.82 (b) 486. (c) 3.362 (d) 1.673 (e) 0.4143
(f) .01444 (g) 9.644 (h) 1.871 (i) 4.179 (j) 27.15
(k) .01341 (l) 21.83

3. (a) 16.59 (b) 201.2 (c) 7.694 (d) .01058 (e) 17920.
(f) .004926 (g) .09049 (h) 3.561 (i) 427.5 (j) 25.94
(k) 2.018×10^{-5} (l) 0.1188 (m) 9.0923 (n) 1.5763 (o) 0.8379
(p) 5239 (q) 4474. (r) .06742 (s) 0.2458×10^{-5}
(t) 3.209 (u) 5.843 (v) 39.01 (w) 3.640 (x) 0.7033
(y) 1.912 (z) 1.856

4. (a) 2 (b) 7.305 (c) 10.29 (d) 0.6008 (e) 21.89
(f) .01455

5. (a) 5.98 (b) 46.8 (c) 168 (d) 30.6 (e) 6670.
(f) 10.09 (g) 2.546 (h) 3.578 (i) 548 (j) 15.08

CHAPTER 5

Exercise 5-1 (Page 110)

1. (a) \$127 (b) \$403.20 (c) \$652.50
(d) \$514.94 (e) \$248.69 (f) \$3292.61

2. (a) 3% (b) 5% (c) $3\frac{1}{2}$% (d) $2\frac{1}{4}$% (e) 3% (f) $3\frac{1}{4}$%

3. (a) 6 yrs. (b) 8 yrs. (c) 12 yrs. (d) $3\frac{1}{2}$ yrs. (e) 4 yrs. (f) $8\frac{1}{2}$ yrs.

Exercise 5-2 (Page 113)

1. $200\ (1.04)^{12}$;
semi-annually, 9 years;
6%, 7 years;
\$1200, semi-annually, 10 years;
$640\ (1.005)^{36}$
\$680, monthly, $1\frac{1}{4}$ years;
\$900, 4%, 8 years;
quarterly, 10 years;
\$200, 5%, n years;
\$$P$, i%, annually;
\$$P$, $4i$%, quarterly.

2. (a) \$320.21 (b) \$545.88 (c) \$721.74
(d) \$728.27 (e) \$1713.90 (f) \$964.91
(g) \$1213.55 (h) \$971.45
(i) \$670.98 calculated as $500\ (1.04)^7\ (1.02)$
(j) \$1459.30 calculated as $800\ (1.03)^{20}\ (1.01)$

3. (a) 5.063% (b) 8.243% (c) 12.683%
(d) 6.136% (e) 6.090% (f) 12.551%
(g) 12.360%

4. $\{[(100)(1.03)+100]\ (1.03)+100\}\ (1.03)^8 = \$391.55$

Exercise 5-3 (Page 116)

2. (i) \$322 (ii) \$315 (iii) \$328 (iv) about 10 years (v) \$6.33
3. (i) \$682 (ii) \$657 (iii) \$652 (iv) approx. 6 (v) \$22
4. (i) \$464 (ii) \$460 (iii) \$643 (iv) about $5\frac{1}{2}$ (v) \$31

Exercise 5-4 (Page 118)

1. (i) \$493.95 (ii) \$210.41 (iii) \$209.86
   (iv) \$308.32 (v) \$17.47 (vi) \$1103.79
   (vii) \$1105 calculated as $\dfrac{1380}{(1.03)^7(1.015)}$ (viii) \$211 calculated as $\dfrac{250}{(1.02)^8(1.01)}$
2. \$8,807.24
3. \$8,879.70
4. \$1961.55

Exercise 5-5 (Page 121)

1. $e^0 = 1$, $e^{.5} \simeq 1.6$, $e^1 \simeq 2.7$, $e^{1.5} \simeq 4.3$, $e^2 \simeq 7.3$
2. 0.6931; 1.0986; 1.3863; 1.6094; 1.7918; 1.9459; 2.0794; 2.1972
3. (a) 13.5° C (b) 16.09 mins.
4. $k = \log_e 2 = 0.6931$

Review Exercise 5-6 (Page 121)

1. (i) \$1190.00 (ii) \$1263.06
2. 8.24%
3. \$572.09
4. \$1345.08
5. (a) \$316.33 (b) \$317.06 (c) \$1575.85
   (d) \$1563.94 (e) \$636.65 (f) \$809.30
6. (a) 19.562% (b) 8.16% (c) 4.06%
   (d) 18.81% (e) 19.252%
7. (a) \$1353.68 (b) \$1229.31 (c) \$497.38
   (d) \$501.93 (e) \$269.41
9. 29 mins.

CHAPTER 6

Exercise 6-1 (Page 128)

4. Use the same construction as in the Example, with $k > 0$ if $a > 0$ and $k < 0$ if $a < 0$.
   Then $\text{QM} = \sqrt{\dfrac{k}{a}} = \text{MP}$.

5. If the lines $x = a$ and $x = -a$ intersect the graph, the ordinates of the points of intersection are $f(a)$ and $f(-a)$ respectively.
   Since $f(a) = f(-a)$, the line segment joining the points $(a, f(a))$ and $(-a, f(-a))$ on the graph is perpendicular to and bisected by the y-axis.
   In (4), $f(x) = ax^2 = a(-x)^2 = f(-x)$ and so the graph whose equation is $y = ax^2$ is symmetric about the y-axis.

6. The equation $Ax+By+C=0$ includes the equations of lines parallel to the y-axis (set $B=0$) whereas the equation $y=mx+b$ does not.

7. The equation of a line parallel to the y-axis is of the form $x=c$. The graph includes an infinite number of points with coordinates (c, y) ,where y is arbitrary.

Exercise 6-2 (Page 131)

Questions 1-10: The axis of symmetry is the y-axis in each case; the vertices are:

1. $(0, 0), (0, 5), (0, -3), (0, \frac{3}{2})$
2. $(0, 0), (0, 3), (0, -\frac{3}{2}), (0, -5)$
3. $(0, 0), (0, 7), (0, \frac{1}{4}), (0, -5)$
4. $(0, 0), (0, 4), (0, -2), (0, -\frac{3}{2})$
5. $(0, -4)$
6. $(0, 5)$
7. $(0, 4)$
8. $(0, -\frac{1}{4})$
9. $(0, \frac{1}{2})$
10. $(0, \frac{3}{2})$

	Vertex	Axis
11.	$(0, 0)$	$x=0$
	$(-2, 0)$	$x=-2$
	$(4, 0)$	$x=4$
	$(\frac{3}{2}, 0)$	$x=\frac{3}{2}$
12.	$(0, 0)$	$x=0$
	$(-1, 0)$	$x=-1$
	$(3, 0)$	$x=3$
	$(-\frac{4}{3}, 0)$	$x=\frac{4}{3}$
13.	$(0, 0)$	$x=0$
	$(-3, 0)$	$x=-3$
	$(3, 0)$	$x=3$
	$(-\frac{5}{2}, 0)$	$x=-\frac{5}{2}$
14.	$(0, 0)$	$x=0$
	$(-4, 0)$	$x=-4$
	$(5, 0)$	$x=5$
	$(\frac{5}{3}, 0)$	$x-\frac{5}{3}$
15.	$(5, 0)$	$x=5$
16.	$(-3, 0)$	$x=-3$
17.	$(5, 0)$	$x=5$
18.	$(-2, 0)$	$x=-2$
19.	$(-\frac{1}{2}, 0)$	$x=-\frac{1}{2}$
20.	$(\frac{3}{2}, 0)$	$x=\frac{3}{2}$

Exercise 6-3 (Page 133)

1. vertices: $(0, 0), (2, 3), (2, -4), (2, \frac{5}{2})$
   axes: $x=0$ for first, $x-2=0$ for others.
2. vertices: $(0, 0), (-3, -2), (-3, 4), (-3, -\frac{7}{4})$
   axes: $x=0$ for first, $x+3=0$ for others.
3. vertices: $(0, 0), (1, 1), (1, -4), (1, \frac{7}{2})$
   axes: $x=0$ for first, $x-1=0$ for others.
4. vertices: $(0, 0), (-\frac{3}{2}, 4), (-\frac{3}{2}, -2), (-\frac{3}{2}, -\frac{5}{2})$
   axes: $x=0$ for first, $x+\frac{3}{2}=0$ for others.
5. vertices: $(0, 0), (-2, 2), (-2, -2), (-2, -\frac{3}{2})$
   axes: $x=0$ for first, $x+2=0$ for others.
6. vertices: $(0, 0), (\frac{3}{2}, \frac{1}{2}), (\frac{3}{2}, 6), (\frac{3}{2}, -\frac{3}{2})$
   axes: $x=0$ for first, $x-\frac{3}{2}=0$ for others.
7. vertices: $(0, 0), (2, 3), (2, -1), (2, -\frac{3}{2})$
   axes: $x=0$ for first, $x-2=0$ for others.
8. vertices: $(0, 0), (-\frac{5}{2}, \frac{1}{2}), (-\frac{5}{2}, -\frac{3}{2}), (-\frac{5}{2}, 6)$
   axes: $x=0$ for first, $x+\frac{5}{2}=0$ for others.

9. vertices: $(0, 0)$, $(-3, -2)$, $(2, 3)$, $(-2, -3)$
axes: $x=0$, $x+3=0$, $x-2=0$, $x+2=0$

10. vertices: $(0, 0)$, $(3, 2)$, $(\pi, -\frac{1}{2})$, (π, π)
axes: $x=0$, $x-3=0$, $x-\pi=0$, $x-\pi=0$

11. $y=x^2$ 12. $y=-4x^2$ 13. $y=\frac{1}{2}x^2$

14. $y=-\frac{1}{3}x^2$ 15. $y=-\frac{2}{5}x^2$ 16. $y=\frac{2}{3}x^2$

Exercise 6-4 (Page 136)

1. $y=(x+\frac{3}{2})^2-\frac{9}{4}$, $y\geq-\frac{9}{4}$, $(0, 0)$
2. $y=(x-\frac{5}{2})^2-\frac{25}{4}$, $y\geq-\frac{25}{4}$, $(0, 0)$
3. $y=(x+\frac{3}{2})^2-\frac{25}{4}$, $y\geq-\frac{25}{4}$, $(0, -4)$
4. $y=(x-\frac{5}{2})^2+\frac{3}{4}$, $y\geq\frac{3}{4}$, $(0, 7)$
5. $y=-(x-2)^2+2$, $y\leq2$, $(0, -2)$
6. $y=-(x+\frac{3}{2})^2+\frac{29}{4}$ $y\leq\frac{29}{4}$, $(0, 5)$
7. $y=(x-\frac{1}{4})^2+\frac{47}{16}$, $y\geq\frac{47}{16}$, $(0, 3)$
8. $y=-(x+\frac{1}{6})^2+\frac{145}{36}$, $y\leq\frac{145}{36}$, $(0, 4)$
9. $y=-(x+3)^2+2$, $y\leq2$, $(0, -7)$
10. $y=2(x+\frac{3}{2})^2-\frac{3}{2}$, $y\geq-\frac{3}{2}$, $(0, 3)$
11. $y=5(x+1)^2-7$, $y\geq-7$, $(0, 3)$
12. $y=4(x+\frac{17}{8})^2-\frac{13}{2}$, $y\geq-\frac{13}{2}$, $(0, 2)$
13. $y=6(x-\frac{5}{12})^2+\frac{23}{24}$, $y\geq\frac{23}{24}$, $(0, 2)$
14. $y=-3(x+1)^2+8$, $y\leq8$, $(0, 5)$
15. $y=-2(x-\frac{3}{4})^2-\frac{7}{8}$, $y\leq-\frac{7}{8}$, $(0, -2)$
16. $y=4(x-\frac{3}{16})^2-\frac{41}{64}$, $y\geq-\frac{41}{64}$, $(0, -\frac{1}{2})$
17. $y=-6(x+1)^2+1$, $y\leq1$, $(0, -5)$
18. $y=-\frac{1}{2}(x+\frac{3}{2})^2+\frac{25}{8}$, $y\leq\frac{25}{8}$, $(0, 2)$

Exercise 6-6 (Page 140)

1. min. -13 2. max. 4 3. min. $-\frac{49}{4}$
4. min. $-\frac{19}{2}$ 5. min. $-\frac{19}{4}$ 6. max. $-\frac{13}{8}$
7. max. -1 8. max. 2 9. max. $\frac{25}{4}$
10. min. $-\frac{4}{3}$ 11. max. $\frac{9}{5}$ 12. min. $-\frac{97}{16}$
13. max. 5 14. min. -7 15. max. $\frac{37}{8}$
16. (a) 75, 75 (b) 150, 150 17. 10, 10
18. 2, -2 19. (a) 3 sec. (b) 144 ft. (c) 6 sec.
20. 75, 150 21. 3 inches deep, 6 inches wide

22. 40,000 ft., 50 sec.
23. AB = 5 inches, altitude = 5 inches
25. 3 inches
26. 3 square inches
27. $\frac{u^2}{64}$ ft., $\frac{u}{32}$ sec.

Exercise 6-7 (Page 145)

1. $(0, 0)$, $y=0$, $x\geq 0$
2. $(0, 0)$, $y=0$, $x\leq 0$
3. $(-2, 0)$, $y=0$, $x\geq -2$
4. $(-\frac{5}{2}, 0)$, $y=0$, $x\geq -\frac{5}{2}$
5. $(\frac{2}{3}, 0)$, $y=0$, $x\leq \frac{2}{3}$
6. $(2, 0)$, $y=0$, $x\leq 2$
7. $(0, 2)$, $y=2$, $x\geq 0$
8. $(0, -1)$, $y=-1$, $x\leq 0$
9. $(-\frac{1}{2}, -\frac{1}{2})$, $y=-\frac{1}{2}$, $x\geq -\frac{1}{2}$
10. $(\frac{3}{4}, 4)$, $y=4$, $x\geq \frac{3}{4}$
11. $(\frac{1}{2}, -2)$, $y=-2$, $x\geq \frac{1}{2}$
12. $(\frac{5}{3}, \frac{3}{2})$, $y=\frac{3}{2}$, $x\leq \frac{5}{3}$
13. $x\geq 2$
14. $x\geq -3$
15. $x\leq 2$
16. $x\leq 3$
17. $x\geq -\frac{3}{2}$
18. $x\leq \frac{3}{2}$
19. $x\geq -3$
20. $x\geq 2$
21. $x\leq 3$
22. $x\geq \frac{1}{2}$
23. $x\leq \frac{2}{3}$
24. $x\geq \frac{1}{4}$
25. $\left(\frac{7}{2}, \frac{\sqrt{2}}{2}\right)$
26. (a) -2 (b) 2 (c) 8.84 (approx.)

Review Exercise 6-8 (Page 147)

2. (a) $(0, 2)$, $x=0$, $(0, 2)$
   (b) $(0, 0)$, $x=0$, $(0, 0)$
   (c) $(-3, 0)$, $x=-3$, $(0, \frac{9}{5})$
   (d) $(1, 3)$, $x=1$, $(0, \frac{16}{5})$
   (e) $(-2, 1)$, $x=-2$, $(0, \frac{1}{5})$
   (f) $(2, -\frac{1}{2})$, $x=2$, $(0, \frac{30}{1})$
3. (a) $(0, -2)$, $x=0$, down, $(0, -2)$
   (b) $(-3, 0)$, $x=-3$, down, $(0, -18)$
   (c) $(0, 0)$, $x=0$, up, $(0, 0)$
   (d) $(-2, -1)$, $x=-2$, down, $(0, -9)$
   (e) $(2, -3)$, $x=2$, down, $(0, -11)$
   (f) $(-1, 5)$, $x=-1$, down, $(0, 3)$
4. $(x+\frac{7}{2})^2-\frac{49}{4}$
5. $(x-\frac{1}{4})^2-\frac{1}{16}$
6. $(x+1)^2-5$
7. $(x-\frac{5}{2})^2-\frac{33}{4}$
8. $-(x+1)^2+4$
9. $-2(x-\frac{1}{4})^2+\frac{9}{8}$
10. $3(x-\frac{1}{12})^2+\frac{95}{48}$
11. $3(x-\frac{1}{18})^2+\frac{211}{540}$
12. $-\frac{1}{2}(x+2)^2+\frac{7}{3}$
13. $y\geq -\frac{49}{4}$, min.
14. $y\geq -\frac{1}{16}$, min.
15. $y\geq -5$, min.
16. $y\geq -\frac{33}{4}$, min.
17. $y\leq 4$, max.
18. $y\leq \frac{9}{8}$, max.
19. $y\geq \frac{95}{48}$, min.
20. $y\geq \frac{211}{540}$, min.
21. $y\leq \frac{7}{3}$, max.
22. $\frac{11}{4}$, min.
23. 5, max.
24. $\frac{23}{8}$, min.
25. $-\frac{71}{12}$, max.
26. $\frac{1}{4}$, min.
27. -5, min.
28. 49, 49
29. $\frac{1}{2}$
30. 5
31. $y^2=-(x-3)$
32. $(y+2)^2=x-2$
33. $(y-4)^2=-2(x-\frac{1}{2})$
34. (a) $x\leq 3$, $y\geq 0$
    (b) $x\geq 2$, $y\geq -2$
    (c) $x\leq \frac{1}{2}$, $y\leq 4$
35. 6.2 (approx.)

CHAPTER 7

Exercise 7-1 (Page 151)

The approximate values of the real roots of the associated quadratic equation are given for (3)-(16)).

3. -2
4. 1
5. none
6. ± 1.4
7. ± 1.4
8. none
9. .6, -4.6
10. none
11. .4, 5.6
12. 7.6, -1.6
13. .3, 3.7
14. .6, -4.6
15. 2.5, -0.5
16. none

17. The following are the solution sets (approximations used) of the associated quadratic equations (two each).

17-1 $\{\pm .7\}$; ϕ
17-2 ϕ; $\{\pm .9\}$
17-3 $\{-1, -3\}$; ϕ
17-4 ϕ; $\{.1, 1.9\}$
17-5 $\{0\}$; ϕ
17-6 $\{\pm 1.1\}$; $\{\pm 1.7\}$
17.7 $\{\pm 1.9\}$; $\{\pm .7\}$
17-8 ϕ; ϕ
17-9 $\{-5.2, 1.2\}$; $\{0, -4\}$
17-10 ϕ; ϕ
17-11 $\{-.2, 6.2\}$; $\{1, 5\}$
17-12 $\{-1.4, 7.4\}$; $\{-2.2, 8.2\}$
17-13 $\{2\}$; $\{-.4, 4.4\}$
17-14 $\{-4.8, .8\}$; $\{.2, -4.2\}$
17-15 $\{-.1, 2.1\}$; $\{-.8, 2.8\}$
17-16 $\{-.8, 1.1\}$; ϕ

In the following we list one set of values for a, b, and c in (18)-(38). Your set may be ka, kb, kc, where $k \neq 0$.

24. 1, 1, -11
25. 1, 0, -9
26. 3, 4, 3
27. 5, -1, -8
28. 3, -2, 0
29. 1, -1, 1
30. 1, -5, 1
31. 6, -12, 1
32. 4, -20, -1
33. 5, -2, -5
34. 2, 0, -21
35. 6, 16, 9
36. 0, 2, 1
37. 1, 0, $-\frac{7}{2}$
38. 2, 0, -1.

39-30 $x \neq 0$
39-31 $x \neq 0$
39-32 $x \neq 0$
39-33 $x \neq \pm 1$
39-34 $x \neq \pm 3$
39-35 $x \neq -1, x \neq -2$
39-36 $x \neq \pm 2$
39-37 $x \neq 0$
39-38 $x \neq 0, x \neq 1$

Exercise 7-2 (Page 155)

11. 3, -2
12. $-\frac{1}{3}, -\frac{5}{2}$
13. $-\frac{3}{2}, -5$
14. $\frac{2}{3}$, --5
15. $-\frac{1}{3}, \frac{9}{2}$
16. $\frac{3}{4}, -\frac{5}{3}$
17. $\frac{1}{6}$, 1
18. $-\frac{1}{2}, -3$
19. $\frac{2}{3}, -1$
20. 8, 3
21. 4, 1
22. $-\frac{1}{4}, \frac{3}{2}$
23. $\frac{2}{3}$, 1
24. 2, 10
25. $(\frac{1}{4}, 3), (-\frac{3}{2}, 3)$
26. $(\frac{3}{2}, -1), (1, -1)$
27. $(\frac{3}{4}, 5), (-\frac{5}{2}, 5)$
28. $(-\frac{3}{2}, -5), (-\frac{5}{3}, -5)$
29. $(-\frac{1}{3}, 8), (2, 8)$
30. none
31. $a-3, a+4$
32. $2m-5, 2m+1$
33. $2\sqrt{2}, \frac{3}{2}\sqrt{2}$
34. $-\frac{5}{2}\sqrt{6}, \frac{1}{2}\sqrt{6}$.

Exercise 7-3 (Page 157)

1. $0, \frac{\pi}{2}$
2. $\frac{\pi}{3}, \frac{\pi}{2}$
3. $\frac{\pi}{2}$
4. $\frac{\pi}{2}$
5. no roots
6. no roots
7. $\frac{\pi}{2}$
8. $\frac{\pi}{2}$
9. no roots
10. 0
11-1 π
11-2 no roots
11-3 $\frac{3\pi}{4}$
11-4 no roots
11-5 $\frac{2}{3}\pi$
11-6 no roots
11-7 no roots
11-8 $\frac{2\pi}{3}$
11-9 no roots
11-10 π
12. 2
13. 0, 4
14. ± 2
15. ± 2
16. 3
17. no roots
18. $-2, -3$
19. 0
20. 0, 1
21. -3
22. 2
23. no roots
24. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

Exercise 7-4 (Page 160)

4. $\{-2 \pm \sqrt{3}\}$
5. $\{-3 \pm \sqrt{13}\}$
6. $\{3 \pm 3\sqrt{2}\}$
7. $\{5 - \sqrt{22}\}$
8. $\{-4 \pm \sqrt{23}\}$
9. $\{\frac{1}{2}(5 \pm \sqrt{17})\}$
10. $\{-6, 2\}$
11. $\{\frac{1}{2}(7 \pm \sqrt{37})\}$
12. $\{\frac{1}{2}(5 \pm \sqrt{65})\}$
13. $\{6 \pm 2\sqrt{11}\}$
14. $\{\frac{1}{2}(11 \pm \sqrt{149})\}$
15. $\{3 \pm 2\sqrt{2}\}$
16. $\{\frac{2}{3}, 1\}$
17. $\{\frac{6}{5}, -2\}$
18. $\{\frac{1}{4}, -2\}$
19. $\{-\frac{3}{4}, 2\}$
20. $\{\frac{1}{4}(3 \pm \sqrt{41})\}$
21. $\{\frac{3}{2}, \frac{4}{3}\}$
22. $\{\frac{3}{2}, -4\}$
23. no real roots
24. $\{\frac{5}{2}, -\frac{2}{3}\}$
25. $\{\frac{1}{2}(1 \pm \sqrt{113})\}$
26. $\{-a, 3a\}$
27. $\left\{\frac{b}{2}(1 \pm \sqrt{5})\right\}$
28. $\{-a \pm \sqrt{a^2+b}\}$
29. $\{6a \pm 3\sqrt{2}a\}$
30. $\left\{\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right\}$, real if $b^2-4ac \geq 0$

Exercise 7-5 (Page 164)

9. $\{\frac{1}{4}(15 \pm \sqrt{393})\}$
10. $\{-\frac{11}{4}, 5\}$
11. $\{-1 \pm \frac{1}{3}\sqrt{15}\}$
12. $\{\frac{1}{7}(1 \pm 2\sqrt{2})\}$
13. $\{-.6, 2.4\}$
14. $\{-4.7, 1.7\}$
15. $\{-.6, 6.6\}$
16. $\{-.7, 2.7\}$
17. $\{0.4, 7.6\}$
18. $\{0.5, 1.0\}$
19. $\{-.7, 1.4\}$
20. $\{-.7, .5\}$
21. $\{-3.4, -.6\}$
22. $\{-7, 1\}$
23. $\{\frac{1}{2}(1 \pm \sqrt{5})\}$
24. $\{\frac{2}{3}, \frac{3}{2}\}$
25. $\{\frac{1}{2}(1 \pm \sqrt{17})\}$
26. $\{\frac{1}{2}(5 \pm \sqrt{65})\}$
27. $\{2, 8\}$
28. no real roots
29. $\{2, 16\}$

30. $\{-3, 4\}$
31. no real roots
32. $\{-\frac{7}{4}, \frac{5}{2}\}$
33. $\{-2, 5\}$
34. $\{\frac{2}{3}\}$
35. $\{\pm 10\sqrt{5}\}$
36. $\{\frac{1}{18}(5\pm\sqrt{457})\}$
37. $\{\frac{1}{4}(5\pm\sqrt{145})\}$
38. $\{-.92, .86\}$ approx.
39. $\{\frac{1}{2}(5\pm\sqrt{105})\}$
40. $\{\frac{1}{6}(1\pm\sqrt{19})\}$
41. $\{\frac{1}{12}(-15\pm\sqrt{105})\}$
42. $\{\frac{1}{3}(2\pm\sqrt{19})\}$
43. $\{5, -6\}$
44. $\{\frac{1}{8}(13\pm\sqrt{393})\}$
45. $\{\frac{1}{18}(-1\pm\sqrt{109})\}$
46. no real roots
47. no roots
48. 58°
49. 13°
50. 34°

Exercise 7-6 (Page 167)

6. no real roots
7. $\{\frac{1}{40}(-1\pm\sqrt{1201})\}$
8. $\{3\pm 3\sqrt{2}\}$
9. no real roots
10. $\{-1, 975\}$
11. $\{-2, \frac{3}{4}\}$
12. $\{\frac{1}{4}(-5\pm\sqrt{673})\}$
13. $\{\sqrt{3}\pm\sqrt{5}\}$
14. $\{\sqrt{2}, -3\sqrt{2}\}$
15. $\{-2\}$
16. $\{\frac{1}{2}(29\pm\sqrt{909})\}$
17. $\{\frac{1}{4}(-3\pm\sqrt{105})\}$
18. $\{2\pm\frac{1}{2}\sqrt{6}\}$
19. $\{-6\pm 2\sqrt{17}\}$
20. $\{1, 5\}$
21. $\left\{-2, \frac{1}{3-h}\right\}$ $(h\neq 3)$
22. $\{-k-h, h-k\}$
23. $\left\{\frac{r}{s}, \frac{s}{r}\right\}$, $r\neq 0$, $s\neq 0$
24. $\left\{\frac{1}{a-b}, \frac{1}{a+b}\right\}$, $a^2\neq b^2$
25. $\{-3, \frac{7}{2}\}$
26. $\{\pm\sqrt{\frac{7}{2}}\}$
27. $\{\pm\sqrt{10}, \pm 2\}$
28. $\{-6, 1, \frac{1}{2}(-5\pm\sqrt{21})\}$
29. $\{0, \pm 1\}$
30. no real roots

Exercise 7-7 (Page 171)

2. unequal, irrat.
3. unequal, irrat.
4. nonreal
5. equal, rat.
6. unequal, irrat.
7. nonreal
8. unequal, irrat.
9. unequal, irrat.
10. unequal, irrat.
11. unequal, irrat. if $k<4$; equal, rat. if $k=4$
12. same as (11)
13. 6
14. 0, 6
15. $\frac{2}{3}$, 2
16. 1, 4
17. 9
18. 3
19. $\pm\frac{2}{\sqrt{3}}$
20. 4
21. -2
22. $\frac{1}{17}(14\pm\sqrt{43})$
23. $1, -\frac{1}{2}$
24. none
25. (a) $\{k\in \mathrm{Re}|k>\frac{9}{16}\}$ (b) $\{k\in \mathrm{Re}|k<\frac{9}{16}\}$
26. (a) $\{k\in \mathrm{Re}||k|<3\}$ (b) $\{k\in \mathrm{Re}||k|>3\}$
27. (a) $\{k\in \mathrm{Re}|k<-\frac{1}{72}\}$ (b) $\{k\in \mathrm{Re}|k>-\frac{1}{72}\}$
28. (a) $\{k\in \mathrm{Re}|k>\frac{5}{4}\}$ (b) $\{k\in \mathrm{Re}|k<\frac{5}{4}\}$
29. (a) $\{k\in \mathrm{Re}|-2\sqrt{5}-2<k<2\sqrt{5}-2\}$
(b) $\{k\in \mathrm{Re}|k<-2\sqrt{5}-2\}\cup\{k\in \mathrm{Re}|k>2\sqrt{5}-2\}$

30. (a) $\{k \in \mathrm{Re} | k > \frac{49}{12}\}$ (b) $\{k \in \mathrm{Re} | k < \frac{49}{12}\}$

31. Real for $\{a \in \mathrm{Re} | a < \frac{1}{4}\}$

33. $\{m | m < \frac{n^2}{4}; m, n \in \mathrm{Re}\}$ 34. No

Exercise 7-8 (Page 173)

1. $\{\frac{1}{2}(-1 \pm i\sqrt{3})\}$ 2. $\{\frac{1}{2}(1 \pm i\sqrt{3})\}$ 3. $\{0 \pm 2i\}$
4. $\{-1 \pm i\sqrt{3}\}$ 5. $\{\frac{1}{2}(3 \pm i)\}$ 6. $\{\frac{1}{4}(1 \pm i)\}$
7. $\{\frac{1}{6}(1 \pm i\sqrt{11})\}$ 8. $\{\frac{1}{16}(1 \pm i\sqrt{95})\}$ 9. $\{\frac{1}{16}(1 \pm \sqrt{97})\}$
10. $\{\frac{1}{20}(-1 \pm i\sqrt{79})\}$ 11. $\{\frac{1}{6}(1 \pm i\sqrt{17})\}$ 12. $\{\frac{1}{8}(3 \pm i\sqrt{23})\}$

Exercise 7-9 (Page 175)

(*Note*: For the purpose of this section, a possible real root that cannot be checked without involving nonreal numbers will be considered as not a root.)

4. $2, \frac{5}{4}$ 5. no solution 6. -1
7. no solution 8. -4 9. no solution
10. no solution 11. -1 12. $1+\frac{2}{3}\sqrt{3}$
13. 9 14. $4+2\sqrt{2}$ 15. no solution
16. 0 17. no solution 18. $\frac{9}{11}$
19. no solution; possible roots are $\frac{11 \pm \sqrt{409}}{24}$, that is, $\frac{11 \pm 20}{24}$ (approx.)
that is, $\frac{5}{4}, -\frac{1}{3}$ (approx.)
20. no solution 21. $0, \frac{2}{3}$ 22. $-2+\frac{2}{7}\sqrt{7} \simeq -1.3$
23. -3 24. no solution 25. no solution
26. $\frac{61}{4}$ 27. $\frac{n-m}{2}$ 28. $3, -7$

Exercise 7-10 (Page 178)

1. 24, 72 2. 3 3. 8, 18 4. 20, 4
5. 2, 14 in. 6. 11, 14 7. 100 sq. in. 8. 15, 17
9. 16, 18 10. no solution 11. 11, 13 12. no solution
13. 5 14. 4 15. length 7 in. 16. base 8 in.
17. no 18. no 19. Mr. Cheer: 6 rows of 6 desks
Mr. Joy: 7 rows of 5 desks
20. 3, 4.5 21. 5 22. a 23. 40 m.p.h.
24. 40 m.p.h. 25. 240 m.p.h. 26. 5 ft. 27. 75
28. $\sqrt{34}$ in. 29. 3 30. $\frac{5}{2}$ and 3 secs. 31. 8

32. either \$40 or \$60 33. 22.75 ft. 34. 87 ft.
35. 55.6 and 75.6 m.p.h. 36. 7.9 ft./sec. 37. 600
38. 32 students 39. 1250 40. 7, 24 inches 41. 67
42. first 20 43. $(-1, 1)$

Exercise 7-11 (Page 183)

10. $\{x \in \mathrm{Re} | x > \frac{3}{2}\} \cup \{x \in \mathrm{Re} | x < -1\}$
11. $\{x \in \mathrm{Re} | -1 < x < \frac{3}{2}\}$
12. $\{x \in \mathrm{Re} | -\frac{3}{2} < x < \frac{2}{3}\}$
13. $\{x \in \mathrm{Re} | \frac{1}{6} < x < \frac{1}{5}\}$
14. $\{x \in \mathrm{Re} | x < -3\} \cup \{x \in \mathrm{Re} | x > \frac{1}{3}\}$
15. $\{x \in \mathrm{Re} | x > -1\}$
16. $\{x \in \mathrm{Re} | x \geq 4\} \cup \{x \in \mathrm{Re} | x \leq 3\}$
17. $\{x \in \mathrm{Re} | 3 \leq x \leq 4\}$
18. $\{x \in \mathrm{Re} | -\frac{5}{2} \leq x \leq \frac{4}{3}\}$
19. (a) $\{k \in \mathrm{Re} | |k| < 6\}$ (b) $\{k \in \mathrm{Re} | |k| > 6\}$
20. (a) $\{k \in \mathrm{Re} | k > 2\} \cup \{k \in \mathrm{Re} | k < \frac{2}{3}\}$ (b) $\{k \in \mathrm{Re} | \frac{2}{3} < k < 2\}$
21. (a) $\{k \in \mathrm{Re} | 1 < k < 4\}$ (b) $\{k \in \mathrm{Re} | k < 1\} \cup \{k \in \mathrm{Re} | k > 4\}$
22. (a) ϕ (b) $\{k \in \mathrm{Re} | k \neq 3\}$
23. (a) $\{k \in \mathrm{Re} | 0 < k < 4\}$ (b) $\{k \in \mathrm{Re} | k < 0\} \cup \{k \in \mathrm{Re} | k > 4\}$
24. (a) $\{k \in \mathrm{Re} | 1 < k < 2\}$ (b) $\{k \in \mathrm{Re} | k < 1\} \cup \{k \in \mathrm{Re} | k > 2\}$
25. (a) ϕ (b) $\{k \in \mathrm{Re} | k \neq -\frac{1}{2}\}$
26. (a) $\{k \in \mathrm{Re} | k \neq 0\}$ (b) ϕ

Review Exercise 7-12 (Page 184)

1. $(0, 9)$; $(\frac{3}{2}, 0)$ 2. $(0, -5)$; $(-\frac{1}{3}, 0)$, $(5, 0)$ 3. $(0, 4)$; none
4-1 $(.6, 3)$, $(2.4, 3)$; none 4-2 $(-.5, 3)$, $(5.2, 3)$; $(-.1, -3)$, $(4.8, -3)$
4-3 none; none
5. $\{\frac{1}{3}, \frac{5}{2}\}$ 6. $\{-\frac{1}{2}, 2\}$ 7. $\{-\frac{5}{2}, \frac{1}{4}\}$
8. $\{\frac{1}{2}(5 \pm i\sqrt{39})\}$ 9. $\{\frac{1}{3}(1 \pm i\sqrt{14})\}$ 10. $\{3 \pm \sqrt{19}\}$
11. $\{-1, 4\}$ 12. $\{2 \pm \frac{3}{2}\sqrt{2}\}$ 13. $\{\frac{1}{6}(-2 \pm \sqrt{10})\}$
14. (a) $\left(\frac{-2+\sqrt{39}}{5}, 4\right), \left(\frac{-2-\sqrt{39}}{5}, 4\right)$ (b) none
15. .27 rads. 16. 2 18. $\{-\frac{2}{7}\}$ 19. $\left\{\pm 3, \pm\sqrt{\frac{13}{2}}\right\}$
20. $\left\{\frac{31 \pm \sqrt{829}}{22}\right\}$ 21. $\left\{\frac{-13 \pm \sqrt{505}}{24}\right\}$ 22. nonreal 23. equal, irrat.
24. unequal, irrat. 25. unequal, irrat.
26. (a) $\frac{49}{8}$ (b) $\{k \in \mathrm{Re} | k < \frac{49}{8}\}$ (c) $\{k \in \mathrm{Re} | k > \frac{49}{8}\}$
27. 2 28. no real solution 29. 0 30. $2 + \frac{2}{3}\sqrt{3}$
31. 7 32. No 33. 13, 16 34. 9, 12

35. 300 36. 10 and 20 or 40 and 50 mph. 37. $\frac{1}{4}$, $\frac{9}{4}$ sec.

38. $(-1, -5)$, $(-1, -1)$, $(2, -2)$, $(2, -1)$ 39. 4 or 8 in.

40. 4 in. 41. $\frac{2}{3}$ in. 42. $\{x \in \mathrm{Re} | x > \frac{1}{3}\} \cup \{x \in \mathrm{Re} | x < \frac{1}{5}\}$

43. $\{x \in \mathrm{Re} | -5 \leq x \leq \frac{3}{2}\}$ 44. $\{k \in \mathrm{Re} | -1 < k < 5\}$

CHAPTER 8

Exercise 8-1 (Page 189)

14. 7; 2; 14; k^2-6k+7; $\frac{1}{x^2}-\frac{6}{x}+7$; $\frac{7x^2-20x+14}{(x-1)^2}$

15. $\frac{1}{2}$; $\frac{7}{2}$; $\frac{1}{2}\sqrt{7}-\sqrt{2}$; 1; $4x^2-x+\frac{1}{2}$ 16. -2; 4; $\frac{3}{4}$; $3-5\sqrt{5}$; t^2-3t-6

17. -1; -7; a^3-3a^2+2a-1; $\frac{1}{x^3}-\frac{3}{x^2}+\frac{2}{x}-1$; $4\sqrt{2}-7$

18. 3; 2; 6; $\frac{1}{y^4}-\frac{2}{y}+3$ 19. $3 \pm \sqrt{2}$ 20. $\frac{4}{3}$, -1

21. $\frac{1}{2} \pm \frac{\sqrt{29}}{2}$ 22. $\frac{1}{10} \pm \frac{\sqrt{21}}{10}$ 23. 0 24. 0, 1, -1

Exercise 8-2 (Page 192)

1. (i) $(x-1)(x-5)+2$ (ii) $(x+3)(x-9)+34$ (iii) $(x+\frac{1}{2})(x-\frac{13}{2})+\frac{41}{4}$

2. (i) $(x+3)(x-16)+84$ (ii) $(x-4)(x-9)+0$ (iii) $(x-\frac{1}{3})(x-\frac{38}{3})+\frac{286}{9}$

3. (i) $(x-2)(4x+2)+7$ (ii) $(x+5)(4x-26)+133$ (iii) $(x+\frac{1}{4})(4x-7)+\frac{19}{4}$

4. (i) $(x+1)(x^2-7x+10)-12$ (ii) $(x-3)(x^2-3x-6)-20$
(iii) $(x-\frac{3}{2})(x^2-\frac{9}{2}x-\frac{15}{4})-\frac{61}{8}$

5. (i) $(x+2)(3x^2-11x+23)-45$ (ii) $(x-1)(3x^2-2x-1)+0$
(iii) $(x+\frac{1}{3})(3x^2-6x+3)+0$

6. (i) $(x-1)(4x^2+4x-2)$ (ii) $(x+1)(4x^2-4x-2)$
(iii) $(x-\frac{1}{2})(4x^2+2x-5)-\frac{1}{2}$

7. (i) $(x+2)(6x^3-12x^2+22x-44)+91$ (ii) $(x-2)(6x^3+12x^2+22x+44)$
(iii) $(x+\frac{1}{3})(6x^3-2x^2-\frac{4}{3}x+\frac{4}{9})+\frac{77}{27}$

8. (i) $(x-1)(2x^4+2x^3+2x^2+2x+2)-\frac{1}{x-1}$
(ii) $(x+2)(2x^4-4x^3+8x^2-16x+32)$
(iii) $(x+\frac{1}{2})(2x^4-x^3+\frac{1}{2}x^2-\frac{1}{4}x+\frac{1}{8})-\frac{49}{16}$

Exercise 8-3 (Page 193)

5. $(x-2)(x^2+1)$ 6. $(x+1)(x^2-2)$ 7. $(x-1)(2x^2+3)$

8. $(x+2)(3x^2-4)$ 9. $(x+1)(x+2)(x^2-x+1)$

10. no integral zeros 11. $(x-1)(x+1)(x-2)(x+2)$

12. $(x-2)(x^3+x+1)$

Exercise 8-4 (Page 195)

6. 6 7. $\frac{13}{8}$ 8. 2 9. $-\frac{5}{2}$ 10. -6 11. $\frac{15}{4}$ 12. 13 13. 2
14. $\frac{10}{3}$ 15. 2 16. $\frac{3}{10}$

Exercise 8-5 (Page 198)

4. irreducible
5. $(t-2+2\sqrt{3})(t-2-2\sqrt{3})$
6. $3(x+1)(x+\frac{2}{3})$
7. $3\left(y+\frac{5}{6}+\frac{\sqrt{13}}{6}\right)\left(y+\frac{5}{6}-\frac{\sqrt{13}}{6}\right)$
8. irreducible
9. $12(a-\frac{1}{2})(a-\frac{1}{6})$
10. $6\left(x-\frac{1}{12}+\frac{\sqrt{97}}{12}\right)\left(x-\frac{1}{12}-\frac{\sqrt{97}}{12}\right)$
11. $12(b-\frac{5}{2})(b-\frac{1}{6})$
12. $6\left(y+\frac{1}{6}+\frac{\sqrt{19}}{6}\right)\left(y+\frac{1}{6}-\frac{\sqrt{19}}{6}\right)$
13. $4\left(x+\frac{1}{4}+\frac{\sqrt{29}}{4}\right)\left(x+\frac{1}{4}-\frac{\sqrt{29}}{4}\right)$
14. irreducible
15. $5(z+1)(z-\frac{4}{10})$

Exercise 8-6 (Page 200)

4. $\left(x-1\right)\left(x+\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(x+\frac{5}{2}-\frac{\sqrt{21}}{2}\right)$
5. $(y-1)(y-3)(y-7)$
6. $(x+1)(x-3)(x-5)$
7. $(x-2)(x^2-2x+3)$
8. $2(z-1)(z-2)(z-9)$
9. $3(x-4+\sqrt{13})(x-4-\sqrt{13})$
10. $\left(t-1\right)\left(t+\frac{1}{2}+\frac{\sqrt{17}}{2}\right)\left(t+\frac{1}{2}-\frac{\sqrt{17}}{2}\right)$
11. $(x-2)(x+4)(2x-1)$
12. $(a+2)(a^2-2a+4)$
13. $(3z-2)(9z^2+6z+4)$
14. $(b+2)(b-2)(b^2+3)$
15. $\left(x-\sqrt{\frac{7-3\sqrt{5}}{2}}\right)\left(x+\sqrt{\frac{7-3\sqrt{5}}{2}}\right)\left(x-\sqrt{\frac{7+3\sqrt{5}}{2}}\right)\left(x+\sqrt{\frac{7+3\sqrt{5}}{2}}\right)$
16. $t(t-1)(t^2+t+1)$
17. $t(t-1)(t+1)(t^2+1)$
18. $(y-1)(y+2)(y-\sqrt{3})(y+\sqrt{3})$
19. $(y^2+3)(y^2+2)$
20. $-1, 3, -3$
21. $2, -3, -3$
22. $1, -2, 4$
23. $-5, \sqrt{6}, -\sqrt{6}$
24. 2
25. 4
26. $-1, -2, 3$
27. $3, 3, -1$

Exercise 8-7 (Page 203)

8. $1, \frac{bc}{a}$
9. $-\frac{1}{4}, -\frac{11}{4}$
10. $\frac{13}{9}, \frac{8}{9}$
11. $\frac{b}{a^2-b^2}, \frac{-ab}{a^2-b^2}$
12. $\frac{-q}{p}, \frac{-r}{p}$
13. $b=0, a=-c$
14. -2
15. 14
16. $\frac{5}{2}$
17. $\frac{2}{3}, -\frac{1}{2}$
19. (a) $\frac{9}{10}$ (b) $-\frac{25}{8}$
20. (a) 0 (b) $\frac{8}{11}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$ (e) 3 (f) $-\frac{1}{2}$ (g) $\frac{3}{2}$
21. No

Exercise 8-8 (Page 205)

5. $x^2-4x+1=0$
6. $4x^2-8x-1=0$
7. $4x^2-2(p+q)x+pq=0$
8. $x^2-2px+p^2-q^2=0$
13. $x^2-8x+13=0$
14. $9x^2-18x+7=0$
15. $4x^2-4px+p^2-q^2=0$
16. $4px^2-(p^2+8)x+2p=0$
17. $x^2-(q+2)x+2q=0$
18. $x^2-qx-6q^2=0$
19. $x^2-8x-q^2+4q+12=0$
20. $q^3x^2-q(q+1)x+1=0$
21. 27
22. $\dfrac{24\sqrt{2}-\sqrt{3}}{\sqrt{3}}=8\sqrt{6}-1$

Exercise 8-9 (Page 207)

1.	(a) $x^2+6x+7=0$	(b) $x^2-22x+49=0$	(c) $7x^2-6x+1=0$
2.	(a) $x^2-9=0$	(b) $x^2-18x+81=0$	(c) $9x^2-1=0$
3.	(a) $y^2+3y+4=0$	(b) $y^2-y+16=0$	(c) $4y^2-3y+1=0$
4.	(a) $2x^2+5x+7=0$	(b) $4x^2+3x+49=0$	(c) $7x^2-5x+2=0$
5.	(a) $-z^2-3z+2=0$	(b) $-z^2+13z-4=0$	(c) $-2z^2+3z+1=0$
6.	(a) $3x^2+4x-1=0$	(b) $9x^2-22x+1=0$	(c) $x^2+4x-3=0$
7.	(a) $t^2+\frac{1}{2}t-\frac{1}{3}=0$	(b) $12t^2-11t-2=0$	(c) $2t^2+3t-6=0$
8.	(a) $5x^2+\frac{1}{2}x+3=0$	(b) $100x^2+119x+36=0$	(c) $6x^2-x+10=0$
9.	(a) $-2x^2-4x=9$	(b) $4x^2+20x+81=0$	(c) $9x^2-4x+2=0$
10.	(a) $3a^2+a-7=0$	(b) $9a^2-43a+49=0$	(c) $7a^2-a-3=0$
11.	(a) $y^2-y+1=0$	(b) $y^2+y+1=0$	(c) $y^2+y+1=0$
12.	(a) $z^2-az+b=0$	(b) $z^2+(2b-a^2)z+b^2=0$	(c) $bz^2+az+1=0$

Review Exercise 8-10 (Page 208)

3. x^3-x+5
4. (c) $(x-\frac{1}{2})(5x-\frac{3}{2})+\frac{5}{4}$
5. (c) $(x+3)(4x^2-11x+34)-108$
6. (c) $(x-1)(x^4+2x^2-4x-1)-2$
10. $1, \frac{1}{2}$
11. (a) no (b) no
12. true
13. true
14. false
15. false
16. true
17. true
18. false
19. true
20. true
21. true
22. true
25. 3, 2, −1
26. $2(4x-3)(x-4)(x-\frac{1}{2}\sqrt{2})(x+\frac{1}{2}\sqrt{2})$
27. $(x-1)(x+1)(x-3)(x^2+3)$
28. $(y+3)(y^2+2y+3)$
29. -4
30. $(x-4)(ax+b)=ax^2+(b-4a)x-4b=0$
31. $[a+\frac{1}{2}(1+\sqrt{29})]\,[a+\frac{1}{2}(1-\sqrt{29})]$
32. (e) $\dfrac{3}{\sqrt{2}}, \dfrac{5}{2\sqrt{2}}$ (f) $-8, -2\sqrt{3}$

33.	(a) $4x^2+5x-5=0$	(b) $16x^2-65x+25=0$	(c) $5x^2+5x-4=0$
34.	(a) $6y^2-7=0$	(b) $3y^2-7=0$	(c) $7y^2-6=0$
35.	(a) $5z^2-z=0$	(b) $25z^2-z=0$	(c) nonexistent
36.	(a) $-3x^2+2x+5=0$	(b) $9x^2-34x+25=0$	(c) $5x^2-2x-3=0$

CHAPTER 9

Exercise 9-1 (Page 212)

7. $P(0)=-3$, $Q(0)=-\frac{3}{4}$, etc. 8. Yes; $P(x)=Q(x)$ for all x

9. $P(0)=-6$, $Q(0)=-3$; $P\left(\frac{5\pm\sqrt{73}}{4}\right)=Q\left(\frac{5\pm\sqrt{73}}{4}\right)=0$

11. The x-intercepts of the curve whose equation is $y=x^3-3x-4$ are the roots of the equation $x^3-3x-4=0$.

Exercise 9-2 (Page 214)

1. x^3+x-1 2. $x^3-13x+16$ 3. $x^3-50x+139$

4. $x^3+\left(q-\frac{p^2}{3}\right)x+\frac{2p^3}{27}-\frac{pq}{3}+r=0$ 5. $x^3-3x+11$ 6. $x^3-47x+126$

7. $x^3-\frac{29}{6}x+\frac{2114}{54}$ 11. .638 (approx.) 12. 0.933 (approx.)

13. -1.846 (approx.)

Exercise 9-3 (Page 217)

4. (a) $\frac{2}{3}$, $\frac{2}{3}$ (b) $-\pi$, $\frac{\sqrt{2}}{3}$ (c) $-\frac{1}{2}$, $\frac{11}{2}$ (d) 3, -3
(e) 0, -3 (f) -2, 1 (g) $\frac{7}{5}$, $\frac{8}{5}$ (h) $-\frac{10}{9}$, $-\frac{13}{9}$

5. $\frac{1}{12}(1\pm i\sqrt{47})$ 6. $\frac{1}{6}(\sqrt{2}\pm i\sqrt{10})$ 7. $\frac{5\pm i\sqrt{23}}{2\sqrt{2}}$ 8. $\frac{1\pm i\sqrt{15}}{2\sqrt{3}}$

Exercise 9-4 (Page 219)

13. $10+0i$ 14. $5+3i$ 15. $-1+2i$ 16. $\frac{11}{2}-i$

17. $\frac{5}{2}+\frac{7}{2}i$ 18. $12+0i$ 19. $-4\sqrt{2}+4i$

20. $(3\sqrt{2}-\pi)+(4-\sqrt{2})i$ 21. $\left(\frac{1}{\sqrt{3}}-2\right)+(1+\sqrt{2})i$

24. $-6-2i$ 25. $6-\frac{5}{2}i$ 26. $(\sqrt{2}-4)+(1-\sqrt{2})i$

27. $6+(\sqrt{2}-6)i$

Exercise 9-5 (Page 222)

1. $22+9i$ 2. $0+30i$ 3. $\frac{27}{4}-\frac{7}{2}i$ 4. $52+0i$

5. $-1-\frac{17}{4}i$ 6. $(-16-\pi^2)+0i$ 7. $(2\sqrt{3}-2)+(3+2\sqrt{2})i$

8. $-17\sqrt{2}+9i$ 9. $14+5\sqrt{2}i$ 17. 37 18. 20

19. $\frac{1}{2}$ 20. 16 21. $\frac{1}{4}$ 22. 1

26. No. The product contains a complex coefficient.

29. $x^2-4x+5=0$ 30. $x^2+6x+13=0$ 31. $x^2+1=0$ 32. $x^2+2\sqrt{2}x+11=0$

33. $x^2-12x+39=0$ 34. $x^2-x+\frac{3}{4}=0$

Exercise 9-6 (Page 224)

1. $\frac{34}{53}+\frac{13}{53}i$
2. $-\frac{6}{5}-\frac{9}{10}i$
3. $-\frac{4}{5}-\frac{1}{10}i$
4. $\frac{27}{25}-\frac{14}{25}i$
5. $\frac{5}{13}-\frac{12}{13}i$
6. $-\frac{11}{29}-\frac{13}{29}i$
7. $\frac{\pi^2-16}{\pi^2+16}+\frac{8\pi i}{\pi^2+16}$
8. $\frac{3\sqrt{2}+2}{13}+\frac{(3-2\sqrt{2})}{13}i$
9. $-\frac{17\sqrt{2}}{11}-\frac{15}{11}i$
13. $\frac{4}{17}-\frac{i}{17}$
14. $\frac{3}{13}+\frac{2}{13}i$
15. $\frac{2}{15}-\frac{i}{15}$
16. $-\frac{3}{20}-\frac{i}{20}$
17. $1+i$
18. $\frac{\sqrt{3}}{4}-\frac{i}{4}$
19. $\frac{1}{3}+\frac{\sqrt{2}}{6}i$
20. $\frac{\sqrt{3}}{5}-\frac{\sqrt{2}}{5}i$
21. $\frac{a}{a^2+b^2}-\frac{bi}{a^2+b^2}$
22. $\frac{1}{2}-\frac{5}{2}i$
23. $-\frac{19}{13}+\frac{4}{13}i$
24. $-\frac{9}{40}+\frac{17}{40}i$
25. $\frac{8}{25}+\frac{6}{25}i$

Exercise 9-7 (Page 228)

1. (iii) $P(x)=(x-\frac{1}{2})6x+5$
2. (iii) $P(x)=(x+\frac{1}{2})(x^2-\frac{11}{2}x+\frac{35}{4})-\frac{51}{8}$
3. (iii) $P(x)=(x+\frac{1}{3})(4x^2-\frac{4}{3}x-\frac{50}{9})+\frac{131}{27}$
4. (iii) $P(x)=(x+\frac{1}{2})(x^3-\frac{13}{2}x^2-\frac{11}{4}x+\frac{35}{8})-\frac{67}{16}$
5. (iii) $P(x)=(x-\frac{1}{2})(x^4+\frac{x^3}{2}-\frac{19}{4}x^2-\frac{19}{8}x+\frac{77}{16})+\frac{77}{32}$
7. (ii) $(x-1)(2x^2-4x-1)-2$
8. (ii) $(x-2)(x^3+2x^2-2x-4)-6$
9. (ii) $(x-1)(5x^3+5x^2+5x+2)+3$
10. (ii) $(x+2)(2x^4-4x^3+4x^2-8x+19)-45$
11. $\frac{2-2a^3+3a^2}{4a}$
12. -3

Exercise 9-8 (Page 230)

1. (iii) $\frac{3}{2}$
2. (iii) $\frac{152}{27}$
3. (iii) $\frac{41}{8}$
4. (iii) $\frac{91}{16}$
5. (iii) $\frac{17161}{3433}$
6. (iii) -121
7. $\frac{2}{3}$, -1
8. (a) 39 (b) 39
9. -1, -2
10. (a) No (b) 0, $\frac{2}{3}$
11. (a) $-\frac{3}{2}$, 1 (b) $-\frac{3}{4}$, $\frac{1}{2}$

Exercise 9-9 (Page 232)

1. 2 or 0
2. 3 or 1
3. 3 or 1
4. 4 or 2
5. 4, 2, or 0
6. 5, 3, or 1
7. 5, 3, or 1
8. 8, 6, 4, 2, or 0
9. 7, 5, 3, or 1
10. 9, 7, 5, or 3
11. 6, 4, 2, or 0
12. 5, 3, or 1

Review Exercise 9-10 (Page 233)

3. (c) $13-11i$ (d) $\frac{17}{10}-\frac{i}{10}$ 4. (c) $0-\frac{145}{4}i$ (d) $\frac{24}{145}+\frac{143}{145}i$

5. (c) $3+0i$ (d) $\frac{1}{3}-\frac{2\sqrt{2}}{3}i$ 6. $19+0i$ (d) $\frac{13}{19}-\frac{8\sqrt{3}}{19}i$

7. (c) $5-12i$ (d) -1

8. (c) $\left(\frac{3}{2}-\sqrt{3}\right)-\left(\frac{3\sqrt{2}}{2}+1\right)i$ (d) $\frac{3+2\sqrt{3}}{26}+\frac{2-3\sqrt{3}}{26}i$

12. The points are the vertices of a square.

13. 2 14. 2 15. $-\frac{1}{8}$ 16. $\frac{2}{13}+\frac{3i}{13}$

17. $-\frac{2}{101}-\frac{20}{101}i$ 18. $\frac{4\sqrt{2}}{17}-\frac{6i}{17}$ 19. $\frac{1}{2}-\frac{1}{2}i$ 20. $\frac{1}{20}+\frac{7}{20}i$

21. $x^2+12x+52=0$

22. No. The product of the roots is a complex number.

23. (c) $(x-\frac{1}{2})(5x-\frac{3}{2})+\frac{5}{4}$ 24. (c) $(x+3)(4x^2-11x+34)-108$

25. (c) $(x-1)(x^4-6x-6)-7$ 26. (c) $(x+\sqrt{2})(2x-7\sqrt{2})+20$

27. (a) 5 (b) 32 (c) 2 28. (a) -215 (b) 225 (c) $\frac{3}{4}\frac{1}{4}$

29. (a) 0 (b) 40 (c) 180 30. (a) $11+4\sqrt{3}$ (b) 1 (c) 13

31. $2\pm\sqrt{7}$ 32. $\frac{1}{2}$, 1 33. $-\frac{1\pm\sqrt{2701}}{18}$ 34. 2 or 0

35. 3 or 1 36. 4 or 2 37. 8, 6, 4, 2, or 0 38. 10, 8, 6, or 4

39. 7, 5, 3, or 1

CHAPTER 10

Exercise 10-1 (Page 239)

5. The lines should intersect at the centre.
The right bisectors would be coincident if the chords were parallel.

Exercise 10-2 (Page 241)

2. $\frac{2}{2}\frac{1}{2}$ sq. in. 3. $\frac{\pi}{4}$

5. (i) 10.5 in., 42 sq. in. (ii) 7 in., $\frac{4}{4}\frac{9}{4}$ sq. in.
(iii) $\frac{5\pi}{8}$ ft., $\frac{2}{3}\frac{5}{2}\pi$ sq. ft. (iv) 2π ft., 6π sq. ft.
(v) 4 ft., 5 sq. ft.

6. (i) 1.25 rad., 72° (ii) $\frac{\pi}{12}$ rad., 15° (iii) $\frac{\pi}{32}$ rad., 5.625°

7. 1.08 sq. in. 8. 6 in. 9. .96 rad., 55° 10. $\frac{11}{2}$ in.

11. 5.00 ft. 12. 5.58 in., 17.9 sq. in. 13. 44°

Exercise 10-3 (Page 244)

1. 5 cm. 2. 5 cm. 3. PR = 32, RQ = 2

6. An infinite number with centre on the right bisector of the line joining the two points.

Exercise 10-4 (Page 246)

4. $\angle ACD = 58°$, $\angle BDA = \angle ACB = 50°$, $\angle DCB = 108°$

5. $\angle ACB = 35°$, $\angle ABC = 113°$ 9. CD = 7

Exercise 10-5 (Page 249)

6. circle with chord BC

Exercise 10-7 (Page 254)

4. 65°

Exercise 10-8 (Page 256)

1. OA = 5 2. AB = 12 3. $6\sqrt{5}$

Exercise 10-9 (Page 258)

3. 100° 4. $\angle ADB = 43°$, $\angle ABD = 46°$.

Review Exercise 10-11 (Page 264)

1. $\angle ACB = \angle ADB = 36°$ 2. OR = 4 4. $\angle EAC = 38°$

5. 1.6 cm. 6. 13 cm. 7. $\angle FEH = 35°$ 8. $\angle A = 105°$

CHAPTER 11

Exercise 11-1 (Page 268)

1. (a) $x^2+y^2=9$ (b) $4x^2+4y^2=9$ (c) $x^2+y^2=2$
   (d) $x^2+y^2=45$ (e) $25x^2+25y^2=12$

2. (a) 6 (b) $\sqrt{5}$ (c) $\frac{5\sqrt{3}}{3}$ (d) $\frac{7}{3}$ (e) $\frac{\sqrt{21}}{2}$ (f) $\frac{\sqrt{105}}{5}$

3. (a) $x^2+y^2=25$ (d) $x^2+y^2=25$ (c) $x^2+y^2=169$
   (d) $x^2+y^2=10$ (e) $x^2+y^2=25$ (f) $x^2+y^2=29$

Exercise 11-2 (Page 273)

1.

	Domain	Range	x-int.	y-int.	Symmetry
(a)	$-6\leq x\leq 6$	$-6\leq y\leq 6$	± 6	± 6	about the origin
(b)	$-8\leq x\leq 8$	$-8\leq y\leq 8$	± 8	± 8	about the origin
(c)	$-\frac{5}{2}\leq x\leq \frac{5}{2}$	$-\frac{5}{2}\leq y\leq \frac{5}{2}$	$\pm \frac{5}{2}$	$\pm \frac{5}{2}$	about the origin
(d)	$-3\sqrt{2}\leq x\leq 3\sqrt{2}$	$-3\sqrt{2}\leq y\leq 3\sqrt{2}$	$\pm 3\sqrt{2}$	$\pm 3\sqrt{2}$	about the origin
(e)	$-\frac{7\sqrt{3}}{3}\leq x\leq \frac{7\sqrt{3}}{3}$	$-\frac{7\sqrt{3}}{3}\leq y\leq \frac{7\sqrt{3}}{3}$	$\pm \frac{7\sqrt{3}}{3}$	$\pm \frac{7\sqrt{3}}{3}$	about the origin
(f)	$-\frac{\sqrt{22}}{2}\leq x\leq \frac{\sqrt{22}}{3}$	$-\frac{\sqrt{22}}{2}\leq y\leq \frac{\sqrt{22}}{2}$	$\pm \frac{\sqrt{22}}{2}$	$\pm \frac{\sqrt{22}}{2}$	about the origin

2. (a) about y-axis
(b) about the x-axis and about the y-axis
(c) about the origin
(d) not symmetrical
(e) about the x-axis
(f) about the x-axis
(g) about the y-axis
(h) about the line $x=-1$
(i) about the x-axis

Exercise 11-4 (Page 276)

1. (a) $4\sqrt{5}$, $x+2y-5=0$
(b) 10, $3x+y+10=0$
(c) 4, $8x+10y-41=0$
(d) $\frac{8}{3}$, $3x-24y-65=0$
(e) 5, $12x+10y-61=0$
(f) no solution

2. The point (3, 2) is outside the circle.

3. $2\sqrt{2}$

4. $x^2+y^2=169$

Exercise 11-5 (Page 278)

1. (a) 8 units
(b) $\sqrt{19}$ units
(c) $\sqrt{55}$ units
(d) 4 units
(e) 4 units
(f) no solution

2. Point lies inside given circle.

3. $\sqrt{a+b^2-r^2}$ units; $a^2+b^2\geq r^2$

Exercise 11-6 (Page 279)

1. (a) $3x+4y-25=0$
(b) $3x-4y-25=0$
(c) $3x-4y+25=0$
(d) $3x+4y+25=0$
(e) $12x-5y+169=0$
(f) $15x-8y-289=0$
(g) $5x+4y+41=0$
(h) $5x+3y-17=0$

3. $ax+by=r^2$

6. $m=0$

7. $m=-1$

Review Exercise 11-7 (Page 280)

1. (a) $x^2+y^2=49$
(b) $x^2+y^2=24$
(c) $3x^2+3y^2=4$

2. (a) 12
(b) $\frac{5}{2}$
(c) $\frac{2\sqrt{7}}{3}$

3. (a) $x^2+y^2=169$
(b) $x^2+y^2=40$
(c) $2x^2+2y^2=29$

4.

	Domain	Range	x-int.	y-int.
(a)	$-8\le x\le 8$	$-8\le y\le 8$	± 8	± 8
(b)	$-10\le x\le 10$	$-10\le y\le 10$	± 10	± 10
(c)	$-3\sqrt{3}\le x\le 3\sqrt{3}$	$-3\sqrt{3}\le y\le 3\sqrt{3}$	$\pm 3\sqrt{3}$	$\pm 3\sqrt{3}$
(d)	$-\frac{7}{2}\le x\le\frac{7}{2}$	$-\frac{7}{2}\le y\le\frac{7}{2}$	$\pm\frac{7}{2}$	$\pm\frac{7}{2}$
(e)	$-\frac{\sqrt{17}}{3}\le x\le\frac{\sqrt{17}}{3}$	$-\frac{\sqrt{17}}{3}\le y\le\frac{\sqrt{17}}{3}$	$\pm\frac{\sqrt{17}}{3}$	$\pm\frac{\sqrt{17}}{3}$
(f)	$-\frac{\sqrt{17}}{\sqrt{5}}\le x\le\frac{\sqrt{17}}{\sqrt{5}}$	$-\frac{\sqrt{17}}{\sqrt{5}}\le y\le\frac{\sqrt{17}}{\sqrt{5}}$	$\pm\frac{\sqrt{17}}{\sqrt{5}}$	$\pm\frac{\sqrt{17}}{\sqrt{5}}$

5. (a) and (e) are inside (b) and (d) are on
(c) and (f) are outside

7. 10, $3x-2y-13=0$ 8. 7 units 9. $4x-5y-41=0$

10. $4x^2+4y^2=61$, $12x+10y+61=0$ 11. $x^2-4x+y^2-2y-11=0$ 12. $\frac{9}{16}$

CHAPTER 12

Exercise 12-1 (Page 284)

1. (a) $s=\frac{1}{\sqrt{2}}$, $c=\frac{1}{\sqrt{2}}$, $t=1$
(b) $s=-\frac{\sqrt{3}}{2}$, $c=\frac{1}{2}$, $t=-\sqrt{3}$
(c) $s=\frac{1}{2}$, $c=\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$
(d) $s=-\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=\sqrt{3}$
(e) $s=-\frac{\sqrt{3}}{2}$, $c=\frac{1}{2}$, $t=-\sqrt{3}$
(f) $s=0$, $c=-1$, $t=0$
(g) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(h) $s=-\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=1$
(i) $s=1$, $c=0$, t is undefined
(j) $s=\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=-1$
(k) $s=-\frac{1}{2}$, $c=-\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$
(l) $s=-\frac{1}{2}$, $c=-\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$

2. (a) $s=.2924$, $c=.9563$, $t=.3057$
(b) $s=.8387$, $c=-.5446$, $t=-1.5399$
(c) $s=-.8290$, $c=-.5592$, $t=1.4826$
(d) $s=.7880$, $c=.6157$, $t=1.2799$
(e) $s=.9703$, $c=.2419$, $t=4.0108$
(f) $s=.3256$, $c=-.9455$, $t=-.3443$
(g) $s=-.3907$, $c=.9205$, $t=-.4245$
(h) $s=-.9563$, $c=.2924$, $t=-3.2709$
(i) $s=-.2588$, $c=-.9659$, $t=.2679$
(j) $s=.5878$, $c=.8090$, $t=.7265$
(k) $s=.6018$, $c=-.7986$, $t=-.7536$
(l) $s=-.9903$, $c=-.1392$, $t=7.1154$

4. (Answers are to the nearest degree.)
(a) 19°, 161° (b) 58°, 302° (c) 36°, 216° (d) 132°, 228°
(e) 112°, 292° (f) 207°, 333° (g) 230°, 310° (h) 40°, 140°
(i) 74°, 286° (j) 58°, 238° (k) 115°, 245° (l) 154°, 334°

Exercise 12-2 (Page 288)

1. (a) sin (b) sin (c) tan (d) cos (e) cos (f) sin
(g) sin (h) sin (i) cos (j) sin, cos, tan (k) tan
(l) sin (m) sin (n) sin (o) tan (p) sin (q) cos
(r) none (s) none (t) none (u) cos (v) sin (w) tan
(x) sin

2. (a) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(b) $s=\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=-1$
(c) $s=-\frac{1}{2}$, $c=-\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$
(d) $s=-\frac{1}{\sqrt{2}}$, $c=\frac{1}{\sqrt{2}}$, $t=-1$
(e) $s=-\frac{1}{\sqrt{2}}$, $c=\frac{1}{\sqrt{2}}$, $t=-1$
(f) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(g) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(h) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(i) $s=-\frac{1}{2}$, $c=\frac{\sqrt{3}}{2}$, $t=-\frac{1}{\sqrt{3}}$
(j) $s=\frac{1}{2}$, $c=\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$
(k) $s=-\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=1$
(l) $s=1$, $c=0$, t is undefined.
(m) $s=\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=-1$
(n) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(o) $s=-\frac{1}{2}$, $c=-\frac{\sqrt{3}}{2}$, $t=\frac{1}{\sqrt{3}}$
(p) $s=1$, $c=0$, t is undefined.
(q) $s=0$, $c=1$, $t=0$
(r) $s=0$, $c=-1$, $t=0$
(s) $s=0$, $c=-1$, $t=0$
(t) $s=0$, $c=-1$, $t=0$
(u) $s=-\frac{\sqrt{3}}{2}$, $c=\frac{1}{2}$, $t=-\sqrt{3}$
(v) $s=\frac{\sqrt{3}}{2}$, $c=-\frac{1}{2}$, $t=-\sqrt{3}$
(w) $s=-\frac{1}{\sqrt{2}}$, $c=-\frac{1}{\sqrt{2}}$, $t=1$
(x) $s=1$, $c=0$, t is undefined.

3. (a) .5150 (b) .5592 (c) .8988 (d) −.9205
(e) .2924 (f) .4540 (g) −.8192 (h) −.7071
(i) .9945 (j) .9511 (k) .4067 (l) .9135
(m) .4848 (n) −.7314 (o) .9563 (p) −.8387
(q) −.9877 (r) −.4067 (s) −.2588 (t) −.5446
(u) −.9659 (v) −.0523 (w) −.7071 (x) −.9816

4. (a) −.8829 (b) −.9135 (c) −.0175 (d) .7547
(e) .8988 (f) −.3256 (g) .3746 (h) .9903
(i) .1908 (j) −.0175 (k) −.4540 (l) −.7986
(m) −.8910 (n) −.3907 (o) −.9945 (p) .8192
(q) −.9455 (r) .7880 (s) .7547 (t) .9613
(u) .0698 (v) .9613 (w) −.9455 (x) .2924

Exercise 12-3 (Page 289)

3. (a) −.3907 (b) .2924 (c) −.7771 (d) .8090
(e) .9063 (f) −.8090 (g) .6157 (h) −.8480
(i) −.7986 (j) −.4226 (k) −.4226 (l) .8910
(m) .7547 (n) −.5150 (o) −.6157 (p) −.6018
(q) −.9205 (r) .5592 (s) .3746 (t) .8090
(u) .1392 (v) −.7193 (w) −.5299 (x) −.9877

4. (a) .9205 (b) −.9563 (c) .6293 (d) −.5878
(e) .4226 (f) −.5878 (g) .7880 (h) .5299
(i) .6018 (j) .9063 (k) .9063 (l) −.4540

	(m)	$-.6561$	(n)	$-.8572$	(o)	$-.7880$	(p) $.7986$
	(q)	$-.3907$	(r)	$-.8290$	(s)	$.9272$	(t) $-.5878$
	(u)	$.9903$	(v)	$.6947$	(w)	$-.8480$	(x) $-.1564$
5.	(a)	$-.4245$	(b)	$-.3057$	(c)	$-.12349$	(d) -1.3764
	(e)	2.1445	(f)	1.3764	(g)	$.7813$	(h) -1.6003
	(i)	-1.3270	(j)	$-.4663$	(k)	$-.4663$	(l) -1.9626
	(m)	-1.1504	(n)	$.6009$	(o)	$.7813$	(p) $-.7536$
	(q)	2.3559	(r)	$-.6745$	(s)	$.4040$	(t) -1.3764
	(u)	$.1405$	(v)	-1.0355	(w)	$.6249$	(x) 6.3138

Exercise 12-4 (Page 291)

1.

θ	$\sin\theta$
-4π	0
$-\frac{15}{4}\pi$	$\frac{1}{\sqrt{2}}$
$-\frac{7}{2}\pi$	1
$-\frac{13}{4}\pi$	$\frac{1}{\sqrt{2}}$
-3π	0
$-\frac{11}{4}\pi$	$-\frac{1}{\sqrt{2}}$
$-\frac{5}{2}\pi$	-1
$-\frac{9}{4}\pi$	$-\frac{1}{\sqrt{2}}$
-2π	0
$-\frac{7}{4}\pi$	$\frac{1}{\sqrt{2}}$
$-\frac{3}{2}\pi$	1
$-\frac{5}{4}\pi$	$\frac{1}{\sqrt{2}}$
$-\pi$	0
$-\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$
$-\frac{1}{2}\pi$	-1
$-\frac{1}{4}\pi$	$-\frac{1}{\sqrt{2}}$
0	0
$\frac{1}{4}\pi$	$\frac{1}{\sqrt{2}}$
$\frac{1}{2}\pi$	1
$\frac{3}{4}\pi$	$\frac{1}{\sqrt{2}}$
π	0
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$

2.

θ	$\cos\theta$
-4π	1
$-\frac{15\pi}{4}$	$\frac{1}{\sqrt{2}}$
$-\frac{7\pi}{2}$	0
$-\frac{13\pi}{4}$	$-\frac{1}{\sqrt{2}}$
-3π	-1
$-\frac{11\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{5\pi}{2}$	0
$-\frac{9\pi}{4}$	$\frac{1}{\sqrt{2}}$
-2π	1
$-\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$
$-\frac{3\pi}{2}$	0
$-\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\pi$	-1
$-\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{\pi}{2}$	0
$-\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$
π	-1
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$

θ	$\sin\theta$
$\frac{3}{2}\pi$	-1
$\frac{7}{4}\pi$	$-\frac{1}{\sqrt{2}}$
2π	0
$\frac{9}{4}\pi$	$\frac{1}{\sqrt{2}}$
$\frac{5}{2}\pi$	1
$\frac{11}{4}\pi$	$\frac{1}{\sqrt{2}}$
3π	0
$\frac{13}{4}\pi$	$-\frac{1}{\sqrt{2}}$
$\frac{7}{2}\pi$	-1
$\frac{15}{4}\pi$	$-\frac{1}{\sqrt{2}}$
4π	0

θ	$\cos\theta$
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$
2π	1
$\frac{9\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{5\pi}{2}$	0
$\frac{11\pi}{4}$	$-\frac{1}{\sqrt{2}}$
3π	-1
$\frac{13\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{7\pi}{2}$	0
$\frac{15\pi}{4}$	$\frac{1}{\sqrt{2}}$
4π	1

3.

$\theta°$	$\sin\theta$
-180	0
-165	$-.2588$
-150	$-\frac{1}{2}$
-135	$-\frac{1}{\sqrt{2}}$
-120	$-\frac{\sqrt{3}}{2}$
-105	$-.9659$
-90	-1
-75	$-.9659$
-60	$-\frac{\sqrt{3}}{2}$
-45	$-\frac{1}{\sqrt{2}}$
-30	$-\frac{1}{2}$
-15	$-.2588$
0	0
15	$.2588$

4.

$\theta°$	$\sin\theta$
0	0
15	$.2588$
30	$\frac{1}{2}$
45	$\frac{1}{\sqrt{2}}$
60	$\frac{\sqrt{3}}{2}$
75	$.9659$
90	1
105	$.9659$
120	$\frac{\sqrt{3}}{2}$
135	$\frac{1}{\sqrt{2}}$
150	$\frac{1}{2}$
165	$.2588$
180	0
195	$-.2588$

$\theta°$	$\sin\theta$	$\theta°$	$\sin\theta$
30	$\frac{1}{2}$	210	$-\frac{1}{2}$
45	$\frac{1}{\sqrt{2}}$	225	$-\frac{1}{\sqrt{2}}$
60	$\frac{\sqrt{3}}{2}$	240	$-\frac{\sqrt{3}}{2}$
75	.9659	255	−.9659
90	1	270	−1
105	.9659	285	−.9659
120	$\frac{\sqrt{3}}{2}$	300	$-\frac{\sqrt{3}}{2}$
135	$\frac{1}{\sqrt{2}}$	315	$-\frac{1}{\sqrt{2}}$
150	$\frac{1}{2}$	330	$-\frac{1}{2}$
165	.2588	345	−.2588
180	0	360	0

5.

$\theta°$	$\cos\theta$	$\theta°$	$\cos\theta$
−180	−1	0	1
−165	−.9659	15	.9659
−150	$-\frac{\sqrt{3}}{2}$	30	$\frac{\sqrt{3}}{2}$
−135	$-\frac{1}{\sqrt{2}}$	45	$\frac{1}{\sqrt{2}}$
−120	$-\frac{1}{2}$	60	$\frac{1}{2}$
−105	−.2588	75	.2588
−90	0	90	0
−75	.2588	105	−.2588
−60	$\frac{1}{2}$	120	$-\frac{1}{2}$
−45	$\frac{1}{\sqrt{2}}$	135	$-\frac{1}{\sqrt{2}}$
−30	$\frac{\sqrt{3}}{2}$	150	$-\frac{\sqrt{3}}{2}$
−15	.9659	165	−.9659
0	1	180	−1
15	.9659	195	−.9659
30	$\frac{\sqrt{3}}{2}$	210	$-\frac{\sqrt{3}}{2}$
45	$\frac{1}{\sqrt{2}}$	225	$-\frac{1}{\sqrt{2}}$
60	$\frac{1}{2}$	240	$-\frac{1}{2}$
75	.2588	255	−.2588
90	0	270	0
105	−.2588	285	.2588
120	$-\frac{1}{2}$	300	$\frac{1}{2}$
135	$-\frac{1}{\sqrt{2}}$	315	$\frac{1}{\sqrt{2}}$
150	$-\frac{\sqrt{3}}{2}$	330	$\frac{\sqrt{3}}{2}$
165	−.9659	345	.9659
180	−1	360	1

6.

θ	$\sin\theta$
0	0
$\frac{\pi}{12}$	.2588
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{12}$	.9659
$\frac{\pi}{2}$	1
$\frac{7\pi}{12}$	.9659
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
$\frac{11\pi}{12}$	.2588
π	0
$\frac{13\pi}{12}$	−.2588
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{17\pi}{12}$	−.9659
$\frac{3\pi}{2}$	−1
$\frac{19\pi}{12}$	−.9659
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
$\frac{23\pi}{12}$	−.2588
2π	0

7.

θ	$\cos\theta$
0	1
$\frac{\pi}{12}$	.9659
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{5\pi}{12}$	.2588
$\frac{\pi}{2}$	0
$\frac{7\pi}{12}$	−.2588
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
$\frac{11\pi}{12}$	−.9659
π	−1
$\frac{13\pi}{12}$	−.9659
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$\frac{4\pi}{3}$	$-\frac{1}{2}$
$\frac{17\pi}{12}$	−.2588
$\frac{3\pi}{2}$	0
$\frac{19\pi}{12}$	.2588
$\frac{5\pi}{3}$	$\frac{1}{2}$
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{23\pi}{12}$	.9659
2π	1

8. (a)

$\theta°$	$\tan\theta$	$\theta°$	$\tan\theta$	θ	$\tan\theta$
-180	0	0	0	0	0
-165	$.2679$	15	$.2679$	$\frac{\pi}{12}$	$.2679$
-150	$\frac{1}{\sqrt{3}}$	30	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$
-135	1	45	1	$\frac{\pi}{4}$	1
-120	$\sqrt{3}$	60	$\sqrt{3}$	$\frac{\pi}{3}$	$\sqrt{3}$
-105	3.7321	75	3.7321	$\frac{5\pi}{12}$	3.7321
-90	undefined	90	undefined	$\frac{\pi}{2}$	undefined
-75	-3.7321	105	-3.7321	$\frac{7\pi}{12}$	-3.7321
-60	$-\sqrt{3}$	120	$-\sqrt{3}$	$\frac{2\pi}{3}$	$-\sqrt{3}$
-45	-1	135	-1	$\frac{3\pi}{4}$	-1
-30	$-\frac{1}{\sqrt{3}}$	150	$-\frac{1}{\sqrt{3}}$	$\frac{5\pi}{6}$	$-\frac{1}{\sqrt{3}}$
-15	$-.2679$	165	$-.2679$	$\frac{11\pi}{12}$	$-.2679$
0	0	180	0	π	0
15	$.2679$	195	$.2679$	$\frac{13\pi}{12}$	$.2679$
30	$\frac{1}{\sqrt{3}}$	210	$\frac{1}{\sqrt{3}}$	$\frac{7\pi}{6}$	$\frac{1}{\sqrt{3}}$
45	1	225	1	$\frac{5\pi}{4}$	1
60	$\sqrt{3}$	240	$\sqrt{3}$	$\frac{4\pi}{3}$	$\sqrt{3}$
75	3.7321	255	3.7321	$\frac{17\pi}{12}$	3.7321
90	undefined	270	undefined	$\frac{3\pi}{2}$	undefined
105	-3.7321	285	-3.7321	$\frac{19\pi}{12}$	-3.7321
120	$-\sqrt{3}$	300	$-\sqrt{3}$	$\frac{5\pi}{3}$	$-\sqrt{3}$
135	-1	315	-1	$\frac{7\pi}{4}$	-1
150	$-\frac{1}{\sqrt{3}}$	330	$-\frac{1}{\sqrt{3}}$	$\frac{11\pi}{6}$	$-\frac{1}{\sqrt{3}}$
165	$.2679$	345	$-.2679$	$\frac{23\pi}{12}$	$-.2679$
180	0	360	0	2π	0

8. (b)

$\theta°$	$\cot\theta$	$\theta°$	$\cot\theta$	θ	$\cot\theta$
-180	undefined	0	undefined	0	undefined
-165	3.7321	15	3.7321	$\frac{\pi}{12}$	3.7321
-150	$\sqrt{3}$	30	$\sqrt{3}$	$\frac{\pi}{6}$	$\sqrt{3}$
-135	1	45	1	$\frac{\pi}{4}$	1
-120	$\frac{1}{\sqrt{3}}$	60	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{3}$	$\frac{1}{\sqrt{3}}$
-105	.2679	75	.2679	$\frac{5\pi}{12}$	.2679
-90	0	90	0	$\frac{\pi}{2}$	0
-75	$-.2679$	105	$-.2679$	$\frac{7\pi}{12}$	$-.2679$
-60	$-\frac{1}{\sqrt{3}}$	120	$-\frac{1}{\sqrt{3}}$	$\frac{2\pi}{3}$	$-\frac{1}{\sqrt{3}}$
-45	-1	135	-1	$\frac{3\pi}{4}$	-1
-30	$-\sqrt{3}$	150	$-\sqrt{3}$	$\frac{5\pi}{6}$	$-\sqrt{3}$
-15	-3.7321	165	-3.7321	$\frac{11\pi}{12}$	-3.7321
0	undefined	180	undefined	π	undefined
15	3.7321	195	3.7321	$\frac{13\pi}{12}$	3.7321
30	$\sqrt{3}$	210	$\sqrt{3}$	$\frac{7\pi}{6}$	$\sqrt{3}$
45	1	225	1	$\frac{5\pi}{4}$	1
60	$\frac{1}{\sqrt{3}}$	240	$\frac{1}{\sqrt{3}}$	$\frac{4\pi}{3}$	$\frac{1}{\sqrt{3}}$
75	.2679	255	.2679	$\frac{17\pi}{12}$	.2679
90	0	270	0	$\frac{3\pi}{12}$	0
105	$-.2679$	285	$-.2679$	$\frac{19\pi}{12}$	$-.2679$
120	$-\frac{1}{\sqrt{3}}$	300	$-\frac{1}{\sqrt{3}}$	$\frac{5\pi}{3}$	$-\frac{1}{\sqrt{3}}$
135	-1	315	-1	$\frac{7\pi}{4}$	-1
150	$-\sqrt{3}$	330	$-\sqrt{3}$	$\frac{11\pi}{6}$	$-\sqrt{3}$
165	-3.7321	345	-3.7321	$\frac{23\pi}{12}$	-3.7321
180	undefined	360	undefined	2π	undefined

9. (a)

$\theta°$	sec θ	$\theta°$	sec θ	θ	sec θ
-180	-1	0	1	0	1
-165	-1.035	15	1.035	$\frac{\pi}{12}$	1.035
-150	$-\frac{2}{\sqrt{3}}$	30	$\frac{2}{\sqrt{3}}$	$\frac{\pi}{6}$	$\frac{2}{\sqrt{3}}$
-135	$-\sqrt{2}$	45	$\sqrt{2}$	$\frac{\pi}{4}$	$\sqrt{2}$
-120	-2	60	2	$\frac{\pi}{3}$	2
-105	-3.864	75	3.864	$\frac{5\pi}{12}$	3.864
-90	undefined	90	undefined	$\frac{\pi}{2}$	undefined
-75	3.864	105	-3.864	$\frac{7\pi}{12}$	-3.864
-60	2	120	-2	$\frac{2\pi}{3}$	-2
-45	$\sqrt{2}$	135	$-\sqrt{2}$	$\frac{3\pi}{4}$	$-\sqrt{2}$
-30	$\frac{2}{\sqrt{3}}$	150	$-\frac{2}{\sqrt{3}}$	$\frac{5\pi}{6}$	$-\frac{2}{\sqrt{3}}$
-15	1.035	165	-1.035	$\frac{11\pi}{12}$	-1.035
0	1	180	-1	π	-1
15	1.035	195	-1.035	$\frac{13\pi}{12}$	-1.035
30	$\frac{2}{\sqrt{3}}$	210	$-\frac{2}{\sqrt{3}}$	$\frac{7\pi}{6}$	$-\frac{2}{\sqrt{3}}$
45	$\sqrt{2}$	225	$-\sqrt{2}$	$\frac{5\pi}{4}$	$-\sqrt{2}$
60	2	240	-2	$\frac{4\pi}{3}$	-2
75	3.864	255	-3.864	$\frac{17\pi}{12}$	-3.864
90	undefined	270	undefined	$\frac{3\pi}{12}$	undefined
105	-3.864	285	3.864	$\frac{19\pi}{12}$	3.864
120	-2	300	2	$\frac{5\pi}{3}$	2
135	$-\sqrt{2}$	315	$\sqrt{2}$	$\frac{7\pi}{4}$	$\sqrt{2}$
150	$-\frac{2}{\sqrt{3}}$	330	$\frac{2}{\sqrt{3}}$	$\frac{11\pi}{6}$	$\frac{2}{\sqrt{3}}$
165	-1.035	345	1.035	$\frac{23\pi}{12}$	1.035
180	-1	360	1	2π	1

9. (b)

θ°	cosec θ	θ°	cosec θ	θ	cosec θ
-180	undefined	0	undefined	0	undefined
-165	-3.864	15	3.864	$\frac{\pi}{12}$	3.864
-150	-2	30	2	$\frac{\pi}{6}$	2
-135	$-\sqrt{2}$	45	$\sqrt{2}$	$\frac{\pi}{4}$	$\sqrt{2}$
-120	$-\frac{2}{\sqrt{3}}$	60	$\frac{2}{\sqrt{3}}$	$\frac{\pi}{3}$	$\frac{2}{\sqrt{3}}$
-105	-1.035	75	1.035	$\frac{5\pi}{12}$	1.035
-90	-1	90	1	$\frac{\pi}{2}$	1
-75	-1.035	105	1.035	$\frac{7\pi}{12}$	1.035
-60	$-\frac{2}{\sqrt{3}}$	120	$\frac{2}{\sqrt{3}}$	$\frac{2\pi}{3}$	$\frac{2}{\sqrt{3}}$
-45	$-\sqrt{2}$	135	$\sqrt{2}$	$\frac{3\pi}{4}$	$\sqrt{2}$
-30	-2	150	2	$\frac{5\pi}{6}$	2
-15	-3.864	165	3.864	$\frac{11\pi}{12}$	3.864
0	undefined	180	undefined	π	undefined
15	3.864	195	-3.864	$\frac{13\pi}{12}$	-3.864
30	2	210	-2	$\frac{7\pi}{6}$	-2
45	$\sqrt{2}$	225	$-\sqrt{2}$	$\frac{5\pi}{4}$	$-\sqrt{2}$
60	$\frac{2}{\sqrt{3}}$	240	$-\frac{2}{\sqrt{3}}$	$\frac{4\pi}{3}$	$-\frac{2}{\sqrt{3}}$
75	1.035	255	-1.035	$\frac{17\pi}{12}$	-1.035
90	1	270	-1	$\frac{3\pi}{12}$	-1
105	1.035	285	-1.035	$\frac{19\pi}{12}$	-1.035
120	$\frac{2}{\sqrt{3}}$	300	$-\frac{2}{\sqrt{3}}$	$\frac{5\pi}{3}$	$-\frac{2}{\sqrt{3}}$
135	$\sqrt{2}$	315	$-\sqrt{2}$	$\frac{7\pi}{4}$	$-\sqrt{2}$
150	2	330	-2	$\frac{11\pi}{6}$	-2
165	3.864	345	-3.864	$\frac{23\pi}{12}$	-3.864
180	undefined	360	undefined	2π	undefined

Exercise 12-5 (Page 293)

4. Symmetric about the lines $\theta = \frac{2n+1}{2}\pi$ and about the points $(n\pi, 0)$ for all integers n.

5. Symmetric about the lines $\theta = n\pi$ and the points $\left(\frac{2n+1}{2}\pi, 0\right)$ for all integers n.

Exercise 12-6 (Page 296)

1. $0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, 5\pi$
   $0°, \pm 180°, \pm 360°, \pm 540°, \pm 720°, 900°$

2. $-\frac{3\pi}{2}, -\frac{7\pi}{2}, -\frac{11\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

3. $-\frac{5\pi}{2}, -\frac{9\pi}{2}, -\frac{13\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$

4. $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc.

5. $0, \pm 2\pi, \pm 4\pi$, etc.

6. $\pm\pi, \pm 3\pi, \pm 5\pi$, etc.

7. (a) $-\frac{15\pi}{4}, -\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$

   (b) $\frac{11\pi}{3}, \frac{5\pi}{3}, -\frac{\pi}{3}, -\frac{7\pi}{3}, -\frac{13\pi}{3}$

   (c) $-733°, -373°, -13°, 347°, 707°$

   (d) $-\frac{17\pi}{4}, -\frac{9\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}$

   (e) $-\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

   (f) $-\frac{23\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$

   (g) $-4\pi, -2\pi, 0, 2\pi, 4\pi$

   (h) $-\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}$

   (i) $-\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

   (j) $-671°, -311°, 49°, 409°, 769°$

   (k) $-\frac{25\pi}{6}, -\frac{13\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}$

   (l) $-692°, -332°, 28°, 388°, 748°$

8. (a) $-\frac{15\pi}{4}, -\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$

   (b) $-\frac{19\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}$

   (c) $-\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

   (d) $-608°, -248°, 112°, 472°, 832°$

   (e) $-\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}$

   (f) $-663°, -303°, 57°, 417°, 777°$

   (g) $-\frac{23\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$

   (h) $-\frac{13\pi}{4}, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}, \frac{19\pi}{4}$

   (i) $-4\pi, -2\pi, 0, 2\pi, 4\pi$

   (j) $-685°, -325°, 35°, 395°, 755°$

   (k) $-591°, 231°, 129°, 489°, 849°$

   (l) $-3\pi, -\pi, \pi, 3\pi, 5\pi$

9. (a) Only for $\sin\theta = 1$, $\sin\theta = -1$, must values of θ differ by 2π.

   (b) Only for $\cos\theta = -1$, $\cos\theta = 1$, must values of θ differ by 2π.

   (c) Yes, the two subsets are $\left\{\theta \,\middle|\, \theta = \frac{\pi}{3} + 2n\pi;\ n \in \mathrm{I}\right\}$, $\left\{\theta \,\middle|\, \theta = \frac{2\pi}{3} + 2n\pi;\ n \in \mathrm{I}\right\}$.

   A similar result is true for $\cos\theta = -\frac{1}{2}$. True for all values of $\sin\theta$, $\cos\theta$ except for values of $\sin\theta = \pm 1$, $\cos\theta = \pm 1$.

Exercise 12-7 (Page 299)

1.

$\theta°$	$2 \sin \theta$
0	0
30	1
60	1.7321
90	2
120	1.7321
150	1
180	0

2. Amplitudes are .5, 1, 3 respectively.

4.

$\theta°$	$1.5 \cos \theta$
0	1.5
30	1.2990
60	.75
90	0

5. Amplitudes are .8, 1, 2 respectively.

Exercise 12-8 (Page 302)

1. (a) π (b) $\frac{\pi}{2}$ (c) 4π (d) 8π

2. 2π 3. (a) 2π (b) $\frac{4\pi}{5}$ (c) $\frac{2\pi}{5}$ (d) 5π 4. 2π

5. The period of the pattern of the pairs is the least common multiple of the individual periods.

Exercise 12-9 (Page 304)

(a = amplitude; p = period)

1. Amplitudes are 1, 3, 3 respectively. Periods are π, π, 2π respectively.

2. Amplitudes are 1, 2, 2 respectively. Periods are $\frac{\pi}{2}$, 2π, $\frac{\pi}{2}$ respectively.

3. (a) $a=3.5,\ p=\pi$ (b) $a=\sqrt{2},\ p=\frac{2\pi}{3}$ (c) $a=\frac{1}{4},\ p=\frac{2\pi}{7}$

(d) $a=\frac{\sqrt{3}}{2},\ p=8\pi$ (e) $a=10,\ p=2$ (f) $a=.2,\ p=\frac{8\pi}{3}$

(g) $a=\sqrt{5},\ p=\frac{2}{5}$ (h) $a=4,\ p=10$ (i) $a=5,\ p=\frac{2\pi}{\sqrt{3}}$

(j) $a=\frac{3}{2},\ p=2\sqrt{5}$ (k) $a=3.2,\ p=24\pi$ (l) $a=\frac{1}{\sqrt{3}},\ p=2\sqrt{3}\pi$

(m) $a=4,\ p=2$ (n) $a=2\pi,\ p=\frac{\pi}{2}$ (o) $a=.16,\ p=\frac{2\pi}{\sqrt{3}}$

(p) $a=5\times10^{5},\ p=2\pi$ (q) $a=.1,\ p=\frac{2\pi}{377.4}$ (r) $a=10^{-6},\ p=1$

Exercise 12-10 (Page 306)

(a = amplitude; d = phase angle)

1. (a) $a=3,\ d=\frac{\pi}{6}$ (b) $a=\frac{1}{2},\ d=-\frac{3\pi}{4}$

(c) $a=1,\ d=25°$ (d) $a=2,\ d=-120°$

2. (a) $a=1.2,\ d=\frac{\pi}{4}$ (b) $a=.8,\ d=-\frac{2\pi}{3}$

(c) $a=2,\ d=-75°$ (d) $a=1,\ d=150°$

3. (a) leads by $\frac{\pi}{3}$; phase shift is $-\frac{\pi}{3}$. (b) lags by $\frac{3\pi}{4}$; phase shift is $\frac{3\pi}{4}$.

(c) lags by 144°; phase shift is 144°. (d) lags by $\frac{\pi}{\sqrt{2}}$; phase shift is $\frac{\pi}{\sqrt{2}}$.

(e) leads by $\sqrt{2}$; phase shift is $-\sqrt{2}$. (f) leads by $\frac{\pi}{4}$; phase shift is $-\frac{\pi}{4}$.

4. (a) $-\frac{11\pi}{3}, -\frac{8\pi}{3}, -\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$

(b) $-\frac{13\pi}{4}, -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

(c) $-\frac{22\pi}{6}, -\frac{16\pi}{6}, -\frac{10\pi}{6}, -\frac{4\pi}{6}, \frac{2\pi}{6}, \frac{8\pi}{6}, \frac{14\pi}{6}, \frac{20\pi}{6}$

(d) $-603°, -423°, -243°, -63°, 117°, 297°, 477°, 657°$

(e) $-670°, -490°, -310°, -130°, 50°, 230°, 410°, 590°$

(f) $-660°, 480°, -300°, -120°, 60°, 240°, 420°, 600°$

5. (a) max. for $\theta = 45°, 405°, 765°$; min. for $\theta = 225°, 585°, 945°$

(b) max. for $\theta = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}$; min. for $\theta = \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}$

(c) max. for $\theta = \frac{5\pi}{4}, \frac{13\pi}{4}, \frac{21\pi}{4}$; min. for $\theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$

(d) max. for $\theta = \frac{5\pi}{3}, \frac{11\pi}{3}, \frac{17\pi}{3}$; min. for $\theta = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}$

(e) max. for $\theta = 125°, 485°, 845°$; min. for $\theta = 305°, 665°, 1025°$

(f) max. for $\theta = \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{31\pi}{6}$; min. for $\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$

Exercise 12-11 (Page 309)

(a = amplitude; p = periodicity; d = phase angle)

1. (a) $a = .5$, $p = 2\pi$, no phase difference

(b) $a = .5$, $p = 2\pi$, $d = \frac{\pi}{4}$, leads by $\frac{\pi}{4}$

(c) $a = 1$, $p = 2\pi$, $d = \frac{\pi}{4}$, leads by $\frac{\pi}{4}$

(d) $a = .5$, $p = \frac{2\pi}{3}$, no phase difference

(e) $a = 1$, $p = \frac{2\pi}{3}$, $d = \frac{3\pi}{4}$, leads by $\frac{\pi}{4}$

(f) $a = .5$, $p = \frac{2\pi}{3}$, $d = \frac{3\pi}{4}$, leads by $\frac{\pi}{4}$

2. (a) $a = 1$, $p = 2\pi$, no phase difference

(b) $a = 2$, $p = 2\pi$, $d = -\frac{2\pi}{3}$, lags by $\frac{2\pi}{3}$

(c) $a = 2$, $p = 2\pi$, no phase difference

(d) $a = 2$, $p = 2\pi$, $d = -\frac{2\pi}{3}$, lags by $\frac{2\pi}{3}$

(e) $a=1$, $p=\frac{\pi}{2}$, no phase difference

(f) $a=2$, $p=\frac{\pi}{2}$, $d=-\frac{8\pi}{3}$, lags by $\frac{2\pi}{3}$

3. $d=-\pi$ or $+\pi$, lags and/or leads by π 4. $d=\pi$, leads by $\frac{\pi}{3}$

Review Exercise 12-12 (Page 310)

(a = amplitude; p = period; d = phase angle)

1. (a) $-\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{\sqrt{2}}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$ (e) $-\frac{1}{\sqrt{2}}$ (f) $\frac{\sqrt{3}}{2}$
   (g) $-\frac{1}{2}$ (h) $-\frac{1}{\sqrt{2}}$ (i) 1 (j) 0 (k) $-\frac{\sqrt{3}}{2}$ (l) $-\frac{1}{\sqrt{2}}$
2. (a) 15 (b) $\frac{1}{3}$ (c) 4.5 (d) .25 (e) 4 (f) 1.25
3. (a) $a=1$, $d=\frac{\pi}{3}$ (b) $a=3$, $d=-\frac{\pi}{2}$
   (c) $a=2.5$, $d=50°=\frac{5\pi}{18}$ (d) $a=.375$, $d=-35°=-\frac{7\pi}{36}$
   (e) $a=7.5$, $d=\frac{3\pi}{4}$ (f) $a=4$, $d=-135°=-\frac{3\pi}{4}$
4. (a) $a=1$, $p=\frac{2\pi}{3}$ (b) $a=2$, $p=4\pi$ (c) $a=3$, $p=\pi$
   (d) $a=.25$, $p=\frac{8\pi}{3}$ (e) $a=1$, $p=24\pi$ (f) $a=7.5$, $p=\frac{8\pi}{25}$
5. (a) $a=2$, $d=-\frac{\pi}{3}$ (b) $a=1.25$, $d=30°=\frac{\pi}{6}$ (c) $a=3$, $d=\frac{5\pi}{6}$
   (d) $a=1$, $d=-120°=-\frac{2\pi}{3}$ (e) $a=3$, $d=-\frac{\pi}{2}$ (f) $a=1.5$, $d=-\frac{3\pi}{4}$
6. (a) $a=2.5$, $p=\frac{2\pi}{3}$, $d=\frac{3\pi}{4}$, leads by $\frac{\pi}{4}$ (b) $a=\frac{1}{5}$, $p=\pi$, $d=-\frac{\pi}{3}$, lags by $\frac{\pi}{6}$
   (c) $a=3$, $p=\pi$, $d=\frac{\pi}{2}$, lags by $\frac{\pi}{4}$
   (d) $a=3.5$, $p=4\pi$, $d=-12.5=-\frac{5\pi}{72}$, lags by 25°
   (e) $a=4$, $p=\frac{2\pi}{3}$, $d=45°=\frac{\pi}{4}$, leads by 15°
   (f) $a=2$, $p=\frac{2\pi}{3}$, $d=-180°=-\pi$, lags by 30°

CHAPTER 13

Exercise 13-1 (Page 313)

1. 2.8 2. 2.9 3. 3.2 4. 53 5. 37 6. 60
7. $\angle A=53°$, $\angle B=37°$, $c=150$ 8. $\angle A=24°$, $\angle B=66°$, $c=88$

9. $\angle C = 35°$, $b = 66$ in., $c = 46$ in.

10. $b = 28$ in., $c = 24$ in., $\angle A = 23°$

11. sides 28 in., angles 58°, 58°, 64°

12. 2.6 in.

13. 8.3 cm., 96°, 51°, 33°; 2.6 cm., 18°, 129°, 33°

14. AD = 12 in., AC = 17 in., $\angle ACB = 45°$, $\angle BAC = 68°$

Exercise 13-2 (Page 316)

2. $a = 51$ cm., $b = 95$ cm.

3. $a = 740$ cm., $c = 110$ cm.

4. $c = 140$

5. $b:c = \dfrac{\sqrt{3}}{2} : \dfrac{1}{\sqrt{2}} = \sqrt{\dfrac{3}{2}} : 1$

6. $a = 4$, $c = 3.8$, $\angle A = 70°$

7. (a) $a = 7.1$ cm., $c = 11$ cm., $\angle B = 65°$
   (b) $p = 12$ in., $r = 9.7$ in., $\angle Q = 40°$
   (c) $\angle X = 75°$, $y = 1.7$ in., $z = 3.6$ in.
   (d) $s = .72$ cm., $t = .73$ cm., $\angle S = 66°$
   (e) $\angle U = 33°$, $v = 48$ cm., $w = 31$ cm.

Exercise 13-3 (Page 321)

1. (i) 11 (ii) 36 (iii) 26 (iv) 13

2. (a) $a = 52$, $\angle B = 58°$, $\angle C = 58°$
   (b) $b = 9.7$, $\angle A = 93°$, $\angle C = 47°$
   (c) $c = 6.9$, $\angle B = 130°$, $\angle A = 18°$
   (d) $a = 100$, $\angle B = 72°$, $\angle C = 52°$
   (e) $b = 5.5$, $\angle A = 38°$, $\angle C = 99°$
   (f) $c = 3.7$, $\angle A = 41°$, $\angle B = 102°$

3. (i) $-.46$ (ii) .75 (iii) .84

4. (i) $\angle A = 29°$, $\angle B = 47°$, $\angle C = 104°$
   (ii) $\angle A = 37°$, $\angle B = 53°$, $\angle C = 90°$
   (iii) $\angle A = 29°$, $\angle B = 47°$, $\angle C = 104°$
   (iv) no triangle

5. (i) $\angle A = 85°$, $\angle B = 65°$
   (ii) $\angle A = 36°$, $\angle C = 99°$
   (iii) $\angle B = 41°$, $\angle C = 79°$
   (iv) $\angle A = 67°$, $\angle C = 23°$

Exercise 13-4 (Page 323)

1. $c = 2.9$, $\angle B = 100°$, $\angle C = 19°$; $c = 5.8$, $\angle B = 80°$, $\angle C = 39°$

2. $c = 2.0$, $\angle B = 111°$, $\angle C = 16°$; $c = 6.4$, $\angle B = 69°$, $\angle C = 58°$

3. $c = 21$, $\angle B = 131°$, $\angle C = 26°$; $c = 44$, $\angle B = 49°$, $\angle C = 108°$

4. no triangle

5. $c = 25$, $\angle B = 51°$, $\angle C = 54°$

6. $c = 16$, $\angle B = 30°$, $\angle C = 27°$

7. no triangle

8. $b = 48$, $\angle A = 109°$, $\angle B = 34°$; $b = 82$, $\angle A = 70°$, $\angle B = 74°$

9. $a = 47$, $\angle B = 11°$, $\angle C = 152°$

10. no triangle

11. $a = 41$, $\angle A = 32°$, $\angle B = 99°$; $a = 58$, $\angle A = 50°$, $\angle B = 81°$

12. $b = 29$, $\angle A = 43°$, $\angle B = 79°$

Exercise 13-5 (Page 326)

1. 2400 ft.

2. 28 mi. from A, 17 mi. from B

3. 14 mi. from X, 19 mi. from Y

4. 820 yd.

5. 60 nautical miles

6. 61°
7. 1700 ft. from A, 1500 ft. from B
8. 170 ft.
9. N86°E
10. 24°
11. 24 mi.
12. 130 ft.
13. 260 yd.
14. 4.2 in. and 6.4 in.

Review Exercise 13-6 (Page 328)

1. $a=11,\ c=15,\ \angle A=40°$
2. $c=21,\ \angle B=21°,\ \angle C=26°$
3. $c=39,\ \angle A=46°,\ \angle B=16°$
4. $\angle A=33°,\ \angle B=92°,\ \angle C=55°$
5. $a=25,\ b=41,\ \angle C=129°$
6. $c=13,\ \angle A=19°,\ \angle C=38°$
7. $b=11,\ c=7.9,\ \angle A=63°$
8. $\angle A=61°,\ \angle B=19°,\ \angle C=100°$
9. $a=23,\ b=12,\ \angle C=31°$
10. $a=9.7,\ \angle B=134°,\ \angle C=10°$
11. $c=25,\ \angle B=27°,\ \angle C=110°$
12. $\angle A=64°,\ \angle B=45°,\ \angle C=71°$
13. no triangle
14. $a=4.1,\ \angle B=81°,\ \angle C=44°$
15. 280 mi.
16. 9.2 mi. and 10.4 mi. respectively
17. 65 ft. from 90 ft. high building
18. 10 m.p.h.
19. 350 m.p.h. at N80°W
20. N54°E or N54°W

CHAPTER 14

Exercise 14-1 (Page 330)

1. (i) 3 (ii) 21 (iii) $n+1$ (iv) $n+2$ (v) $2k+4$ (vi) $5r$
2. (i) 3 (ii) 30 (iii) $3n+3$ (iv) 12 (v) 300 (vi) $3k$
3. (i) 50 (ii) $4n$ (iii) $4k+10$ (iv) 12 (v) $2k$ (vi) $4k+10$
4. (i) -1 (ii) 1 (iii) -1 (iv) 1 (v) $(-1)^n$ (vi) 1
   (vii) -1 (viii) -1
   Range $=\{1, -1\}$; Sequence $=\{-1, 1, -1, \ldots, (-1)^n, \ldots\}$
5. (i) 4 (ii) 2500 (iii) k^2 (iv) $k^2+10k+25$ (v) 289
   (vi) k^2-2k+1
6. $3, 2, \frac{5}{3}$
7. (a) 0, 1, 3 (b) 3, 16, 39 (c) $-1, 1, -1$ (d) 16, 32, 64
   (e) 4, 9, 16 (f) cos 1, cos 2, cos 3
8. (a) $g=\{(x, y)|y=2x;\ x\in I^+\}$ (b) $g=\{(x, y)|y=5^x;\ x\in I^+\}$
   (c) $g=\{(x, y)|y=\sqrt{3}(\sqrt{7})^{x-1};\ x\in I^+\}$
9. 1, 3, 5, 13
10. (a) 2 (b) $\frac{n+2}{5n}\leq\frac{n+2n}{5n}$ if $n\geq 1$
11. 250; 125; $1000(2^{-n})$
12. 4; $\frac{4}{3}$; $12(3^{-n})$

13. 200; 102,400; $100(2^n)$

14. (a) 89, 144, 233, 377 (b) (i) 34, 55 (ii) $-8, -13$ (iii) $-1, -2$

15. (a) 1, 2, 3, 4, 5, 6 (b) $1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}$ (c) 2, 4, 12, 48, 240, 1440

16. (a) $P(1+i), P(1+2i), P(1+3i), P(1+4i)$
(b) $P(1+i), P(1+i)^2, P(1+i)^3, P(1+i)^4$

17. $f(1)=2$; $f(n)=f(n-1)+(-2)^{4-n}$ for $n\geq 2$.

18. $f(1)=1$, $f(2)=2$, $f(n)=[f(n-1)]^2+[f(n-2)]^2$ for $n\geq 3$

Exercise 14-2 (Page 333)

1. $n+1$ 2. $2n+1$ 3. 4^n 4. n^3

5. $(\frac{1}{2})^n$ 6. 5^n 7. $(\sqrt{7})^n$ 8. $n(n+1)$

9. $\dfrac{n+2}{n+1}$ 10. $\dfrac{2^{n-1}}{3^{n-1}}$ 11. $\left(\dfrac{1}{n}\right)^3$ 12. $2+(\frac{1}{3})^n$

13. $4+(\frac{1}{2})^n$ 14. $7+\dfrac{1}{n}$ 15. $\dfrac{n(n+2)}{n+1}$ 16. 2^n+1

17. 3^n+1 18. 2^n-1 19. 3^n-1 20. 3^n+2^n

21. 3^n-2^n 22. $\dfrac{3^n+2^n}{6^n}$

23. (i) $n+1+(n-1)(n-2)(n-3)(n-4)$ (ii) $2n+1+(n-1)(n-2)(n-3)(n-4)$

Exercise 14-3 (Page 336)

2. (a) 43 (b) $5n-7$ (c) $5n-2$ (d) $10n-7$

3. (a) 1 (b) 99 (c) $4n-1$ (d) 2

4. 20 5. 729 6. No

7. (a) -25 (b) -2.5 9. 252

10. 457 11. $22x-3y$ 12. No; $3\frac{99}{100}$

13. No essentially different form.

14. $-3, 5, 13$; $8n-11$ 15. $x, x+y, x+2y, x+3y$

Exercise 14-4 (Page 339)

2. (a) $\frac{1}{5}$ (b) $20\ (10^{1-n})$

3. (a) 9 (b) 144 (c) $9(4^{2n-1})$ (d) 4

4. (a) $\frac{8}{7}, \frac{16}{49}$ (b) $\frac{128}{25}, \frac{32}{5}$ (c) $\frac{3}{5}, \frac{25}{9}$ (d) $20\sqrt{5}, 100\sqrt{2}$

5. $2500\sqrt{5}$ 7. $\frac{3}{2}(3^{n-1})$ 8. $\pm\frac{4}{3}$

10. $1000, 1000(1.05), 1000(1.05)^2$; $1000(1.05)^{n-1}$

11. \$1050, \$1102.50, \$1157.63; $1000(1.05)^n$ 12. -1 or ± 4

13. (a) 1 (b) $\frac{27}{16}$ 14. $\{15, 5, \frac{5}{3}\}$ or $\{\frac{5}{3}, 5, 15\}$

Exercise 14-5 (Page 342)

1. 20, n
2. 0, $\frac{1+(-1)^{n-1}}{2}$
3. 600, $n(n+1)+9n$
4. 420, $n(n+1)$
5. $\frac{20}{21}$, $\frac{n}{n+1}$
6. $\frac{20}{41}$, $\frac{n}{2n+1}$
7. $\frac{20}{101}$, $\frac{n}{5n+1}$
8. $\frac{20}{121}$, $\frac{n}{6n+1}$
9. 16
10. n^2+10n
11. n^2+n
12. $14n-2n^2$
13. $n(a-2)-2n^2$
15. $\frac{n^2(n+1)^2}{4}$
16. 16
17. 5

Exercise 14-6 (Page 345)

1. 880
2. -85
3. 945
4. $11a+66x$
5. 1000; 50,500
6. $x-19b$; $20x-190b$
7. 11
8. 12
9. $a=-\frac{659}{40}$, $d=\frac{121}{20}$
10. 189
12. (a) $2\frac{3}{4}$ miles (b) 30 miles
13. $a=10$, $d=-5$
14. 435 bricks
15. $-n$
16. \$147,800
17. 8, 18, 28, 38
18. 1280
19. 204
20. $21b$
21. $a^n r^{\frac{1}{2}n(n-1)}$

Exercise 14-7 (Page 348)

1. 189
2. 129
3. $\frac{.03[1-(.1)^n]}{.9}$
4. $-\frac{1}{6}[1-(-\frac{1}{2})^k]$
5. $-122\sqrt{3}-123$
6. $\frac{(x-y)^{2n}-1}{x-y-1}$
7. 83.2
8. 139
9. 2
10. 12.49
11. 29.9
12. 9.31
13. $2^{64}-1$ cents; approx. $\$1.84\times10^{17}$
14. {4, 8, 16} or {16, 8, 4}
15. {2, 6, 18} or {18, 6, 2}
17. $\frac{3}{128}$
18. $\frac{3}{2}(3^n-1)-n$
19. 5, 15, 45

Exercise 14-8 (Page 352)

1. \$3686.50
2. \$4617.50
3. \$2558.10
4. \$1507.50
5. \$54.79
6. \$597.24
7. \$922.58
8. \$199.07
9. \$13,682.00
10. 15.0 yr.

Exercise 14-9 (Page 355)

1. \$2428.40
2. \$5065.00
3. \$5546.00
4. \$319.32
5. $11.119x$ where x is the payment
6. \$129.50
7. \$773.93
8. \$1295.00

Exercise 14-10 (Page 357)

1. \$1491.80
2. $5.4478x$ where x is the amount of annuity
3. \$2174.00
4. \$3593.90
5. \$6944.60
6. \$1253.40
7. \$1005.80
8. \$822.23

Review Exercise 14-12 (Page 361)

1. 25; n^2; n^2+2n+1; $4k^2+12k+9$
2. (a) 6
3. (a) -133; $17-3n$ (b) 73
5. 960; $2n^2+8n$
6. 12
8. 7651; $\frac{7}{2}(3^n-1)$
9. $4(\frac{3}{2})^{n-1}$
10. $-\frac{2315}{128}$
11. 2
12. $3(2^n-1)$
14. \$1456.00
15. \$1188.10
16. \$6716.70
17. \$985.11
18. \$1875.60
19. \$20698.00

APPENDIX B

Exercise B-3 (Page 374)

1. $\{(2n+1)\pi,\ n\in \mathrm{I}\}$
2. $\{(2n+\frac{1}{4})\pi,\ (2n+\frac{3}{4})\pi,\ n\in \mathrm{I}\}$
3. $\{(2n+\frac{1}{6})\pi,\ (2n+\frac{7}{6})\pi,\ n\in \mathrm{I}\}$
4. $\{(2n+\frac{3}{4})\pi,\ (2n+\frac{7}{4})\pi,\ n\in \mathrm{I}\}$
5. $\{(2n+\frac{1}{3})\pi,\ (2n+\frac{2}{3})\pi,\ n\in \mathrm{I}\}$
6. $\{(2n+\frac{1}{6})\pi,\ (2n+\frac{11}{6})\pi,\ n\in \mathrm{I}\}$
7. $\{(85°+n(120°)),\ (95°+n(120°)),\ n\in \mathrm{I}\}$
8. $\{(26°+n(180°)),\ (154°+n(180°)),\ n\in \mathrm{I}\}$
9. $\{(51°+n(120°)),\ (111°+n(120°)),\ n\in \mathrm{I}\}$
10. $\{(26°+n(72°)),\ (46°+n(72°)),\ n\in \mathrm{I}\}$
11. $\{(37°+n(360°)),\ (217°+n(360°)),\ n\in \mathrm{I}\}$
12. $\{(27°+n(360°)),\ (207°+n(360°)),\ n\in \mathrm{I}\}$
13. $\{(76°+n(360°)),\ (284°+n(360°)),\ n\in \mathrm{I}\}$
14. $\{(165°+n(360°)),\ (345°+n(360°)),\ n\in \mathrm{I}\}$
15. $\left\{\frac{\pi}{8}+k\frac{\pi}{2},\ k\in \mathrm{I}\right\}$
16. $\{55°+n(90°),\ n\in \mathrm{I}\}$
17. $\{30°+n(360°),\ 150°+n(360°),\ 19°+n(360°),\ 161°-n(360°),\ n\in \mathrm{I}\}$
18. $\{(2n+1)\pi,\ 76°+n(360°),\ 284°+n(360°),\ n\in \mathrm{I}\}$
19. $\{27°+n(360°),\ 207°+n(360°),\ 64°+n(360°),\ 244°+n(360°),\ n\in \mathrm{I}\}$
20. $\{60°+n(360°),\ 120°+n(360°),\ n\in \mathrm{I}\}$

21. $\left\{\frac{\pi}{4}+n\pi,\ n\in \mathrm{I}\right\}$

22. ϕ 23. ϕ 24. ϕ

25. (a) $|\sin x|$ cannot have values greater than one for x real.
(b) The solution for $\tan x$ yields equal roots.
(c) (22) $|\cos x|$ cannot have values greater than one for x real.
(23) The values for $\cot x$ are not real.
(24) Equation reduces to $\sin x=1$ if $\cos x\neq 0$.
But $\sin x$ can equal 1 only if $\cos x=0$ since $\sin^2 x+\cos^2 x=1$.

APPENDIX C

Exercise C-1 (Page 377)

1. 1.6 ft./sec.
2. 11 rad./sec.; 105 r.p.m.
3. 59 rad./sec.; 560 r.p.m.
4. 600π rad./min.; 600π rad./min.; 450π ft./min.; 300π ft./min.
5. $1080/2\pi$ r.p.m.
6. 100π ft./min.; 105 rad./min.
7. (a) 100π ft./sec. (b) 100π ft./sec.; 200π rad./min. $=100$ r.p.m.
8. 4°/min.; 18000 m.p.h.

INDEX

A

Abel, Niels Henrik, 213
Abscissa, 10
Ambiguous case, in solution of triangles, 321
Amount, graphs of an, 114
Amplitude, change in, 308
Amplitude of $a \sin \theta$, 297
Analytic geometry of the circle, 267
Angle, between chord and tangent, 257; inscribed, 245; sector, 239; subtended, 245
Angular velocity, 375, 378
Annuities, 349; amount of, 350; deferred, 356; present value of, 353
Antilogarithm, 84, 88
Arc, of a circle, definition, 238
Area, of circle, 366; of sector, 241
Argument of a function, definition, 291
Arithmetic sequence, 334
Arithmetic series, sum of general, 343
Associativity, 2

B

Base, 49; in logarithms, other than ten, 96; real, 56, 59
Binary relations, 12

C

Cardano, Girolamo, 213
Cartesian coördinates, 10
Cartesian product sets, 10
Chord, definition, 237
Circle, area of, 366; definitions, 237; determined by three points, 237; equation of, 267
Circumcircle, of a triangle, 243
Circumference, of a circle, 239, 363
Closure, 2
Collinearity, 42
Commutativity, 2
Complementary identities, 283, 370
Completing the square, 133
Complex numbers, 215; addition of, 218; conjugate, 221, 233; division of, 223; multiplication of, 220
Compound interest, 111
Computer notation, 64
Concyclic points, definition, 247
Consistent equations, 21
Construction of circles, 260
Coördinates, 3
Cosecant, definition, 282
Cosine, definition, 282
Cosine function, domain and range, 290; graph of, 292; periodicity of, 294; zeros of, 294
Cosine law, 317
Cotangent, definition, 282
Curves, concave upward and downward, 125; symmetry of, 125
Cyclic quadrilaterals, 250; exterior angle of, 251; opposite angles supplementary, 250

D

Definitions, summary of circle, 262, 263
Degree, definition, 281
Dependent equations, 20
Diameter, of a circle, 237
Discriminant, in quadratic equations, 168
Distributivity, 2
Division, synthetic, 225
Division, using logarithms, 88
Domain, of a relation, 13; of sine and cosine functions, 290

E

Epsilon, $\in$, 119
Equation of circle, centre at origin, 279
Equations, consistent, 21; dependent, 20; equivalent, 6, 9; inconsistent, 21; indeterminate, 8; involving fractional coefficients, 6; involving literal coefficients, 7; involving logarithmic functions, 94; involving radicals, 173; linear, 17; polynomial, 187; roots of, 3; straight line, 16
Equivalent equations, 6, 19
Exponential functions, 49; 67
Exponents, 49, 56; integral, 56; rational, 59; real, 66
Exterior region, of a circle, 273

F

Factors, of polynomials, 192, 198; of quadratics, 153
Factor theorem, 194

Family, of lines, 45; of parabolas, 127, 129
Ferrari, Ludovico, 213
Ferro, Scipione del, 213
Fibonacci, 331
Floating point, arithmetic, 65
Functional notation, 36
Functions, 31, 33; exponential, 49, 67; linear, 38; logarithmic, 77, 81; of $a \sin k\theta$, $a \cos k\theta$, 302; of $a \sin (k\theta+d)$ and $a \cos (k\theta+d)$, 307

G

Geometric sequence, 337
Geometric series, sum of general, 346
Geometry, symbols used in, 236
Graph, change in amplitude, 308; change in period, 308; change in phase angle, 308
Graphs, of cosine function, 292; of cubic equation, 231; of harmonic motion, 379; of linear equations, 149; of parabola, 126, 127, 130, 144; of sine function, 292
Growth rates of exponential functions, 69

H

Harmonic motion, simple, 378

I

Identities, complementary, 283; reciprocal, 282; supplementary, 283, 370
Identity element, 2
Inclination, angle of, 40
Inconsistent equations, 21
Index, or exponent, 49
Inequalities, 24, 69; quadratic, 136, 181
Inscribed angle, 245
Integral exponents, 56
Intercepts, of a circle, 270
Interest, simple and compound, 109
Interior region, of a circle, 273
Interpolate, 38
Interpolation, linear, 84
Intersection, of straight lines, 18
Inverse element, 2
Inverse operation, 50

L

Law of cosines, 317
Law of sines, 314
Linear equations, 17
Linear functions, 38
Linear interpolation, 84
Line, slope of, 40; equations of, 44
Logarithmic functions, 81
Logarithms, 84; multiplication and division using, 88; on the number line, 100; to bases other than ten, 96

M

Major arc, definition, 238
Major segment, definition, 238
Mathematical induction, proof by, 358
Maximum and minimum values, of quadratic function, 139; 146
Minor arc, definition, 238
Minor segment, definition, 238
Monic polynomial equations, 211
Monic quadratic equations, 204, 205
Multiplication, by logarithms, 88
Multiplicative inverse, 7

N

Notation, computer, 64, 74
Notation, scientific, 63, 64, 74, 77, 89
Number line, 3
Number system, chart of, 1; integers, 1; irrationals, 1; rationals, 1; real, 1

O

Ordered pairs, 12, 13

P

Parabolas, 129
Parameter, 45
Period, change in, 308; of sine function, 301
Periodicity, of cosine function, 294; of cosine $k\theta$, 300; of sine function, 294; of sine $k\theta$, 300
Phase angle, 305; change in, 308
Phase, of $a \sin (\theta+d)$, 304
Plane, 136
Polynomials, 190, 191; zeros and factors of, 192, 198
Polynomial equations, 187; roots of 213
Powers, 49; by logarithms, 92; products and quotients of, 53
Present value, 117
Principal root, 62

Q

Quadratic equations, 149, 151, 152, 165, 187; character of roots of, 168; general, 165; problems solved by, 175; solution by completing the square, 157; solution by factoring, 152; solution by formula, 161; solved by factor theorem, 196; special, 156, 157; sum and product of roots of, 201; with given roots, 204, 206

Quadratic formula, 161
Quadratic functions, 123; maximum and minimum values of, 139
Quadratic inequalities, 136, 181

R

Radian, 240; definition, 281; 375
Radicals, in equations, 173; second order, 173
Radius, of circle, definition, 237
Range, of a relation, 13; of sine and cosine functions, 290
Real bases, 56
Reciprocal identities, 282
Relations, binary, 12; defined by inequalities, 24; domain of, 13; functions as, 31; range of, 13
Remainder theorem, 229
Right triangles, solution of, 311
Root, of an equation, 3; real, 152, 159, 168, 170; nonreal 151, 159, 168, 170, 172
Root, principal, 62
Roots, by logarithms, 92; by slide rule, 106; character of, in quadratics, 169; number of, 230; sum and product of, 201

S

Scientific notation, 63
Secant, definition, 237; extended definition, 252; as a trigonometric function, 282
Sector, 239
Sector angle, 239
Segment, of a circle, 238
Semicircle, 238
Sequence, 329; arithmetic, 334; finite, 329; geometric, 37; infinite, 329; terms of, 329
Series, 329; common ratio, 348; sum of, 340, 343, 344; general arithmetic, 343; general geometric, 346
Set, empty, 154
Significant digits, 112, 311
Simple interest, 109
Sine function, domain of, 290; maximum and minimum values, 294; periodicity, 294; range of, 290; zeros of, 294
Sine law, 314
Slide rule, 103
Slope, of a line, 40
Solution of triangles, summary, 323, 327
Solution set, 3, 4, 150, 158, 189
Solving open sentences, 3
Straight line, equations of, 16
Subtended angle, 245
Supplementary identities, 283
Symbols used in geometry, 236
Symmetry, of a figure, 271; of circle equation, 270
Synthetic division, 225

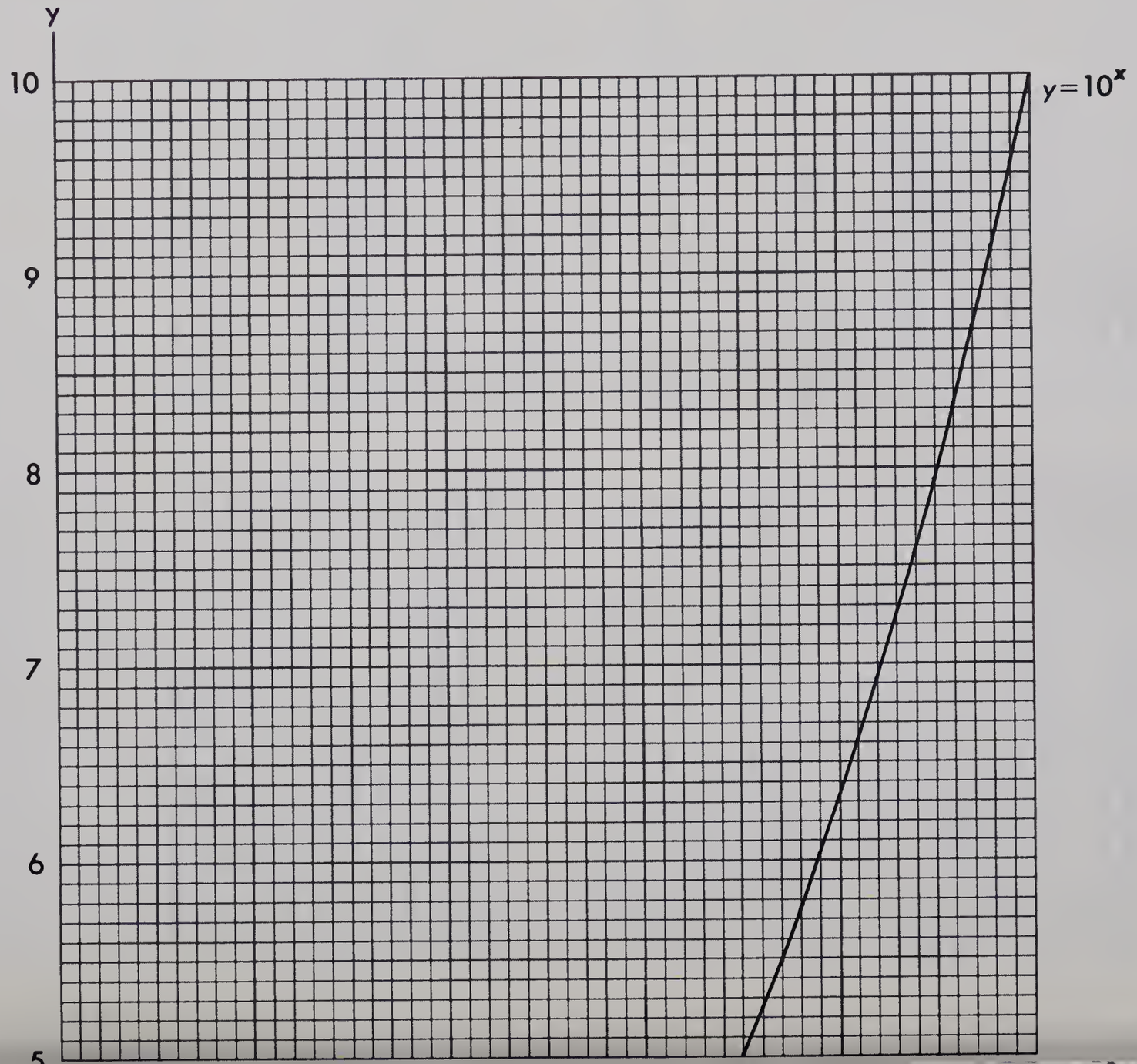
T

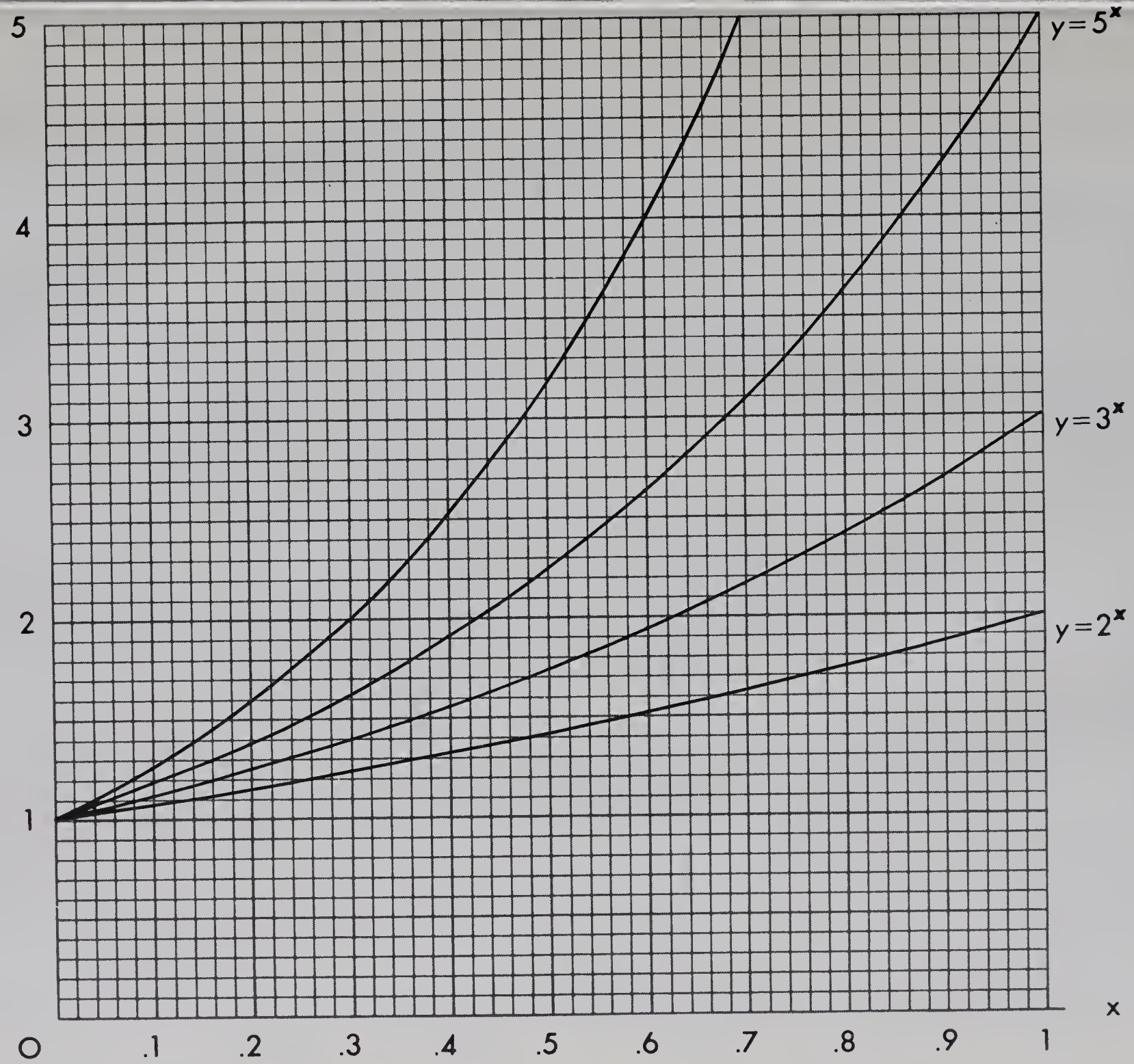
Tables of logarithmic functions, 84
Tangent, definition, 252; equation of a, 278; from an external point, 255; length of a, 277
Tangent, as a trigonometric function, definition, 282
Tartaglia, 213
Theory of equations, 211
Triangles, solution of right, 311
Trigonometric equations, 372
Trigonometric functions, 281; of any angle, 285; of positive and negative angles, 289
Trigonometric identities, 368
Trigonometry, applications of, to problems, 323

V

Variable, 3
Velocity, linear, 375
Verification of trigonometric identities, 368

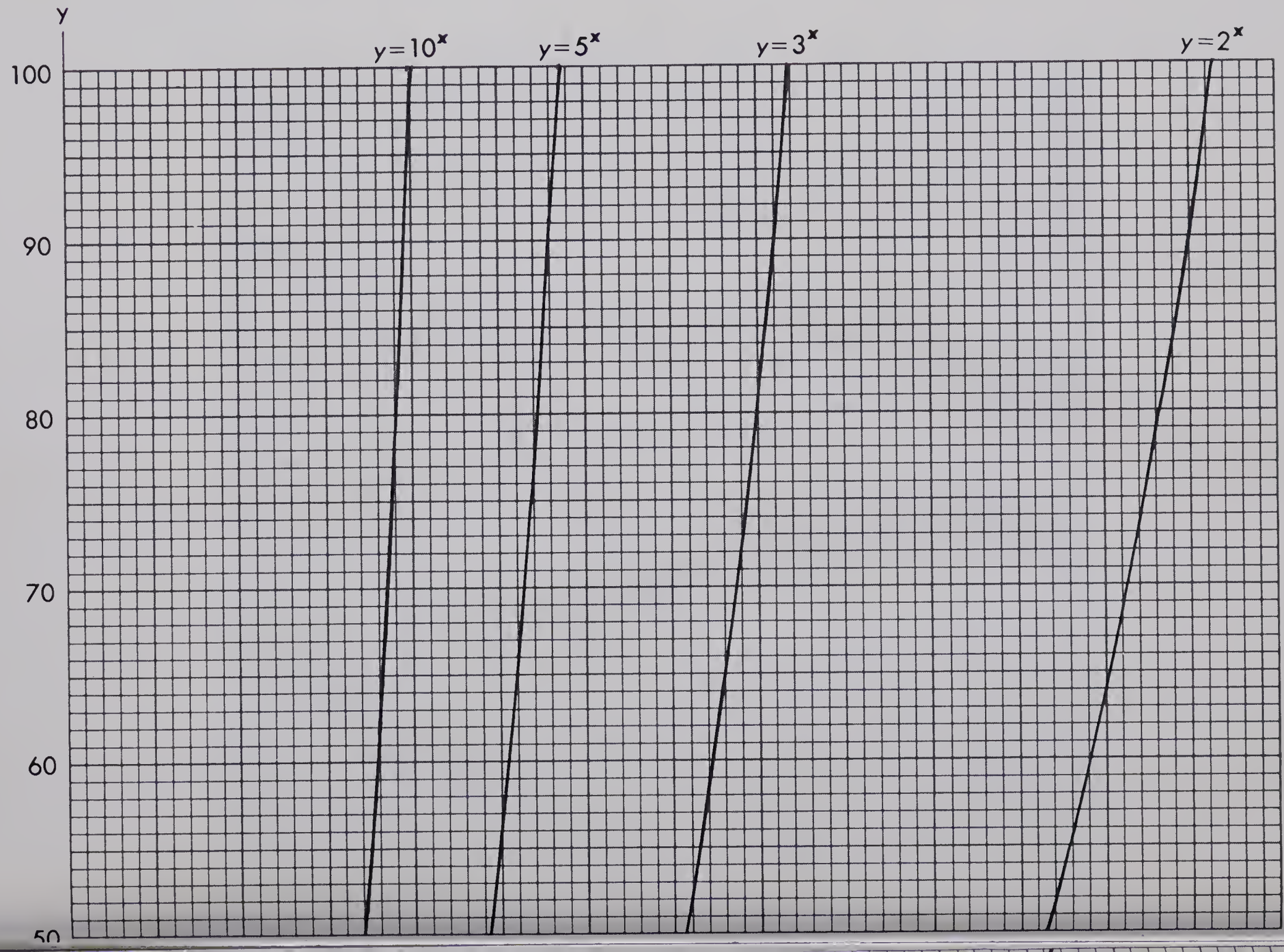
Graphs of $y=a^x$ in Range $0 \leq y \leq 10$ *

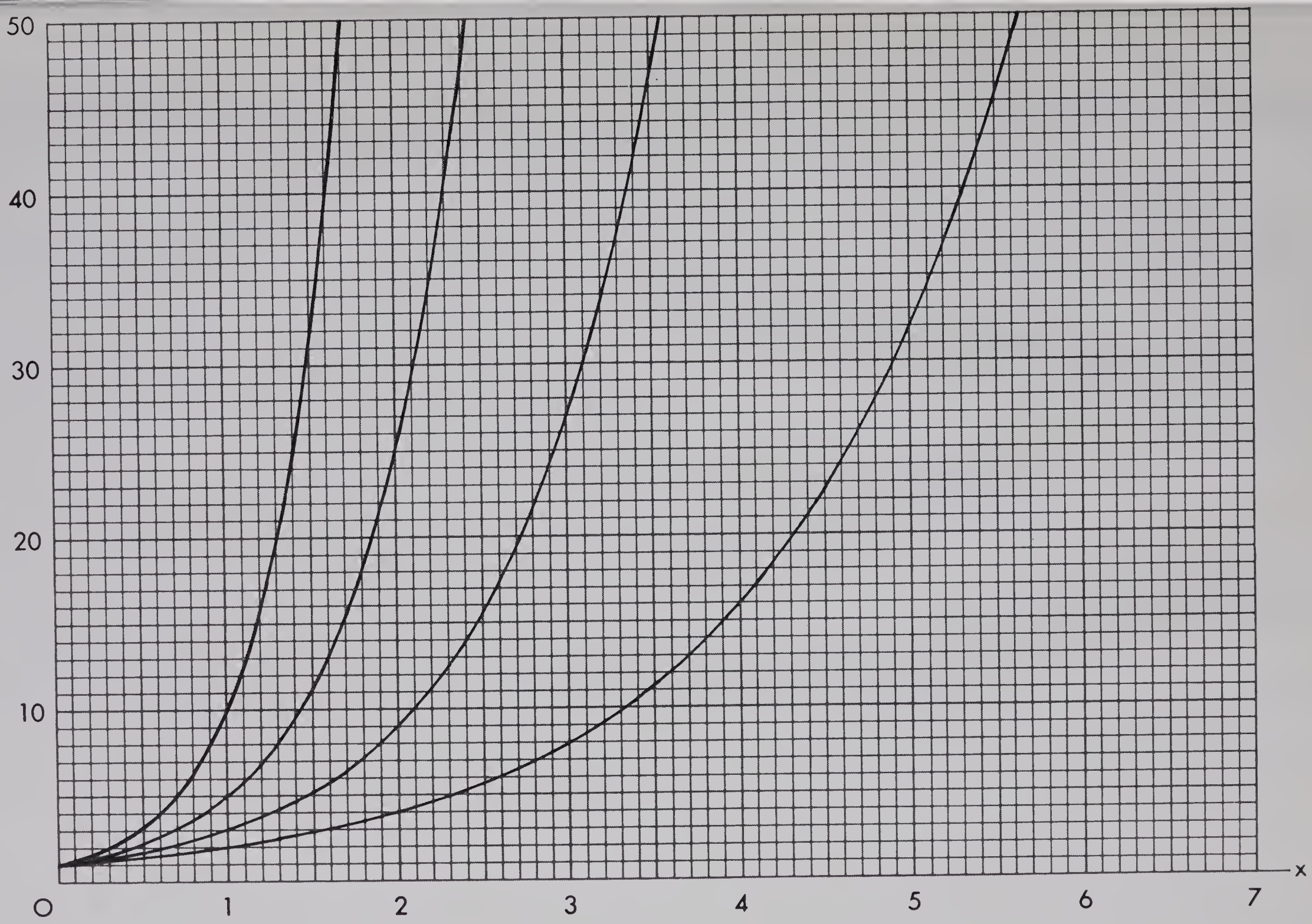




*Refer to the following two pages for the same graphs in range $0 \leq y \leq 100$.

Graphs of $y=a^x$ in Range $0 \leq y \leq 100$ *





*Refer to the preceding two pages for the same graphs in range $0 \leq y \leq 10$.

DATE DUE SLIP

DUE EDUC DEC 4 '83

DUE EDUC DEC 1 1 '83

DUE EDUC DEC 1 8 '83

DEC 1 8 RETURN

DUE EDUC MAY 1 1 '87

DUE EDUC MAY 1 8 '87

DUE EDUC MAY 26 '87

JUN 0 3 RETURN

DUE EDUC MAR 1 6 '88

DUE EDUC MAR 2 3 '88

MAR 2 2 RETURN

F. 255

QA 37-2 S47 1965
SENIOR MATHEMATICS 2/

39501986 CURR

0000201321 55

QA 37.2 S47 1965
Senior mathematics 2 /

39501986 CURR

B2979 A